MPEJ

MATHEMATICAL PHYSICS ELECTRONIC JOURNAL

ISSN 1086-6655
Volume 1, 1995

Paper 4
Received: August 25, 1995, Revised: November 16, 1995, Accepted: November 22, 1995
Editor: A. Kupiainen

Nonintersecting Planar Brownian Motions

Gregory F. Lawler [1]
Department of Mathematics
Box 90320
Duke University
Durham, NC 27708-0320

November 16, 1995

Abstract

In this paper we construct a measure on pairs of Brownian motions starting at the same point conditioned so their paths do not intersect. The construction of this measure is a start towards the rigorous understanding of nonintersecting Brownian motions as a conformal field. Let B^1, B^2 be independent Brownian motions in $\mathbf{R}^2$ starting at distinct points on the unit circle. Let T_r^j be the first time that the jth Brownian motion reaches distance r and let D_r be the event

$$D_r = \{B^1[0, T^1_{e^r}] \cap B^2[0, T^2_{e^r}] = \emptyset\}.$$

We construct the measure by considering the limit of the measure induced by Brownian motions conditioned on the event D_r.

[1]Research supported by the National Science Foundation and NSERC

1 Introduction

Nonintersecting Brownian paths give an easily described interacting field with nontrivial behavior. They have been studied from a purely mathematical perspective [4, 5], but there is also significant interest to mathematical physicists because certain techniques, such as renomalization group [2] and conformal field theory [11] can be applied to this model. Conformal field theory, in fact, has been used by Duplantier and Kwon [11] to make a nonrigorous prediction for a certain quantity, the two-dimensional intersection exponent. The starting point for their analysis is the ansatz that there exists a conformal field whose two point function is given by a Green's function for pairs of planar Brownian motions starting at one point, ending at another point, and conditioned so that their paths do not intersect. In this paper we show that this Green's function is well-defined by defining an appropriate measure on pairs of Brownian motions conditioned not to intersect.

Let B_t^1, B_t^2 be independent Brownian motions taking values in $\mathbf{R}^2$, which we also consider as the complex numbers $\mathbf{C}$, starting at x_1, x_2. For $r > 0$, let

$$T_r^j = \inf\{t : |B_t^j| = r\},$$

and let $D_r = D_r(x_1, x_2)$ be the event

$$D_r = \{B^1[0, T_{e^r}^1] \cap B^2[0, T_{e^r}^2] = \emptyset\}.$$

For $r \geq 1$, let

$$q(r) = \sup\{\mathbf{P}[D_r(x_1, x_2)] : |x_1| = |x_2| = 1\}.$$

By the strong Markov property and Brownian scaling, it is easy to see that

$$q(r + s) \leq q(r)q(s),$$

and hence, using the the subadditivity of $\log q$ (see [16]), there exists a $\zeta > 0$, sometimes called the intersection exponent, such that

$$q(r) \approx e^{-2\zeta r}, \quad r \to \infty, \tag{1}$$

where $\approx$ indicates that the logarithms of both sides are asymptotic. Moreover, $q(r) \geq e^{-2\zeta r}$. The value of ζ is not known rigorously. It has been conjectured [11] using a nonrigorous conformal field theory argument that $\zeta = 5/8$, and simulations [11, 6, 19] are consistent with this conjecture. The best rigorous bounds on ζ are [5]

$$\frac{1}{2} + \frac{1}{8\pi} \leq \zeta < \frac{3}{4}.$$

The probability of intersection of mean zero, finite variance random walks can also be estimated in terms of ζ [4, 9].

In this paper we consider the measure on Brownian paths induced by pairs of paths that do not intersect. We show that a limiting measure exists which we can call "nonintersecting Brownian motion". From the construction it will be clear that this can also be considered as the distribution of a single Brownian motion at a "typical" cut point. We call $t \in (0, 1)$ a cut time and $B(t)$ a cut point for a Brownian motion B on [0,1] if $B[0, t) \cap B(t, 1] = \emptyset$. Although each t is a cut time with probability zero, two dimensional Brownian motions do have cut points [3]; in fact [17], the Hausdorff dimension of the set of cut times is $1 - \zeta$. The local distribution of a Brownian motion near a cut point is the same as the local distribution of two Brownian motions (the "past" and the "future") starting at a point conditioned not to intersect.

As a corollary of the work we improve the estimate in (1) by showing there is a constant $c > 0$ such that

$$q(r) \sim ce^{-2\zeta r}, \tag{2}$$

where $\sim$ means asymptotic. (Throughout this paper $c, c_1, c_2, \ldots$ denote positive constants. The values of c, c_1, c_2 may change from place to place, but the values of $c_3, c_4, \ldots$ do not.) A somewhat different argument was given in [17] to show that $e^{-2\zeta r} \leq q(r) \leq c_2 e^{-2\zeta r}$. That argument had the advantage that it works for three dimensions as well as two. We will not use the results from that paper, however, because they consider only the case of two random walks. The methods in this paper can be used to estimate the nonintersection or disconnection probability for any number of walks (see Section 7). The methods here apply only to two dimensions, however.

Let us describe the main result. For $0 \leq \epsilon < 1$ and $b > 0$, let $\mathcal{X}_\epsilon^b$ denote the set of continuous functions $\gamma : [0, b] \rightarrow \mathbf{C}$ satisfying: $|\gamma(0)| = \epsilon; |\gamma(b)| = 1$; and $|\gamma(t)| < 1, t < b$. We let $\mathcal{X}_\epsilon = \cup_b \mathcal{X}_\epsilon^b$ and for $\gamma \in \mathcal{X}_\epsilon$ we write $b(\gamma)$ or b_γ for the unique b with $\gamma \in \mathcal{X}_\epsilon^b$. We will let d denote the metric on continuous functions that says that

$$d(\gamma, \eta) < u,$$

if and only if there exists an increasing, bijective time change $\sigma : [0, b_\gamma] \rightarrow [0, b_\eta]$ such that for all $t \in [0, b_\gamma]$,

$$|t - \sigma(t)| < u,$$

and

$$|\gamma(t) - \eta(\sigma(t))| < u.$$

Let $\mathcal{A}_\epsilon$ denote the set of ordered pairs $\bar{\gamma} = (\gamma^1, \gamma^2)$ with $\gamma^1, \gamma^2 \in \mathcal{X}_\epsilon$. We will write $\mathcal{X}$ and $\mathcal{A}$ for $\mathcal{X}_0$ and $\mathcal{A}_0$, respectively. If $\delta < \epsilon$ there is a function

$$\Phi_\epsilon : \mathcal{X}_\delta \rightarrow \mathcal{X}_\epsilon,$$

defined as follows. Let $\gamma \in \mathcal{X}_\delta^b$. Then set

$$\rho = \inf\{t : |\gamma(t)| = \epsilon\},$$

$$b(\Phi_\epsilon \gamma) = b - \rho,$$

$$\Phi_\epsilon \gamma(t) = \gamma(t + \rho), \quad 0 \leq t \leq b - \rho.$$

Similarly, $\Phi_\epsilon : \mathcal{A}_\delta \rightarrow \mathcal{A}_\epsilon$ is defined component-wise. Note that Φ_ϵ is a projection in the sense that if $\epsilon_1 < \epsilon_2$, $\Phi_{\epsilon_2}\Phi_{\epsilon_1} = \Phi_{\epsilon_2}$.

Wiener measure, W, on $\mathcal{A}$ is the measure obtained by starting independent Brownian motions B_t^1, B_t^2 at the origin; letting T_1^j be as defined above; and setting

$$\gamma^j(t) = B^j(t), \quad 0 \leq t \leq T_1^j.$$

Wiener measure, W_ϵ on $\mathcal{A}_\epsilon$ is obtained by projecting W on $\mathcal{A}_\epsilon$ by the mapping Φ_ϵ. Equivalently, we can choose x_1, x_2 independently from the uniform distribution on $\{x : |x| = \epsilon\}$; start independent Brownian motions B_t^1, B_t^2 at x_1, x_2, respectively; and set

$$\gamma^j(t) = B^j(t), \quad 0 \leq t \leq T_1^j.$$

Let $\mathcal{D} = \mathcal{D}_0$ be the set of $(\gamma^1, \gamma^2) \in \mathcal{A}$ such that

$$\gamma^1(0, b(\gamma^1)] \cap \gamma^2(0, b(\gamma^2)] = \emptyset.$$

Similarly, for $\epsilon > 0$, let $\mathcal{D}_\epsilon \subset \mathcal{A}_\epsilon$ be the set of ordered pairs $\bar{\gamma} = (\gamma^1, \gamma^2)$ such that

$$\gamma^1[0, b(\gamma^1)] \cap \gamma^2[0, b(\gamma^2)] = \emptyset.$$

It is well known [14] that $W(\mathcal{D}) = 0$, although $W_\epsilon(\mathcal{D}_\epsilon) > 0$ for each $\epsilon > 0$. For any $\epsilon > 0$ let f_ϵ denote the density with respect to W_ϵ of the probability measure on $\mathcal{A}_\epsilon$ induced by conditioning on $\mathcal{D}_\epsilon$, i.e.,

$$f_\epsilon = aI(\mathcal{D}_\epsilon),$$

where I denotes indicator function and the constant a has been chosen so that

$$\int_{\mathcal{A}_\epsilon} f_\epsilon \, dW_\epsilon = 1.$$

For $\delta < \epsilon$, let $f_{\epsilon,\delta}$ be the density, with respect to W_ϵ, of the measure induced by projecting $f_\delta \, dW_\delta$ on $\mathcal{A}_\epsilon$ by Φ_ϵ. We will prove the following theorem.

Theorem 1.1 *There exist c_3, c_4 such that for every $\epsilon > 0$ there is a function $\bar{f}_\epsilon \geq 0$ on $\mathcal{A}_\epsilon$ such that for every $\delta < \epsilon$ satisfying $|\log \delta| \geq (\log \epsilon)^2$,*

$$\int_{\mathcal{A}_\epsilon} |f_{\epsilon,\delta} - \bar{f}_\epsilon| \, dW_\epsilon \leq c_3 \exp\{-c_4 \sqrt{|\log \delta|}\}.$$

In particular, the measures $f_{\epsilon,\delta} \, dW_\epsilon$ converge to the measure $\mu_\epsilon = \bar{f}_\epsilon \, dW_\epsilon$ as $\delta \to 0$. Since Φ_ϵ is a projection, it is easy to check that the measures $\{\mu_\epsilon\}$ must be consistent, i.e., for each $\delta < \epsilon$ the projection of μ_δ on $\mathcal{A}_\epsilon$ by Φ_ϵ is μ_ϵ. We will also show that there exists an $M < \infty$ such that for all $\epsilon > 0$,

$$\int_{\mathcal{A}_\epsilon} |b(\gamma^1) + b(\gamma^2)| \, d\mu_\epsilon \leq M. \tag{3}$$

Hence, by standard arguments, we see that the measures $\{\mu_\epsilon\}$ induce a probability measure μ on $\mathcal{A}$ which is supported on $\mathcal{D}$. [The Kolmogorov construction (see, e.g., [10, 2.I.10]) is used to show that their is a measure μ with the appropriate finite dimensional distributions, and then (3) is used to show that the paths under this measure have a finite lifetime with probability one.] Since μ is supported on $\mathcal{D}$, μ is singular with respect to W. We call the set of paths $\bar{\gamma}$ under the measure μ, "nonintersecting Brownian motion."

The measure μ can also be considered as an eigenstate for a Markov chain. Consider the following continuous time, time homogeneous Markov chain with state space $\mathcal{A}$. Let $\bar{\gamma} = (\gamma^1, \gamma^2) \in \mathcal{A}$ with $b^j = b(\gamma^j), x_j = \gamma^j(b^j)$. Start independent Brownian motions B_t^1, B_t^2 at x_1, x_2, respectively. Let $X_0 = \bar{\gamma}$ and for $s > 0$ let X_s be the element of $\mathcal{A}$ obtained by attaching $B^j[0, T_{e^s}^j]$ to γ^j and then scaling to $\{x : |x| \leq 1\}$. To be precise, we define $X_s = (\gamma_s^1, \gamma_s^2)$ where

$$b(\gamma_s^j) = e^{-2s}[b^j + T_{e^s}^j],$$

$$\gamma_s^j(t) = \begin{cases} e^{-s}\gamma^j(e^{2s}t), & 0 \leq t \leq e^{-2s}b^j, \\ e^{-s}B^j(e^{2s}t - b^j), & e^{-2s}b^j \leq t \leq e^{-2s}[b^j + T_{e^s}^j]. \end{cases}$$

Note that Wiener measure on $\mathcal{A}$ is the invariant measure for this chain, and for any $\bar{\gamma} \in \mathcal{A}$, the measure on X_s conditioned that $X_0 = \bar{\gamma}$ approaches Wiener measure as $s \to \infty$. Note also that $\mathcal{D}^c$ is an absorbing set for this chain. What we will show is that if we start with $X_0 = \bar{\gamma} \in \mathcal{D}$ and we consider the distribution given by X_s conditioned that $X_0 = \bar{\gamma}$ and $X_s \in \mathcal{D}$, then this distribution converges to μ. In particular, if we start with initial probability distribution μ on X_0, the conditional distribution of X_s given $X_s \in \mathcal{D}$ is μ. By definition of ζ, it is clear that if we start with distribution μ on X_0, then

$$\mathbf{P}\{X_s \in \mathcal{D}\} = e^{-2\zeta s} .$$

Let $\bar{\Gamma} = (\Gamma^1, \Gamma^2)$ be an ordered pair of disjoint closed subsets of $\{x : |x| \leq 1\}$ such that Γ^j contains exactly one point of norm 1, which we denote as x_j. We will refer to such a $\bar{\Gamma}$ as an initial configuration. If $\bar{\gamma} = (\gamma^1, \gamma^2) \in \mathcal{D}_\epsilon$ we will also write $\bar{\gamma}$ for the initial configuration (Γ^1, Γ^2) with $\Gamma^j = \gamma^j[0, b(\gamma_j)]$. Start independent Brownian motions, B_t^1, B_t^2, at x_1, x_2, respectively, and let

$$\Gamma_s^j = \Gamma^j \cup B^j[0, T_{e^s}^j].$$

Let $D_s = D_s(\bar{\Gamma})$ be the event

$$D_s = \{\Gamma_s^1 \cap \Gamma_s^2 = \emptyset\}.$$

We will prove the following.

Theorem 1.2 *For any initial configuration* $\bar{\Gamma}$,

$$\lim_{s \to \infty} e^{2\zeta s} \mathbf{P}(D(\bar{\Gamma})) = \psi(\bar{\Gamma}),$$

exists. Moreover there exist c_5, c_6 *such that if we define* $\delta_s = \delta_s(\bar{\Gamma})$ *by*

$$\mathbf{P}(D(\bar{\Gamma})) = \psi(\bar{\Gamma}) e^{-2\zeta s}(1 + \delta_s),$$

then

$$|\delta_s| \leq c_6 e^{-c_5 \sqrt{s}}.$$

Also,

$$c_5 \mathbf{P}[D_1(\bar{\Gamma})] \leq \psi(\bar{\Gamma}) \leq c_6 \mathbf{P}[D_1(\bar{\Gamma})].$$

The outline of this paper is as follows. In section 2 we conformally transform $\mathbf{C}$ into an infinite cylinder by the logarithm. It is more convenient (although not really necessary) to consider the Brownian motions taking values on the cylinder. The logarithm transforms the radial direction to the real axis of the cylinder and power law decay is transformed into exponential decay. The real axis can now be handled in the same way that a "time" variable is considered, and this converts the problem to a question of convergence of a Markov chain whose state space is a space of paths. This technique is sometimes referred to as radial quantization (see [15, 9.1.5]). The necessary estimates for Brownian motions are derived in Section 3, and the main convergence result is proved in the next two sections. A Markov chain on "excursions" is analyzed in Section 4, and a the rate of convergence to equilibrium is estimated. This is used to derive the main result in Section 5. Sections 3-5 are done for Brownian motions on the cylinder, but the results can easily be transformed by the exponential map to results about Brownian motions in $\mathbf{C}$.

In Section 6 we define the "h-process" associated with Brownian paths conditioned never to intersect. (The term h-process is used by probabilists (see [10]) for Brownian motions conditioned on certain events of probability zero. These are often defined in terms of a harmonic function h from which the name comes. We will not need to use any results from the theory of h-processes, but it seems appropriate to use this term for our construction.) This gives a measure on pairs of paths going to ∞ and never intersecting. By conformal transformation we can use this measure to define a measure on pairs of paths starting at one point z_1, conditioned to end at another point z_2, and conditioned so that the paths do not intersect. The starting point for the nonrigorous conformal field theory predictions for ζ is the assumption that a "conformal field" exists whose two-point function is given by the Green's function of two Brownian motions starting at one point, ending at another point, conditioned so that the paths have no intersection. (See [8, 15] for discussions of the methods of conformal field theory and [11] for the particular example of nonintersecting Brownian motions.) While we are far from constructing a field with this two-point function, we have made a start by constructing a measure on nonintersecting Brownian motions which allows one to define such a Green's function.

We have restricted our discussion in these sections to pairs of Brownian motions. However, the same techniques can be used to derive results about other configurations of Brownian motions restricted either to have no intersections or to not "disconnect" points. We discuss these generalizations in Section 7 without proofs (the proofs follow the same basic line as for pairs of nonintersecting Brownian motions). In Section 8 we discuss how these limiting measures can be used to derive an inequality for exponents. This paper was written while the author was visiting the University of British Columbia.

2 Cylinder

It is more convenient to transform the complex plane $\mathbf{C}$ to an infinite cylinder. Let $\bar{\mathbf{C}}$ denote the infinite cylinder derived from the complex numbers $\bar{\mathbf{C}}$ by the equivalence relation $z_1 \sim z_2$ if $z_1 - z_2 = 2\pi j i$ for some integer j. We will use $\Re(z)$ and $\Im(z)$ to denote the real and imaginary parts of z; if $z \in \bar{\mathbf{C}}$, $\Im(z)$ is defined only up to integer multiples of 2π. Note that the exponential function gives a conformal transformation of $\bar{\mathbf{C}}$ onto $\mathbf{C} \setminus \{0\}$. Let F denote the logarithm function, i.e., the inverse of the exponential, which takes $\mathbf{C} \setminus \{0\}$ to $\bar{\mathbf{C}}$. If $\gamma : [0, b] \rightarrow \bar{\mathbf{C}}$, we define the function $G(\gamma) = G_\gamma$ as follows. Let

$$\sigma(t) = \int_0^t \exp\{2\Re(\gamma(s))\} ds.$$

Then $G_\gamma : [0, \sigma(b)] \rightarrow \mathbf{C}$,

$$G_\gamma(t) = \exp\{\gamma(\sigma^{-1}(t))\}.$$

Similarly, if $\gamma : (-\infty, 0] \rightarrow \bar{\mathbf{C}}$ is a continuous function with

$$\int_{-\infty}^0 \exp\{2\Re(\gamma(s))\} \, ds = b_\gamma < \infty,$$

we let $G_\gamma : [0, b_\gamma] \rightarrow \mathbf{C}$ be defined by

$$\sigma(t) = \int_t^0 \exp\{2\Re(\gamma(s))\} \, ds,$$

$$G_\gamma(t) = \begin{cases} \exp\{\gamma(\sigma^{-1}(t))\}, & t > 0 \\ 0, & t = 0. \end{cases}$$

The scaling of time is not arbitrary. It is chosen so that the following result holds. See [13] for a proof.

Proposition 2.1 *If B_t is a standard Brownian motion taking values in $\bar{\mathbf{C}}$, then $Y_t = G_B(t)$ is a standard Brownian motion taking values in $\mathbf{C}$.*

Let $R_a = \{z \in \bar{\mathbf{C}} : \Re(z) = a\}$. Let $\mathcal{Y}$ denote the set of continuous functions $\gamma : (-\infty, 0] \to \bar{\mathbf{C}}$ satisfying

$$\gamma(0) \in R_0,$$

$$\Re(\gamma(t)) < 0, \quad t < 0,$$

$$\int_{-\infty}^{0} \exp\{2\Re(\gamma(t))\}\, dt < \infty. \tag{4}$$

In particular, $\Re(\gamma(t)) \to -\infty$ as $t \to -\infty$. There is a one-to-one correspondence between $\mathcal{Y}$ and $\mathcal{X}$ given by $\gamma \mapsto G_\gamma$. Let $\mathcal{G}$ be the set of ordered pairs $\bar{\gamma} = (\gamma^1, \gamma^2)$ with $\gamma^j \in \mathcal{Y}$. For any $t > 0$, let $\mathcal{Y}_a^b$ be the set of functions $\gamma : [0, b] \to \bar{\mathbf{C}}$ with

$$\gamma(0) \in R_{-a}, \quad \gamma(b) \in R_0,$$

$$\Re(\gamma(t)) < 0, \quad 0 \le t < b.$$

Let $\mathcal{Y}_a = \cup_b \mathcal{Y}_a^b$ and let $\mathcal{G}_a = (\mathcal{Y}_a)^2$. Note that $\mathcal{G}, \mathcal{Y}_a, \mathcal{G}_a$ correspond to $\mathcal{A}, \mathcal{X}_{e^{-a}}, \mathcal{A}_{e^{-a}}$, respectively. For any $\gamma \in \mathcal{Y}$ we will let $\Psi_a \gamma$ be defined by the operation $G^{-1}[\Phi_{e^{-a}} G(\gamma)]$. In other words, let

$$b = \sup\{t : \gamma(-t) \in R_{-a}\},$$

and let $\Psi_a \gamma : [0, b] \to \bar{\mathbf{C}}$,

$$\Psi_a \gamma(t) = \gamma(t - b), \quad 0 \le t \le b.$$

If $r < a$ and $\gamma \in \mathcal{Y}_r$, we can define $\Psi_a \gamma$ similarly.

Wiener measure on $\mathcal{G}_a$, which we denote by W_a, can be derived from Wiener measure on $\mathcal{A}_{e^{-a}}$ by the map G^{-1}. Equivalently, we can take $x_1, x_2 \in R_{-a}$ chosen independently according to the uniform distribution on R_{-a}; let B_t^1, B_t^2 be independent Brownian motions taking values in $\bar{\mathbf{C}}$ starting at x_1, x_2, respectively; and set

$$\tau^j = \tau_0^j = \inf\{t : B_t^j \in R_0\},$$

$$\gamma^j(t) = B_t^j, \quad 0 \le t \le \tau^j.$$

Let $\mathcal{H} = G(\mathcal{D})$, i.e., $\mathcal{H}$ is the set of $\bar{\gamma} = (\gamma^1, \gamma^2) \in \mathcal{G}$ such that

$$\gamma^1(-\infty, 0] \cap \gamma^2(-\infty, 0] = \emptyset,$$

and similarly $\mathcal{H}_a = G(\mathcal{D}_{e^{-a}})$.

We consider the Markov chain on $\mathcal{G}$ defined as follows. Suppose $\gamma \in \mathcal{G}$. Let B^1, B^2 be independent Brownian motions in $\bar{\mathbf{C}}$ starting at $x_1 = \gamma^1(0), x_2 = \gamma^2(0)$, respectively. Let

$$\tau_s^j = \inf\{t : B_t^j \in R_s\},$$

$$\gamma_s^j(t) = \begin{cases} \gamma(t+\tau_s^j) - s, & -\infty < t \leq -\tau_s^j \\ B^j(t+\tau_s^j) - s, & -\tau_s^j \leq t \leq 0, \end{cases}$$

$$Y_s = (\gamma_s^1, \gamma_s^2).$$

This is just the conformal transformation of the Markov chain discussed in the previous section.

3 Estimates for Brownian motion

In this section we consider independent Brownian motions B, B^1, B^2 taking values in the cylinder $\bar{\mathbf{C}}$. We let

$$\tau_a = \inf\{t : B_t \in R_a\},$$

$$\tau_a^j = \inf\{t : B_t^j \in R_a\}.$$

It is well known that if $a, b > 0$ and $B_0 \in R_0$, then

$$\mathbf{P}\{\tau_b < \tau_{-a}\} = \frac{a}{a+b}.$$

We will refer to this fact as the gambler's ruin estimate. We state two lemmas without proof. The first is a version of the Beurling projection theorem (see [1]) and the second can be derived from the first by using an appropriate conformal mapping.

Lemma 3.1 *There exists a $c_7 < \infty$ such that if $a > 0$, $\gamma : [0,1] \to \bar{\mathbf{C}}$ is a continuous function with $\gamma(0) \in R_0, \gamma(1) \in R_a$ and $B(0) \in R_0$, then*

$$\mathbf{P}\{B[0,\tau_a] \cap \gamma[0,1] = \emptyset\} \leq c_7 e^{-a/2}.$$

Moreover,

$$\mathbf{P}\{B[0,\tau_{-a}] \cap \gamma[0,1] \cap \{\Re(z) \geq \frac{a}{2}\} \neq \emptyset\} \geq 1/c_7.$$

Lemma 3.2 *For any $\epsilon, \delta > 0$ and $r \in \mathbf{R}$, let L_1 be the line segment connecting $-\epsilon i$ to $1 + (r-\delta)i$ and L_2 the line segment connecting ϵi to $1 + (r+\delta)i$. Let $D = D(\epsilon, \delta, r)$ be the bounded open domain bounded by R_1, L_1, L_2, and*

$$\{z \in \bar{\mathbf{C}} : |z| = \epsilon, \Re(z) \leq 0\}.$$

Let $\sigma = \sigma(\epsilon, \delta, r)$ be the first time that Brownian motion leaves the region D. Then for every $\delta > 0$ there exist $u > 0$ and $\alpha < \infty$ such that if $\epsilon > 0$, $r \in [-2\pi, 2\pi]$, and B is a Brownian motion starting at the origin, then

$$\mathbf{P}\{B(\sigma) \in R_1\} \geq u\epsilon^{\alpha}.$$

Let $\bar{\Gamma} = (\Gamma^1, \Gamma^2)$ be an initial configuration, i.e., Γ^1, Γ^2 are disjoint closed subsets of $\{z \in \bar{\mathbf{C}} : \Re(z) \leq 0\}$ such that $\Gamma^j \cap R_0$ consists of a single point, which we denote x_j. Let

$$Y = Y(\bar{\Gamma}) = \min\{\mathrm{dist}(x_1, \Gamma^2), \mathrm{dist}(x_2, \Gamma^1)\}.$$

Note that $Y > 0$ since the Γ^j are closed and disjoint. Let B^1, B^2 be independent Brownian motions starting at x_1, x_2, and for $a > 0$, let $\Gamma_a^j = \Gamma^j \cup B^j[0, \tau_a^j]$ and let $D_a = D_a(\bar{\Gamma})$ be the event

$$D_a = \{\Gamma_a^1 \cap \Gamma_a^2 = \emptyset\}.$$

Let

$$q(a) = \sup \mathbf{P}(D_a),$$

where the supremum is over all $x_1, x_2 \in R_0$ and $\Gamma^j = \{x_j\}$. (We will make the convention that when no initial configuration is given then the assumption is that $\Gamma^j = \{x_j\}$.) It can be shown [9] that the supremum is taken on when $|x_1 - x_2| = \pi$, but we will not need to use that fact here. The strong Markov property immediately gives that

$$q(a+b) \leq q(a)q(b).$$

Hence, there exists a λ such that as $a \to \infty$,

$$q(a) \approx e^{-\lambda a}.$$

Moreover, $q(a) \geq e^{-\lambda a}$. It is easy to see that $q(a)$ is the same as the $q(a)$ defined in Section 1 and hence $\lambda = 2\zeta$ where ζ is the intersection exponent. The following lemma is an easy corollary of Lemma 3.2.

Lemma 3.3 *There exist $c_8 > 0$ and $\alpha > 0$ such that if $\bar{\Gamma}$ is any initial configuration with*

$$Y(\bar{\Gamma}) \geq \epsilon,$$

then

$$\mathbf{P}(D_1) \geq c_8 \epsilon^{\alpha}.$$

The next lemma is a very important technical lemma. It gives a uniform estimate for the intuitive statement "Brownian motions which are conditioned not to intersect have a good chance of being a reasonable distance apart." Let

$$Y_a = \min\{\mathrm{dist}(B^1(\tau_a^1), \Gamma_a^2), \mathrm{dist}(B^2(\tau_a^2), \Gamma_a^1)\}.$$

Lemma 3.4 *There exists a $c_9 > 0$ such that for any initial configuration $\bar{\Gamma}$,*

$$\mathbf{P}(D_1; Y_1 \geq \frac{1}{2}) \geq c_9 \mathbf{P}(D_1).$$

Proof. For every $1/2 \leq \rho \leq 1$, let $V(\rho)$ be the event

$$V(\rho) = \{D_1; Y_a \geq \frac{1}{2}, \rho \leq a \leq 1\}.$$

For any $\epsilon > 0$, it is easy to see by direct construction of an event that there is a $u_\epsilon > 0$ such that for any initial configuration with $Y = Y_0 \geq \epsilon$,

$$\mathbf{P}[V(1/2)] \geq u_\epsilon.$$

Choose integer N sufficiently large so that

$$\sum_{n=N}^{\infty} n^2 2^{-n} \leq \frac{1}{8}.$$

For $n > N$, let

$$h_n = \frac{1}{2} + \sum_{m=N+1}^{n} m^2 2^{-m}.$$

Let

$$r(n) = \inf \frac{\mathbf{P}[V(h_n)]}{\mathbf{P}[D(h_n)]},$$

where the infimum is over all initial configurations with $Y_0 \geq 2^{-n}$. By the comment above, $r(n) > 0$ for each fixed n. We will show below that there is a $c_2 > 0$ such that for all $n > N$,

$$r(n) \geq (1 - \frac{c_2}{n^2}) r(n-1), \tag{5}$$

and hence there exists a $c_1 > 0$ such that $r(n) \geq c_1$ for all n. This clearly gives the lemma.

By the Beurling estimates (Lemma 3.1), there exists a $\beta < 1$ such that for any $u > 0$ and any initial configuration with $Y_0 \leq 2u$,

$$\mathbf{P}(D_u) \leq \beta. \tag{6}$$

Choose $n > N$, and assume the initial configuration satisfies $Y_0 \geq 2^{-n}$. Let

$$\sigma = \sigma_n = \inf\{a \geq 0 : Y_a \geq 2^{-(n-1)}\}.$$

Let $q = q_n = n^2 2^{-n}$. By iterating (6), we see that

$$\mathbf{P}(\sigma \geq q; D_q) \leq \beta^{n^2} = 2^{-tn^2},$$

for some $t > 0$. But, by Lemma 3.3,

$$\mathbf{P}(D_q) \geq \mathbf{P}(D_1) \geq c_8 2^{-n\alpha}.$$

Hence there exists a $c_2 > 0$ such that

$$\frac{\mathbf{P}(D_q; \sigma \leq q)}{\mathbf{P}(D_q)} \geq 1 - \frac{c_2}{n^2}.$$

However, by definition of $r(n)$,

$$\mathbf{P}[V(h_n) \mid \sigma \leq q; D_q] \geq r(n-1)\mathbf{P}[D(h_n) \mid \sigma \leq q]. \quad \Box$$

We will need the following slightly stronger version of the lemma which can be proved in the same way.

Lemma 3.5 *There exists a $c_9 > 0$ such that for any $s > 1$ and any initial configuration $\bar{\Gamma}$,*

$$\mathbf{P}(D_1; Y_1 \geq \frac{1}{2}; \tau^1_{-s} > \tau^1_1; \tau^2_{-s} > \tau^2_1) \geq c_9 \mathbf{P}(D_1; \tau^1_{-s} > \tau^1_1; \tau^2_{-s} > \tau^2_1).$$

It seems that there should be a constant $c > 0$ such that for all initial configurations and all $s \geq 1$,

$$\mathbf{P}(D_1; \tau^1_{-s} > \tau^1_1; \tau^2_{-s} > \tau^2_1) \geq c\mathbf{P}(D_1),$$

but we do not have a proof. Fortunately, we will be able to prove what we need with Lemmas 3.4 and 3.5. It follows immediately from Lemma 3.4 that if we start with any initial configuration, and $a > 1$,

$$\mathbf{P}\{Y_a \geq \frac{1}{2} \mid D_a\} \geq c_9.$$

However, it is easy to see that if $Y_a \geq 1/2$, the paths can be extended with positive probability to R_{a+1} without intersection. Hence Lemma 3.6 is a corollary of Lemma 3.4.

Lemma 3.6 *There exists $c_{10} > 0$ such that for all a,*

$$q(a+1) \geq c_{10} q(a).$$

Lemma 3.7 *There exists $c_{11} > 0$ such that if $x_1, x_2 \in R_0$ and B^1, B^2 are independent Brownian motions starting at x_1, x_2, respectively, then for $a > 1$,*

$$\mathbf{P}(D_a) \leq c_{11} q(a) |x_1 - x_2|^{1/2}.$$

Proof. By the Beurling estimate (Lemma 3.1) and Brownian scaling, it is easy to see that

$$\mathbf{P}(D_1) \leq c|x_1 - x_2|^{1/2}.$$

But, using Lemma 3.6,

$$\mathbf{P}(D_a) = \mathbf{P}(D_1)\mathbf{P}(D_a \mid D_1) \leq \mathbf{P}(D_1) q(a-1) \leq c\mathbf{P}(D_1) q(a). \quad \square$$

Lemma 3.8 *For $a, \epsilon > 0$ let $V^j = V^j(\epsilon, a)$ be the event*

$$V^j = \{B^j[0, \tau^j_a] \subset \{\Re(z) > 0\} \cup \{z : |B^j(0) - z| \leq \epsilon\}\}.$$

There exist $\epsilon \in (0, 1/10)$ and $u > 0$ such that for all $a > 0$,

$$\sup \mathbf{P}[V^1 \cap V^2 \cap D_a] \geq u q(a),$$

where the supremum is over all $x_1, x_2 \in R_0$, $|x_1 - x_2| \geq 10\epsilon$, and $B^j(0) = x_j$.

Proof. It suffices to prove the result for $a \geq 1$. We will first prove that there is a $u_1 > 0$ such that for all ϵ sufficiently small, $a \geq 1$

$$\sup \mathbf{P}[V^1 \cap D_a] \geq u_1 \epsilon q(a), \tag{7}$$

where the supremum is over all $x_1, x_2 \in R_0$, $|x_1 - x_2| \geq 10\epsilon$.

By Lemma 3.7, we can find a $\delta > 0$ such that if $|x_1 - x_2| \leq \delta$, then $\mathbf{P}(D_a) \leq q(a)/2$. Choose $a \geq 1$ and choose $x_1, x_2 \in R_0$, which may depend on a, that maximize $\mathbf{P}(D_a)$. Clearly $|x_1 - x_2| \geq \delta \geq 10\epsilon$ if we choose ϵ sufficiently small. Let

$$\rho = \rho_1 = \inf\{t : B^1_t \in \{\Re(z) \leq 0\} \setminus \{z : |z - x_1| < \epsilon\}\},$$

$$\sigma = \sigma_1 = \inf\{t : B^1_t \in R_{-\epsilon}\},$$
$$\eta = \eta_1 = \inf\{t \geq \rho : B^1_t \in R_0\}.$$

It is easy to verify that there is a $v_1 > 0$ (independent of ϵ, a) such that

$$\mathbf{P}\{\sigma < \eta \mid \rho < \tau^1_a\} \geq v_1.$$

It can be seen using the gambler's ruin estimate that there is a $v_2 > 0$ (depending on δ but not on ϵ) such that for any $y \in R_{-\epsilon}$,

$$\mathbf{P}\{|B^1(\eta) - x_2| \leq \delta \mid \sigma \leq \eta \leq \tau^1_a; B^1(\sigma) = y\} \geq v_2\epsilon.$$

Hence,

$$\mathbf{P}\{|B^1(\eta) - x_2| \leq \delta \mid \rho < \tau^1_a\} \geq v_1 v_2 \epsilon.$$

By using the strong Markov property at the stopping time η we see that

$$\mathbf{P}(D_a; \rho < \tau^1_a) \leq \mathbf{P}(D_a \mid \rho < \tau^1_a) \leq [1 - v_1 v_2 \epsilon] q(a) + v_1 v_2 \epsilon [q(a)/2],$$

and hence

$$\mathbf{P}(V^1 \cap D_a) = \mathbf{P}(D_a; \rho > \tau^1_a) \geq \frac{v_1 v_2 \epsilon}{2} q(a).$$

This gives (7).

For $\epsilon > 0$, $a > 0$ choose x_1, x_2, which may depend on a, which maximize

$$\mathbf{P}(V^1 \cap D_a). \tag{8}$$

Note that for any $x_1, x_2 \in R_0$,

$$\mathbf{P}(V^1 \cap D_a) \leq \mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\mathbf{P}(A_2 \mid A_1)\mathbf{P}(A_3 \mid A_1 \cap A_2),$$

where

$$A_1 = \{B^1[0, \tau^1_1] \cap \{\Re(z) \leq \epsilon\} = \emptyset\},$$
$$A_2 = \{B^2[0, \tau^2_1] \cap B^1[0, \tau^1_1] = \emptyset\},$$
$$A_3 = \{B^1[\tau^1_1, \tau^1_a] \cap B^2[\tau^2_1, \tau^2_a] = \emptyset\}.$$

The gambler's ruin estimate gives $\mathbf{P}(A_1) \leq c\epsilon$. The Beurling estimate (Lemma 3.1) estimates $\mathbf{P}(A_2 \mid A_1)$, and Lemma 3.6 estimates $\mathbf{P}(A_3 \mid A_1 \cap A_2)$. Hence we get

$$\mathbf{P}(V^1 \cap D_a) \leq c\epsilon |x_1 - x_2|^{1/2} q(a).$$

In particular, if ϵ is sufficiently small and $|x_1 - x_2| \leq 10\epsilon$,

$$\mathbf{P}(V^1 \cap D_a) \leq u_1 \epsilon q(a)/2,$$

and the maximum in (8) is taken on by x_1, x_2 with $|x_1 - x_2| \geq 10\epsilon$. Fix such a small ϵ and let $\delta = 10\epsilon$. Then proceed as above using stopping times ρ_2, σ_2, η_2. □

The next lemma can be proved easily by direct construction of an appropriate event. We omit the proof.

Lemma 3.9 *For any $y_1, y_2 \in R_1$ and $\epsilon, \delta > 0$ let $V = V(\epsilon, \delta, y_1, y_2)$ be the event*

$$V = \{\mathrm{dist}(B^j(\tau_1^j), B^{3-j}[0, \tau_1^{3-j}]) \geq 2\epsilon, |B^j(\tau_1^j) - y_j| \leq \delta,\ j = 1, 2\}.$$

For every $0 < \delta < \epsilon < 1/10$, $\epsilon_1 > 0$ there exists $u > 0$ such that if $x_1, x_2 \in R_0$, $y_1, y_2 \in R_1$, $|x_1 - x_2| \geq \epsilon_1$, $|y_1 - y_2| \geq 5\epsilon$,

$$\mathbf{P}(V \cap D_1) \geq u.$$

Once we have Lemmas 3.8 and 3.9 we can use the strong Markov property to conclude the following.

Corollary 3.10 *For every $\epsilon > 0$ there is a $u > 0$ such that if $\bar{\Gamma}$ is any initial configuration with $Y(\bar{\Gamma}) \geq \epsilon$, then*

$$\mathbf{P}[D_a(\bar{\Gamma})] \geq uq(a).$$

From Lemma 3.4 and Corollary 3.10 we see that there is a $c_{12} > 0$ such that for all a, b,

$$q(a + b) \geq c_{12} q(a) q(b).$$

In particular $f(a) = \log q(a) + \log c_{12}$ is a superadditive function. It follows from the standard theory that for all a,

$$\frac{f(a)}{a} \leq \lim_{t \to \infty} \frac{f(t)}{t} = \lim_{t \to \infty} \frac{\log q(t)}{t} = -2\zeta,$$

i.e., $q(a) \leq (1/c_{12}) e^{-2\zeta a}$. In particular, in all the lemmas above we may now replace $q(a)$ with $ce^{-2\zeta a}$. With another use of Lemmas 3.4, 3.6, and 3.8 we can deduce the following.

Proposition 3.11 *There exists a $c_{13} > 0$ such that for all $a > 0$,*

$$e^{-2\zeta a} \leq q(a) \leq c_{13} e^{-2\zeta a}.$$

Morevoer, there exist $0 < c_{14} < c_{15} < \infty$ such that if $\bar{\Gamma}$ is any initial configuration, and $a \geq 1$,

$$c_{14} e^{-2\zeta a} \mathbf{P}(D_1) \leq \mathbf{P}(D_a) \leq c_{15} e^{-2\zeta a} \mathbf{P}(D_1).$$

In proving the main convergence result, we will need the following estimate which states intuitively that Brownian motions conditioned not to intersect are transient. Let B^1, B^2 be independent Brownian motions starting on R_0. For any $0 < m < a$, let

$$\sigma^j(m, a) = \sup\{t \leq \tau_a^j : B_t^j \in R_m\},$$

$$J^j(m, a) = \sup\{\Re(B_t^j) - m : t \leq \sigma^j(m, a)\},$$

$$Z^j(a) = \sup\{J^j(m, a) : m \leq a\},$$

$$Z(a) = \max\{Z^1(a), Z^2(a)\}.$$

Lemma 3.12 *There exists $c_{16} < \infty$ such that for every $x_1, x_2 \in R_0$ and every $0 < r < a$,*

$$\mathbf{P}(D_a; Z(a) \geq r) \leq c_{16} a r^{-1} e^{-2\zeta a} e^{-r/4}.$$

Proof. We will show that for $r_n = nr/2, n = 0, 1, 2, \ldots; nr/2 \leq a$,

$$\mathbf{P}(D_a; J^1(r_n, a) \geq \frac{r}{2}) \leq ce^{-2\zeta a}e^{-r/4}, \tag{9}$$

and hence

$$\mathbf{P}(D_a; J^1(r_n, a) \geq \frac{r}{2} \text{ for some } r_n \leq a) \leq car^{-1}e^{-2\zeta a}e^{-r/4}.$$

But it is easy to verify that if $Z^1(a) \geq r$, then $J^1(r_n, a) \geq r/2$ for some r_n. An identical argument works for $Z^2(a)$.

Let n be given and set $b = (n+1)r/2$. Assume $b \leq a$, otherwise (9) is trivial. Let

$$\rho = \inf\{t \geq \tau_b^1 : B_t^1 \in R_{nr/2}\},$$

$$\eta = \inf\{t \geq \rho : B_t^1 \in R_b\},$$

Define events

$$U = \{\rho \leq \tau_a^1; B^1[\tau_b^1, \rho] \cap B^2[0, \tau_b^2] = \emptyset\},$$
$$V = \{B^1[\eta, \tau_a^1] \cap B^2[\tau_b^2, \tau_a^2] = \emptyset\}.$$

Then

$$D_a \cap \{J^1(r_n, a) \geq \frac{r}{2}\} \subset D_b \cap U \cap V,$$

and hence

$$\mathbf{P}(D_a; J^1(r_n, a) \geq \frac{r}{2}) \leq \mathbf{P}(D_b)\mathbf{P}(U \mid D_b)\mathbf{P}(V \mid D_b \cap U).$$

By Proposition 3.11,

$$\mathbf{P}(D_b) \leq ce^{-2\zeta b}, \quad \mathbf{P}(V \mid D_b \cap U) \leq ce^{-2\zeta(a-b)},$$

and the Beurling estimate (Lemma 3.1) gives

$$\mathbf{P}(U \mid D_b) \leq ce^{-r/4}. \quad \Box$$

Suppose we start two Brownian motions with any initial configuration. By Lemma 3.4 we know that by the time the paths reach R_1 there is a good chance they are reasonably far apart (given that no intersection has occurred). Once they are far apart we can attach almost any kind of configuration which lies basically between R_1 and R_2. If the configuration is such that the paths still stay reasonably far apart than we can use Corollary 3.10 to say that a solid proportion of these paths will have this configuration as we go to infinity. We give an example now of one such "configuration".

Suppose $\gamma : [a, b] \to \bar{\mathbf{C}}$ is a continuous function. There is a unique $\tilde{\gamma} : [a, b] \to \mathbf{C}$ that projects to γ under the equivalence relation such that $\Im(\tilde{\gamma}(a)) \in [-\pi, \pi)$. Let

$$w(\gamma) = |\Im(\tilde{\gamma}(b)) - \Im(\tilde{\gamma}(a))|,$$

$$\bar{w}(\gamma) = \sup\{|\Im(\tilde{\gamma}(t)) - \Im(\tilde{\gamma}(a))| : a \leq t \leq b\}.$$

We will use the following simple topological fact. Suppose γ^1, γ^2 are continuous functions from $[0, 1]$ to $\bar{\mathbf{C}}$ with $\gamma^1(0), \gamma^2(0) \in R_s; \gamma^1(1), \gamma^2(1) \in R_{s+1}$. Suppose

$$w(\gamma^1) \geq 6\pi,$$

$$\bar{w}(\gamma^2) \leq \pi.$$

Then

$$\gamma^1(0,1) \cap \gamma^2(0,1) \neq \emptyset.$$

Let U_1, U_2 be the events

$$U_1 = \{w(B^1[T_1^1, T_2^1]) \geq 6\pi\},$$

$$U_2 = \{\bar{w}(B^1[T_1^2, T_2^1]) \leq \pi\}.$$

The following lemma can be proved using the idea in the previous paragraph. We omit the details.

Lemma 3.13 *There exists a constant c such that if Γ is any initial configuration, and $s \geq 2$,*

$$\mathbf{P}(D_s \cap U_1) \geq ce^{-2\zeta s}\mathbf{P}(D_1),$$

$$\mathbf{P}(D_s \cap U_2) \geq ce^{-2\zeta s}\mathbf{P}(D_1).$$

It follows that if $\bar{\gamma}_1 = (\gamma_1^1, \gamma_1^2), \bar{\gamma}_2 = (\gamma_2^1, \gamma_2^2)$ are chosen independently from the conditional distribution given no intersection up to time s and initial configuration $\bar{\Gamma}$, there is a positive probability (independent of $\bar{\Gamma}$ and s) that the paths γ_1^1 and γ_2^1 intersect. This lemma has implications on the measure that we will define on the behavior of Brownian motion near a cut point. At a typical cut point, a Brownian path winds infinitely often around the point.

4 Markov Chain on Excursions

It will be convenient to consider excursions from R_{s_1} to R_{s_2} where $s_1 < s_2$. These are Brownian motions starting on R_{s_1} conditioned not to revisit R_{s_1} before reaching R_{s_2}. These are basically one dimensional excursions, or more precisely, complex Brownian motions doing an excursion in the real part and acting like a standard Brownian motion in the imaginary part. The use of excursions in understanding one dimensional Brownian motions is standard so we will be a little informal in our discussion.

Let $\tilde{\mathcal{Y}}_s$ be the set of excursions from R_{-s} to R_0, i.e., the set of $\gamma \in \mathcal{Y}_s$ with

$$-s < \Re(\gamma(t)) < 0, \quad 0 < t < b(\gamma),$$

and let $\tilde{\mathcal{G}}_s = (\tilde{\mathcal{Y}}_s)^2$. We will also use $\tilde{\mathcal{Y}}_s$ and $\tilde{\mathcal{G}}_s$ to denote the set of excursions from R_{s_1} to R_{s_1+s} for any s_1. Let $\mathcal{L}_s = \cup_b \mathcal{L}_s^b$ where $\mathcal{L}_s^b$ is the set of "loops" of length b which start and end on R_{-s} without reaching R_0, i.e., the set of $\omega : [0,b] \to \bar{\mathbf{C}}$ with

$$\omega(0), \omega(b) \in R_{-s}.$$

$$\mathcal{R}(\omega(t)) < 0, \quad 0 \leq t \leq b.$$

We write b_ω for the unique b with $\omega \in \mathcal{L}_s^b$. Note that we do not assume that ω start and end at the same point; these are loops in the real part only. We can also consider $\mathcal{L}_s$ as the set of loops starting and ending on R_{s_1} that never reach R_{s_1+s}; hence $\mathcal{L}_r \subset \mathcal{L}_s$ for $r < s$.

Let B be a Brownian motion starting on R_{-s}. Let

$$\sigma = \sigma_s = \sup\{t \leq \tau_0 : B_t \in R_{-s}\}.$$

Then B naturally splits into a loop $\omega \in \mathcal{L}_s$ and an excursion $\gamma \in \tilde{\mathcal{Y}}_s$,

$$\omega(t) = B_t, \quad 0 \le t \le \sigma,$$

$$\lambda(t) = B_t, \quad \sigma \le t \le \tau_0.$$

Wiener measure $\tilde{W}_s$ on excursions will be the measure obtained by chosing the starting point on R_{-s} at random (from the uniform distribution on R_{-s}) and considering the measure induced by λ as above. We will use Q_s to denote the measure on $\mathcal{L}_s$ generated by ω, again choosing the initial point from the uniform distribution. We can easily adapt $\tilde{W}_s$ or Q_s if we want a measure on paths starting at a particular point on R_{-s} rather than at a randomly chosen point. Wiener measure $\tilde{W}_s$ on $\tilde{\mathcal{G}}_s$ is defined by taking Wiener measure on each component. It is not difficult using the gambler's ruin estimate to check that if $r < s$ and I_r denotes the indicator function of the set $\mathcal{L}_r$, then

$$dQ_r = (s/r) I_r \, dQ_s.$$

Lemma 4.1 *Let $\mathcal{L}_s^1$ denote the set of $\omega \in \mathcal{L}_s$ such that*

$$|\omega(t) - \omega(0)| \le 1/20, \quad 0 \le t \le b_\omega,$$

and let $\mathcal{L}_s^2$ denote the set of $\omega \in \mathcal{L}_s$ such that $\omega[0, b_\omega]$ does not disconnect R_{-s-1} from R_{-s+1}. (Here $\mathcal{L}_s$ is the set of loops starting and ending on R_{-s}.) Then there exist c_1, c_2 such that for all $s \ge 1$,

$$Q_s(\mathcal{L}_s^1) \ge c_1 s^{-1},$$

$$Q_s(\mathcal{L}_s^2) \le c_2 s^{-1}.$$

Proof. The first inequality is easy. Let B be a Brownian motion starting on R_{-s} and let

$$\rho_1 = \inf\{t : |B_t - B_0| \ge 1/20\},$$

$$\rho_2 = \inf\{t \ge \rho_1 : B_t \in R_{-s}\}.$$

It is easy to see that $P\{\tau_{-s+1} < \rho_2\} \ge c$, and hence, using the gambler's ruin estimate,

$$\begin{aligned} Q_s(\mathcal{L}_s^1) &= \mathbf{P}\{\tau_0 < \rho_2\} \\ &= \mathbf{P}\{\tau_{-s+1} < \rho_2\}\mathbf{P}\{\tau_0 < \rho_2 \mid \tau_{-s+1} < \rho_2\} \\ &\ge cs^{-1}. \end{aligned}$$

For the second inequality define random times by $\rho_0 = 0$ and for $k > 0$,

$$\eta_k = \inf\{t \ge \rho_{k-1} : B_t \in R_{-s+1} \cup R_{-s-1}\},$$

$$\rho_k = \inf\{t \ge \eta_k : B_t \in R_{-s}\}.$$

We will write "A DND" for "the set A does not disconnect R_{-s-1} and R_{-s+1}." Then

$$Q_s(\mathcal{L}_s^2) = \mathbf{P}\{B[0,\sigma] \text{ DND}\},$$

where, as before,

$$\sigma = \sigma_s = \sup\{t \le \tau_0 : B_t \in R_{-s}\}.$$

Note that there is a $u < 1$ such that

$$\mathbf{P}\{B[0,\eta_1] \text{ DND}\} \leq u,$$

and hence

$$\mathbf{P}\{B[0,\rho_1] \text{ DND} \mid \rho_1 < \tau_0\} \leq u.$$

By the strong Markov property,

$$\mathbf{P}\{B[\rho_1,\sigma] \text{ DND} \mid \rho_1 < \tau_0\} = \mathbf{P}\{B[0,\sigma] \text{ DND}\},$$

and hence

$$\mathbf{P}\{B[0,\sigma] \text{ DND}; \rho_1 < \tau_0\} \leq u\mathbf{P}\{B[0,\sigma] \text{ DND}\},$$

and

$$\mathbf{P}\{B[0,\sigma] \text{ DND}; \rho_1 > \tau_0\} \geq (1-u)\mathbf{P}\{B[0,\sigma] \text{ DND}\}.$$

But by the gambler's ruin estimate,

$$\mathbf{P}\{B[0,\sigma] \text{ DND}; \rho_1 > \tau_0\} \leq \mathbf{P}\{\rho_1 > \tau_0\} \leq cs^{-1}. \quad \Box$$

If $\gamma \in \tilde{\mathcal{Y}}_s$, we define the reversed path γ^R by

$$\gamma^R(t) = -s - \gamma(b_\gamma - t), \quad 0 \leq t \leq b_\gamma,$$

and similarly we define $\bar{\gamma}^R$ for $\bar{\gamma} \in \tilde{\mathcal{G}}_s$. Note that Wiener measure $\tilde{W}_s$ is invariant under the mapping $\bar{\gamma} \mapsto \bar{\gamma}^R$. Let $\tilde{\mathcal{H}}_s$ be the set of ordered pairs of excursions that do not intersect, i.e., the set of $(\gamma^1,\gamma^2) \in \tilde{\mathcal{G}}_s$ such that

$$\gamma^1[0,b(\gamma^1)] \cap \gamma^2[0,b(\gamma^2)] = \emptyset.$$

Excursions are somewhat more likely than unrestricted Brownian motions to avoid each other. The next lemma makes this precise.

Lemma 4.2 *There exist c_1, c_2 such that for $s \geq 1$,*

$$c_1 s^2 e^{-2\zeta s} \leq \tilde{W}_s(\tilde{\mathcal{H}}_s) \leq c_2 s^2 e^{-2\zeta s}.$$

Proof. Let B^1, B^2 be independent Brownian motions starting at R_0 with the initial point chosen according to the uniform distribution. Let

$$\sigma^j = \sigma^j_s = \sup\{t \leq \tau^j_s : B_t \in R_0\}.$$

Then the lemma is equivalent to the statement

$$c_1 s^2 e^{-2\zeta s} \leq \mathbf{P}\{B^1[\sigma^1,\tau^1_s] \cap B^2[\sigma^2,\tau^2_s] = \emptyset\} \leq c_2 s^2 e^{-2\zeta s}.$$

For the upper bound define random times (depending on s) by $\rho^1_0 = \rho^2_0 = 0$ and for $k > 0$,

$$\eta^j_k = \inf\{t > \rho^j_{k-1} : B^j_t \in R_1\},$$

$$\rho^j_k = \inf\{t > \eta^j_k : B^j_t \in R_0\}.$$

Let $U(k,m) = U_s(k,m)$ be the event

$$U(k,m) = \{\eta_k^1 < \tau_s^1; \eta_m^2 < \tau_s^2; B^1[\eta_k^1, \tau_s^1] \cap B^2[\eta_m^1, \tau_s^2] = \emptyset\}.$$

By the gambler's ruin estimate, Proposition 3.11, and the strong Markov property, we can see that

$$\mathbf{P}[U(k,m)] \leq c(\frac{s-1}{s})^{k+m} e^{-2\zeta s}.$$

But it is easy to check that

$$\{B^1[\sigma^1, \tau_s^1] \cap B^2[\sigma^2, \tau_s^2] = \emptyset\} \subset \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} U(k,m).$$

Hence

$$\mathbf{P}\{B^1[\sigma^1, \tau_s^1] \cap B^2[\sigma^2, \tau_s^2] = \emptyset\} \leq \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \mathbf{P}[U(k,m)] \leq cs^2 e^{-2\zeta s}.$$

For the lower bound let $V(k,m) = V_s(k,m)$ be the event

$$V_s(k,m) = \{\tau_s^1 > k; \tau_s^2 > m; B^1[k, k+1] \subset \{\Im(z) \in (-\frac{\pi}{4}, \frac{\pi}{4})\};$$

$$B^2[m, m+1] \subset \{\Im(z) \in (\frac{3\pi}{4}, \frac{5\pi}{4})\}; \Re(B_k^1), \Re(B_m^2) \leq 0; \Re(B_{k+1}^1), \Re(B_{m+1}^2) \geq 1\}.$$

It is easy to check that there is a c such that for all $1 \leq k, m \leq s^2$,

$$\mathbf{P}[V(k,m)] \geq cs^{-2}. \tag{10}$$

Let $\hat{V}(k,m)$ be the event

$$\hat{V}(k,m) = V(k,m) \cap \{B^1[k, \tau_1^s] \cap B^2[m, \tau_2^s] = \emptyset; \sigma^1 < k+1; \sigma^2 < m+1\}.$$

Then the $\hat{V}(k,m)$ are disjoint and by (10), the strong Markov property, and Lemma 3.8,

$$\mathbf{P}[\hat{V}(k,m)] \geq cs^{-2} e^{-2\zeta s}, \quad 1 \leq k, m \leq s^2.$$

If we sum over all $k, m \leq s^2$ we get

$$\mathbf{P}\{B^1[\sigma^1, \tau_s^1] \cap B^2[\sigma^2, \tau_s^2] = \emptyset\} \geq cs^2 e^{-2\zeta s}. \ \Box$$

We now define a discrete time, subMarkov chain on excursions $\bar{\gamma} \in \tilde{\mathcal{H}}_s$ (going from R_{-s} to R_0). Assume $X_0 = \bar{\gamma} \in \tilde{\mathcal{H}}_s$. Start Brownian motions B^1, B^2 with initial configuration $\bar{\gamma}$. On the event $[D_{s+4}(\bar{\gamma})]^c$, we kill the process. Otherwise, we set

$$\hat{\sigma}^j = \sup\{t \leq \tau_{s+4}^j : B_t^j \in R_4\},$$

and set $X_1 = \bar{\gamma}_1 = (\gamma_1^1, \gamma_1^2)$ where

$$\gamma_1^j(t) = B^j(t + \hat{\sigma}^j) - (s+4), \quad 0 \leq t \leq \tau_{s+4}^j - \sigma^j.$$

We can continue this procedure to produce a time homogeneous, discrete time chain with state space $\tilde{\mathcal{H}}_s$. Let $g(\bar\gamma_1, \bar\gamma_2)$ be the density for the transition probabilities with respect to Wiener measure given that $X_0 = \bar\gamma_1$, i.e., for any $A \subset \tilde{\mathcal{H}}_s$,

$$\mathbf{P}\{X_1 \in A \mid X_0 = \bar\gamma_1\} = \int_A g(\bar\gamma_1, \bar\gamma_2)\, d\tilde{W}_s(\bar\gamma_2). \tag{11}$$

If $\bar\gamma \in \tilde{\mathcal{G}}_s$ we define

$$\phi(\bar\gamma) = \mathbf{P}(D_1),$$

$$\tilde\phi(\bar\gamma) = \tilde\phi_s(\bar\gamma) = \mathbf{P}(D_1; \tau^1_{-s} > \tau^1_1; \tau^2_{-s} > \tau^2_1),$$

where, as before, we are starting Brownian motions with initial configuration $\bar\gamma$. One can check that ϕ and ϕ_s are continuous functions on $\tilde{\mathcal{G}}_s$ (with respect to the metric d defined in the first section) at all $\bar\gamma$ with $\phi(\bar\gamma), \tilde\phi(\bar\gamma) > 0$.

Lemma 4.3 *There exist c_{19}, c_{20} such that for all $s \geq 4$, $\bar\gamma_1, \bar\gamma_2 \in \tilde{\mathcal{H}}_s$,*

$$c_{19}\phi(\bar\gamma_1)s^{-2}\tilde\phi(\bar\gamma_2^R) \leq g(\bar\gamma_1, \bar\gamma_2) \leq c_{20}\phi(\bar\gamma_1)s^{-2}\tilde\phi(\bar\gamma_2^R).$$

The statement of the lemma is a little imprecise since densities are defined only up to sets of measure zero. We can restate the conclusion of the lemma as follows: for every $\bar\gamma_2$ with $\tilde\phi(\bar\gamma_2^R) > 0$ and every neighborhood U (in the metric d defined in Section 1) of $\bar\gamma_2^R$ sufficiently small,

$$c_1\phi(\bar\gamma_1)s^{-2}\tilde\phi(\bar\gamma_2)\tilde{W}_s(U) \leq \mathbf{P}\{X_1 \in U \mid X_0 = \bar\gamma_1\} \leq c_2\phi(\bar\gamma_1)s^{-2}\tilde\phi(\bar\gamma_2)\tilde{W}_s(U).$$

We will sketch the proof of Lemma 4.3 although we will not give all the details. Assume we start independent Brownian motions B^1, B^2 on R_0 with initial configuration $\bar\gamma_1 = (\gamma_1^1, \gamma_1^2)$ and let them go until they reach R_{s+4}. As before, let

$$\Gamma_a^j = \gamma_1^j \cup B^j[0, \tau_a^j],$$

$$D_a = D_a(\bar\gamma_1) = \{\Gamma_a^1 \cap \Gamma_a^2 = \emptyset\}.$$

Let

$$\sigma = \sigma_s^j = \sup\{t \leq \tau_{s+4}^j : B_t \in R_2\}.$$

We split the Brownian motions into three pieces

$$B^j[0, \tau_2^j],\ B^j[\tau_2^j, \sigma^j],\ B^j[\sigma^j, \tau_{s+4}^j].$$

In order to get $B^1[0, \tau_{s+4}^1], B^2[0, \tau_{s+4}^2]$ we can choose the three pieces from the appropriate spaces: the first piece from $\mathcal{G}_2$ using W_2; the second from $\mathcal{L}_{s+2} \times \mathcal{L}_{s+2}$ using $Q_{s+2} \times Q_{s+2}$; and the third piece from $\tilde{\mathcal{G}}_{s+2}$ using $\tilde{W}_{s+2}$. (These pieces need to be translated appropriately.)

Let

$$\tilde{D}_2 = D_2 \cap \{B^1[0, \tau_2^1] \cap \{\Re(z) \geq \frac{3}{2}\} \subset \{\Im(z) \leq \frac{1}{20}\};$$

$$B^2[0, \tau_2^2] \cap \{\Re(z) \geq \frac{3}{2}\} \subset \{|\Im(z) - \pi| \leq \frac{1}{20}\}\ \}.$$

It is each to check, using Lemma 3.4, that

$$c_1\phi(\bar\gamma_1) \le \mathbf{P}(\tilde D_2) \le \mathbf{P}(D_2) = \phi(\bar\gamma_1).$$

Suppose A is a subset of $\tilde{\mathcal{H}}_{s+2}$ (considered as nonintersecting excursions from R_2 to R_{s+4}). Let

$$\tilde A = \{(\gamma^1,\gamma^2) \in A : \gamma^1[0,b(\gamma^1)] \cap \{\Re(z) \le \frac{5}{2}\} \subset \{|\Im(z)| \le \frac{1}{20}\}$$

$$\gamma^2[0,b(\gamma^2)] \cap \{\Re(z) \le \frac{5}{2}\} \subset \{|\Im(z) - \pi| \le \frac{1}{20}\}\ \}. \tag{12}$$

For $A \subset \tilde{\mathcal{H}}_{s+2}$, let $V(A)$ denote the event

$$V(A) = \{(B^1[\sigma^1, \tau^1_{s+4}], B^2[\sigma^2,\tau^2_{s+4}]) \in A\},$$

where we have abused notation slightly by writing $B[a,b]$ for the function $B : [a,b] \to \bar{\mathbf{C}}$. Suppose we have chosen the first and third pieces, $B^j[0,\tau^j_2], B^j[\sigma^j,\tau^j_{s+4}]$ If we want D_{s+4} to hold then $B^1[\tau^1_2,\sigma^1]$ and $B^2[\tau^2_2,\sigma^2]$ must be in $\mathcal{L}^2_{s+2}$ as defined in Lemma 4.1. From this we can conclude that

$$\mathbf{P}[V(A) \cap D_{s+4}] \le c_2 s^{-2}\phi(\bar\gamma_1)\tilde W_{s+2}(A). \tag{13}$$

Similarly, by appropriate attaching of paths (details omitted), we can see by considering $\mathcal{L}^1_{s+2}$ that

$$c_1 s^{-2}\phi(\bar\gamma_1)\tilde W_{s+2}(\tilde A) \le \mathbf{P}[V(\tilde A) \cap D_{s+4}]. \tag{14}$$

We now consider the third piece which is a pair of excursions from R_4 to R_{s+4}. This can also be thought of as a pair of excursions from R_{s+4} to R_4. Let $\hat B^1, \hat B^2$ be independent Brownian motions starting on R_{s+4} (with the uniform distribution) and let

$$\tilde\sigma^j = \sup\{t \le \tau^j_2 : \hat B^j_t \in R_{s+4}\},$$

$$\gamma^j(t) = \hat B^j[\sigma^j, \tau^j_2]^R,$$

that is to say,

$$\gamma^j(t) = \hat B^j(\tau^j_2 - \sigma^j - t), \quad 0 \le t \le \tau^j_{s-2} - \sigma^j.$$

The measure induced by (γ^1,γ^2) is Wiener measure $\tilde W_{s+2}$ on $\tilde{\mathcal{G}}_{s+2}$. Let

$$\hat\rho = \inf\{t \ge \tilde\sigma^j : \hat B^j_t \in R_4\},$$

and let

$$\hat\gamma^j = \hat B^j[\sigma^j, \hat\rho^j]^R.$$

Then it is easy to check that the measure induced by $(\hat\gamma^1,\hat\gamma^2)$ is, in fact, Wiener measure $\tilde W_s$. Let $\bar\gamma_2$ be given with $\tilde\phi(\bar\gamma_2) > 0$ and let U be an open neighborhood about $\bar\gamma_2$ in the metric d. By the continuity of $\tilde\phi$, if U is chosen sufficiently small then

$$\frac{1}{2}\tilde\phi(\bar\gamma_2) \le \tilde\phi(\bar\gamma) \le 2\tilde\phi(\bar\gamma_2), \quad \bar\gamma \in U. \tag{15}$$

If $\gamma \in \mathcal{Y}_{s+2}$, considered as an excursion from R_2 to R_{s+4}, write $\Lambda_s\gamma$ for the excursion from R_4 to R_{s+4} obtained for γ by considering the path starting at the last visit to R_4 and proceeding along γ

to R_{s+4}. Define $\Lambda_s\bar\gamma$ for $\bar\gamma \in \tilde{\mathcal{G}}_{s+2}$ similarly. Let A_U be the subset of $\mathcal{H}_{s+2}$ (considered as excursions from R_2 to R_{s+4}),

$$A_U = \{\bar\gamma \in \mathcal{H}_{s+2} : \Lambda\bar\gamma \in U\},$$

and define $\tilde A_U$ as in (12). Then by the strong Markov property, Lemma 3.4, and (15) we can see that

$$c_1 \tilde W_s \phi(\bar\gamma_2) \le \tilde W_{s+2}(\tilde A_U) \le \tilde W_{s+2}(A_U) \le c_2 \tilde W_s \phi(\bar\gamma_2).$$

Combining this with (13) and (14), we see that

$$c_1 \phi(\bar\gamma_1) s^{-2} \tilde\phi(\bar\gamma_2) \tilde W_s(U) \le \mathbf{P}\{X_1 \in U \mid X_0 = \bar\gamma_1\} \le c_2 \phi(\bar\gamma_1) s^{-2} \tilde\phi(\bar\gamma_2) \tilde W_s(U).$$

From the definition of g (see (11)), it is easy to derive Lemma 4.3.

The next lemma is a standard result about rate of convergence of subMarkov chains. We have included a proof because it will be important for our purposes that the constants u_1, u_2 depend only on a_1, a_2, a_3.

Lemma 4.4 *For every $0 < a_1, a_2, a_3 < \infty$ there exist $u_1 = u_1(a_1, a_2, a_3) > 0, u_2 = u_2(a_1, a_2, a_3) < \infty$ such that the following holds. Suppose (X, μ) is a probability space with a nonnegative kernel $K(x, y)$ defined on $X \times X$. Suppose there exist $M < \infty$ and functions ϕ, ψ on X with $0 \le \phi \le 1, 0 \le \psi \le M$ and*

$$a_1 \phi(x)\psi(y) \le K(x, y) \le a_2 \phi(x)\psi(y). \tag{16}$$

Let A be the operator on $L^1(X, \mu)$ with kernel K,

$$Ag(y) = \int g(x)K(x, y)\, d\mu(x).$$

Let f be the unique positive eigenfunction for A, i.e., the unique $f \ge 0$ satisfying

$$\int f(x)\, d\mu(x) = 1,$$

$$Af(x) = \beta f(x),$$

for some $\beta > 0$. Assume that

$$\int f(x)\phi(x)\, d\mu(x) \ge a_3. \tag{17}$$

Let $\|\cdot\|$ denote the L^1 norm,

$$\|g\| = \int |g(x)|\, d\mu(x).$$

Let $h_n = h_n(g) = A^n g/\|A^n g\|$. Then for all $g \ge 0$ with $\|g\| < \infty$,

$$\|h_n - f\| \le u_2 e^{-n u_1}.$$

Proof. Without loss of generality we may assume $\beta = 1$ for otherwise we consider $\tilde K = \beta^{-1} K, \tilde\psi = \beta^{-1}\psi$. Note that (16) implies

$$\begin{aligned} f(y) &= \int f(x)K(x, y)\, d\mu(x) \\ &\le a_2 \int f(x)\phi(x)\psi(y)\, d\mu(x) \\ &\le a_2 \psi(y) \int f(x)\, d\mu(x) = a_2 \psi(y). \end{aligned}$$

Similarly, using (17) as well,

$$f(y) \geq a_1\psi(y) \int f(x)\phi(x)\, d\mu(x) \geq a_1 a_3 \psi(y).$$

Suppose $g \geq 0$. Then for all $n \geq 1$,

$$\begin{aligned}
\|A^n g\| &= \int A^{n-1}(Ag(x))\, d\mu(x) \\
&= \int A^{n-1}[\int g(y)K(y,x)\, d\mu(y)]\, d\mu(x) \\
&\leq a_2 \int A^{n-1}[\int g(y)\phi(y)\psi(x)\, d\mu(y)]\, d\mu(x) \\
&\leq \frac{a_2}{a_1 a_3}[\int g(y)\phi(y)\, d\mu(y)] \int A^{n-1} f(x)\, d\mu(x) \\
&= \frac{a_2}{a_1 a_3}[\int g(y)\phi(y)\, d\mu(y)] \qquad (18) \\
&\leq \frac{a_2}{a_1 a_3}\|g\|. \qquad (19)
\end{aligned}$$

Let $g_0 = g$ and

$$\alpha_0 = \int g_0(x)\phi(x)\, d\mu(x).$$

Note that

$$\begin{aligned}
Ag_0(x) &= \int g_0(y)K(y,x)\, d\mu(y) \\
&\geq a_1 \int g_0(y)\phi(y)\psi(x)\, d\mu(y) \\
&\geq \frac{a_1\alpha_0}{a_2} f(x).
\end{aligned}$$

Hence we can write

$$Ag_0(x) = \frac{a_1\alpha_0}{a_2} f(x) + g_1(x),$$

where $g_1 \geq 0$. Also, by (18),

$$\|Ag_0\| \leq \frac{a_2\alpha_0}{a_1 a_3}.$$

Hence, by (19), for every k,

$$\|A^k g_1\| \leq \|A^{k+1} g_0\| \leq (\frac{a_2}{a_1 a_3})^2 \alpha_0.$$

Since $\|A^k f\| = \|f\|$, we see that this implies that for all k,

$$\frac{\|A^k g_1\|}{\|A^{k+1} g_0\|} \leq \rho,$$

where $\rho < 1$ is given by

$$\rho = \frac{(a_2/a_1 a_3)^2}{(a_2/a_1 a_3)^2 + (a_1/a_2)}.$$

Similarly, we define inductively,

$$\alpha_n = \int g_n(x)\phi(x)\, d\mu(x),$$

and $g_{n+1} \geq 0$ by

$$Ag_n(x) = \frac{a_1\alpha_n}{a_2} f(x) + g_{n+1}(x).$$

As above we establish for each k,

$$\frac{\|A^k g_{n+1}\|}{\|A^{k+1} g_n\|} \leq \rho.$$

In particular, by induction, we see that

$$\frac{\|g_n\|}{\|A^n g_0\|} \leq \rho^n.$$

But,

$$A^n g_0 = r_n f + g_n,$$

for an appropriate number r_n and hence

$$h_n = t_n f + u_n$$

where $\|u_n\| \leq \rho^n$ and t_n is a constant, $1 - \rho^n \leq t_n \leq 1$. Therefore

$$\|h_n - f\| = \|(t_n - 1)f + u_n\| \leq 2\rho^n.$$

This proves the lemma. $\square$

For each $s \geq 4$, let f_s denote the nonnegative eigenfunction of the Markov chain on excursions, i.e., f_s is the nonnegative function on $\tilde{\mathcal{H}}_s$ such that

$$\int f_s(\bar{\gamma})\, d\tilde{W}_s(\bar{\gamma}) = 1,$$

and

$$\int f_s(\bar{\gamma}_1) g(\bar{\gamma}_1, \bar{\gamma}_2)\, d\tilde{W}_s(\bar{\gamma}_1) = \beta_s f_s(\bar{\gamma}_2),$$

for some $\beta_s > 0$. Here g is as defined in (11). We will apply Lemma 4.4 to this chain with $K(\bar{\gamma}_1, \bar{\gamma}_2), \phi(\bar{\gamma}_1), \psi(\bar{\gamma}_2)$ in the lemma corresponding to $g(\bar{\gamma}_1, \bar{\gamma}_2), \phi(\bar{\gamma}_1), s^{-2}\tilde{\phi}(\bar{\gamma}_2^R)$, respectively. Note that Lemma 3.4 can be used to show that for any f, if

$$Af(\bar{\gamma}_2) = \int f(\bar{\gamma}_1) g(\bar{\gamma}_1, \bar{\gamma}_2)\, d\tilde{W}_s(\bar{\gamma}_1),$$

and $h = \|Af\|/\|f\|$, then

$$\int h(\bar{\gamma}_1)\phi(\bar{\gamma}_1)\, d\tilde{W}_s(\bar{\gamma}_1) \geq c,$$

and hence the same must hold for the eigenfunction f_s. This establishes condition (17) of Lemma 4.4. Lemma (4.3) establishes that (16) holds. In both cases the constants are independent of s. Hence we have the following corollary.

Lemma 4.5 *There exist $c_{20} < \infty$ and $c_{21} > 0$ such that the following holds. Suppose $s \geq 4$ and Q is a any probability measure on X_0 such that $\mathbf{P}(D_1) > 0$. Let $g_n = g_{n,s}$ denote the density, with respect to Wiener measure, of X_n given that X_0 is distributed according to Q and let $h_n = g_n/\|g_n\|$ be the normalized density. Then if f_s is defined as above,*

$$\int |f_s - h_n| \, d\tilde{W}_s \leq c_{20} e^{-c_{21} n}.$$

We emphasize again that the constants in the lemma do not depend on s. Suppose $r \leq s$. Any measure on $\tilde{\mathcal{H}}_s$ gives a measure on $\mathcal{H}_r$ by projection by Ψ_r, as described in Section 2. This measure will be absolutely continuous with respect to Wiener measure W_r on $\mathcal{H}_r$. Let $f_{r,s}$ denote the density of the measure obtained by projecting the measure with density f_s. Then the following is an immediate corollary of Lemma 4.5.

Lemma 4.6 *There exist $c_{20} < \infty$ and $c_{21} > 0$ such that the following holds. Suppose $s \geq 4$ and $0 < r < s$. Consider the Markov chain on $\tilde{\mathcal{H}}_s$ and let Q be a probability measure on X_0. Let $g_n = g_{n,r}$ be the density, with respect to Wiener measure, of the projection onto $\mathcal{H}_r$ of the measure on X_n, given that X_0 is distributed according to Q. Let $h_n = h_{n,r} = g_n/\|g_n\|$ be the normalized density. Then for all n,*

$$\int |f_{r,s} - h_n| \, dW_r \leq c_{20} e^{-c_{21} n}.$$

5 Convergence

In this section we prove the main result. Start with any initial configuration $\bar{\Gamma} = (\Gamma^1, \Gamma^2)$. As before, let

$$\tau_r^j = \inf\{t : B_t^j \in R_r\}.$$

Let $\bar{\gamma}_r = (\gamma_r^1, \gamma_r^2) \in \mathcal{G}_r$ be defined by

$$\gamma_r^j(t) = B^j(t) - r, \quad 0 \leq t \leq \tau_r^j,$$

and let $\bar{\Gamma}_r = (\Gamma_r^1, \Gamma_r^2)$ be defined by

$$\Gamma_r^j = (\Gamma^j - r) \cup \gamma^j[0, \tau_r^j].$$

As before, we define the event

$$D_r = D_r(\bar{\Gamma}) = \{\Gamma_r^1 \cap \Gamma_r^2 = \emptyset\}.$$

Fix $s > 4$ for the time being; we will eventually let s go to infinity. Let $D^n = D_{n(s+4)}(\bar{\Gamma})$. For $n > 0$, let

$$T_n^j = T_{n,s}^j = \tau_{n(s+4)}^j,$$

$$S_n^j = S_{n,s}^j = \sup\{t \leq T_n^j : B_t^j \in R_{n(s+4)-s}\}.$$

For $n > 0$, let F_n be $B[S_n^j, T_n^j] - n(s+4)$, i.e., $F_n = \bar{\eta}_n = (\eta_n^1, \eta_n^2) \in \tilde{\mathcal{G}}_s$ where

$$\eta_n^j(t) = B^j(S_n^j + t) - n(s+4), \quad 0 \leq t \leq T_n^j - S_n^j.$$

Let

$$Z_n = \begin{cases} F_n, & \text{if } D^n \text{ holds,} \\ \Delta, & \text{otherwise,} \end{cases}$$

where Δ is an absorbing cemetery state. By Lemma 3.4, Lemma 3.8, and Proposition 3.11, there exists c_1, c_2 with

$$c_1 e^{-2\zeta n(s+4)} \mathbf{P}[D_1(\bar{\Gamma})] \leq \mathbf{P}[D^n] \leq c_2 e^{-2\zeta n(s+4)} \mathbf{P}[D_1(\bar{\Gamma})]. \tag{20}$$

Let K^n be the event

$$D^1 \cap \{B^1[S^1_{k-1}, T^1_k] \cap B^2[S^2_{k-1}, T^2_k] = \emptyset,\ k = 2, 3, \ldots, n\}.$$

Note that $D^n \subset K^n$, but by Lemma 3.12 and (20), if $n \leq 30s$,

$$\mathbf{P}[K^n \setminus D^n] \leq c_2 e^{-cs} \mathbf{P}[D^n]. \tag{21}$$

Let X_n be defined by

$$X_n = \begin{cases} F_n, & \text{if } K^n \text{ holds,} \\ \Delta, & \text{otherwise,} \end{cases}$$

Note that $X_1, \ldots, X_n$ has the law of the subMarkov chain on $\tilde{\mathcal{H}}_s$ described in the previous section (with X_1 giving the initial distribution). Let g_n denote the density with respect to Wiener measure $\tilde{W}_s$ of X_n, and $h_n = g_n/\|g_n\|$, the normalized density. Then by Lemma 4.5,

$$\int |f_s - h_n| \, d\tilde{W}_s \leq c_2 e^{-cn},$$

where f_s is the normalized invariant density for this chain. Let u_n denote the normalized density for Z_n given $Z_n \neq \Delta$. Then by (21), for $n \leq 30s$,

$$\int |u_n - h_n| \, d\tilde{W}_s \leq c_2 e^{-cn},$$

and hence for $n \leq 30s$,

$$\int |f_s - u_n| \, d\tilde{W}_s \leq c_2 e^{-cn},$$

for appropriately chosen c_2, c. Since this holds for any initial configuration we can conclude that for all $n \geq s/8$,

$$\int |f_s - u_n| \, d\tilde{W}_s \leq c_2 e^{-cs}. \tag{22}$$

If $r < s$, let $h_{r,n}$ denote the density of the measure on $\mathcal{G}_r$ obtained by projecting $u_n \, d\tilde{W}_s$ by the mapping Ψ_r as defined in Section 2. Then (22) implies that for $r \leq s/8 \leq n$,

$$\int |f_{r,s} - h_{r,n}| \, dW_r \leq c_2 e^{-cs}, \tag{23}$$

where $f_{r,s}$ denotes the density of the measure on $\mathcal{G}_r$ induced by projecting $f_s \, d\tilde{W}_s$.

We are now ready to prove the main theorem. Suppose we start with any initial configuration $\bar{\Gamma}$ and start independent Brownian motions B^1, B^2 with initial configuration $\bar{\Gamma}$. For $r < s/2 < \sqrt{u}$, define $(\gamma^1_u, \gamma^2_u), (\eta^1_{u,s}, \eta^2_{u,s}) \in \mathcal{G}_r$ as follows:

$$\sigma^j = \sup\{t \leq \tau^j_u : B^j_t \in R_{u-s}\},$$

$$\rho^j = \inf\{t \geq \sigma^j : B_t^j \in R_{u-r}\},$$
$$\gamma_u^j(t) = B_t^j - u, \quad 0 \leq t \leq \tau_u^j - \tau_{u-r}^j,$$
$$\eta_{u,s}^j(t) = B_t^j - u, \quad 0 \leq t \leq \tau_u^j - \rho^j.$$

Let $h^{r,u}$ and $\tilde{h}^{r,u,s}$ be the normalized densities with respect to W_r on $\mathcal{G}_r$ of the distributions induced by (γ_u^1, γ_u^2) and $(\eta_{u,s}^1, \eta_{u,s}^2)$, respectively, given the event $D_u(\bar{\Gamma})$. By Lemma 3.12,

$$\int |h^{r,u} - \tilde{h}^{r,u,s}| \, dW_r \leq c_2 e^{-cs}.$$

But by (23),

$$\int |f_{r,s} - \tilde{h}^{r,u,s}| \, dW_r \leq c_2 e^{-cs},$$

and hence

$$\int |f_{r,s} - h^{r,u}| \, dW_r \leq c_2 e^{-cs}.$$

In particular, for $2r \leq s \leq s' \leq \sqrt{u}$,

$$\int |f_{r,s} - f_{r,s'}| \, dW_r \leq \int |f_{r,s} - h^{r,u}| \, dW_r + \int |f_{r,s'} - h^{r,u}| \, dW_r \leq c_2 e^{-cs}.$$

If we now fix r, and let s go to infinity, we see that $\{f_{r,s}\}$ is a Cauchy sequence in $L^1(\mathcal{G}_r, W_r)$ and hence has a limit which we denote f^r. We have proved the following, and as a consequence have proved Theorem 1.1.

Theorem 5.1 *There exist constants c_{22}, c_{23} such that for all $u \geq r^2$,*

$$\int |f^r - h^{r,u}| \, dW_r \leq c_{22} \exp\{-c_{23}\sqrt{u}\}.$$

Let Q_s be a probability measure on $\mathcal{G}$ such that the projection of Q_s on $\mathcal{G}_s$ given by Ψ_s is $f^s \, dW_s$. We will take the limit of Q_s as $s \to \infty$. One small point is to make sure that the limit lies on curves that satisfy (4). In other words, we want to show that $\{Q_s\}$ is a tight collection of measures on $\mathcal{G}$. If one starts a Brownian motion on R_0, it is known that $\mathbf{E}(\tau_1) = \infty$. The next lemma implies that this expectation is finite if we require the Brownian motion to avoid an infinite curve.

Lemma 5.2 *There exist c_{24}, c_{25} such that the following holds. Let $\gamma : (-\infty, 0] \to \bar{\mathbf{C}}$ be a continuous curve with $\gamma(0) \in R_0$ and $\Re(\gamma(t)) \to -\infty$ as $t \to -\infty$. Suppose B is a Brownian motion starting at R_0. Then for every $y > 0$,*

$$\mathbf{P}\{\tau_1 > y; B[0, \tau_1] \cap \gamma(-\infty, 0] = \emptyset\} \leq c_{24} e^{-c_{25} y}.$$

Proof. Let

$$\sigma = \inf\{t : B_t \in \gamma(-\infty, 0]\},$$

and $T = \tau_1 \wedge \sigma$. By the Beurling estimates (Lemma 3.1) one can easily see that there exists a $\beta < 1$ such that

$$\mathbf{P}\{T > y + 1 \mid T > y\} \leq \beta,$$

and hence for integer r.

$$\mathbf{P}\{T > r\} \leq \beta^r.$$

But

$$\{\tau_1 > y; B[0,\tau_1] \cap \gamma(-\infty,0] = \emptyset\} \subset \{T > y\}. \quad \Box$$

For any $\bar{\gamma} = (\gamma^1,\gamma^2) \in \mathcal{G}$ let

$$\sigma_s^j = -\sup\{t : \gamma^j(t) \in R_{-s}\}.$$

By Lemma 5.2, we can see that for any $r \geq s$, $y > 0$,

$$Q_r\{\sigma_s^j - \sigma_{s-1}^j > y,\ j = 1,2\} \leq ce^{-c_{25}y}.$$

Hence if we define μ to be the weak limit of the measures Q_r, we have for every s, y,

$$\mu\{\sigma_s^j - \sigma_{s-1}^j > y,\ j = 1,2\} \leq ce^{-c_{25}y}. \tag{24}$$

By the Borel-Cantelli Lemma, there exists a c_{26} such that

$$\mu\{\sigma_s^j - \sigma_{s-1}^j \leq c_{26}\log s \text{ for all sufficiently large } s\} = 1. \tag{25}$$

In particular, μ is supported on (γ^1,γ^2) such that γ^j satisfies (4), i.e., μ is a probability measure on $\mathcal{G}$.

For $s > 0$ let

$$\alpha_s = \mathbf{P}(D_1),$$

where the initial configuration is selected from the distribution $f^s\ dW_s$ on $\mathcal{G}_s$. It follows from Theorem 5.1 that for any initial configuration $\bar{\Gamma}$,

$$\mathbf{P}[D_{s+1}(\bar{\Gamma}) \mid D_s(\bar{\Gamma})] = \alpha_s[1 + O(e^{-c_{23}\sqrt{s}})].$$

In particular, the α_s have a limit α and

$$\alpha_s = \alpha[1 + O(e^{-c_{23}\sqrt{s}})].$$

It is not difficult to see that α must equal $e^{-2\zeta}$. For any initial configuration $\bar{\Gamma}$, and $s > 0$, let

$$\psi_s = e^{2\zeta s}\mathbf{P}[D_s(\bar{\Gamma})].$$

We have just demonstrated the following, which proves Theorem 1.2.

Corollary 5.3 *There exist c_{27}, c_{28} such that the following holds. For any initial configuration $\bar{\Gamma}$, the limit*

$$\psi(\bar{\Gamma}) = \lim_{s\to\infty} \psi_s(\bar{\Gamma})$$

exists. Moreover, if we define $\delta_s = \delta_s(\bar{\Gamma})$ by

$$\psi(\bar{\Gamma}) = \psi_s(\bar{\Gamma})(1 + \delta_s),$$

then

$$|\delta_s| \leq c_{27}e^{-c_{28}\sqrt{s}}.$$

Moreover,

$$c_{28}\psi_1(\bar{\Gamma}) \leq \psi(\bar{\Gamma}) \leq c_{27}\psi_1(\bar{\Gamma}).$$

Corollary 5.4 *If $q(r)$ is defined as in Section 1, then there is a constant c_{29} such that*

$$q(r) \sim c_{29}e^{-2\zeta r}.$$

6 Properties of the Measure

We now will return to the spaces $\mathcal{A}$ and $\mathcal{D}$ on $\mathbf{C}$ as defined in Section 1. We will write μ for the measure obtained on $\mathcal{D}$ by transforming the measure μ of the previous section by the exponential map. Note that by (24),

$$\mathbf{E}_\mu[b(\gamma^1) + b(\gamma^2)] < \infty.$$

For any $\gamma \in \mathcal{X}$ and $r \in \mathbf{R}$ let $\Theta_r\gamma$ denote the function obtained by Brownian scaling by a factor of e^r,

$$\Theta_r\gamma(t) = e^r\gamma(e^{-2r}t), \quad 0 \leq t \leq e^{2r}b(\gamma).$$

Note that $\Theta_r\gamma \in \mathcal{X}(r)$ where

$$\mathcal{X}(r) = \{\gamma : \Theta_{-r}\gamma \in \mathcal{X}\},$$

and $\Theta_r\mu$ gives a measure on $\mathcal{D}(r)$, where $\mathcal{D}(r)$ is defined in the obvious way. For $\gamma \in \mathcal{X}(r)$, $r \geq s$, define $\Upsilon_s\gamma \in \mathcal{X}$ in the following way. Let

$$\sigma = \inf\{t : |\gamma(t)| = e^s\},$$

$$\Upsilon_s\gamma(t) = \gamma(t), \quad 0 \leq t \leq \sigma.$$

For $r > 0$, let μ^r be the measure on $\mathcal{D}$ given by $\Upsilon_0\Theta_r\gamma$. The measure μ^r is not the same as μ. In fact, it is not difficult to show that

$$d\mu^r = \psi_r \, d\mu,$$

where ψ_r is (the transformation by the exponential map of) the ψ_r of the previous section. To be more precise, if $\bar{\Gamma} = (\Gamma^1, \Gamma^2)$ is an initial configuration in $\mathbf{C}$, (a pair of disjoint closed subsets of the closed unit disk with exactly one point of each set of norm one, denoted x_1, x_2) and B^1, B^2 are independent Brownian motions starting at x_1, x_2, respectively, then

$$\psi_r(\bar{\Gamma}) = e^{2\zeta r}\mathbf{P}\{(\Gamma^1 \cup B^1[0, T^1_{e^r}]) \cap (\Gamma^2 \cup B^2[0, T^2_{e^r}]) \subset \{0\}\}.$$

As $r \to \infty$, the measures μ^r approach a measure μ^∞ with

$$d\mu^\infty = \psi \, d\mu.$$

This is the measure of paths conditioned never to intersect.

The measure μ^∞ is the invariant measure for a particular Markov chain. Suppose we start with an initial configuration $\bar{\Gamma} = (\Gamma^1, \Gamma^2)$. As usual, start Brownian motions at x_1, x_2. For any $s > 0$ let $\bar{\Gamma}_s = (\Gamma^1_s, \Gamma^2_s)$ where

$$\Gamma^j_s = e^{-s}(\Gamma^j \cup B^j[0, T^j_{e^s}]). \tag{26}$$

We weight $\bar{\Gamma}_s$ by

$$e^{2\zeta s} I\{\Gamma^1_s \cap \Gamma^2_s \subset \{0\}\} \frac{\psi(\bar{\Gamma}_s)}{\psi(\bar{\Gamma}_0)}.$$

More precisely, let $g_s(\bar{\Gamma}_1, \bar{\Gamma}_2)$ be the transition density of the chain (26) with respect to Wiener measure and let the new chain have transition density

$$e^{2\zeta s} I\{\Gamma^1_s \cap \Gamma^2_s \subset \{0\}\} \frac{\psi(\bar{\Gamma}_s)}{\psi(\bar{\Gamma}_0)} g_s(\bar{\Gamma}_1, \bar{\Gamma}_2).$$

Note that

$$\mathbf{E}[I\{\Gamma_s^1 \cap \Gamma_s^2 \subset \{0\}\}\psi(\bar{\Gamma}_s)] = e^{-2\zeta s}\psi(\bar{\Gamma}_0),$$

and hence this weighting gives a Markov chain, which can be considered as the "h-process" associated with nonintersecting Brownian motions. It is easy to check that μ^∞ is an invariant measure for this Markov chain.

Let $\hat{\mathcal{X}}$ denote the set of continuous functions $\gamma : [0, \infty) \to \mathbf{C}$ with $\gamma(0) = 0$ and $|\gamma(t)| \to \infty$ as $t \to \infty$. Let $\hat{\mathcal{A}} = (\hat{\mathcal{X}})^2$ and let $\hat{\mathcal{D}}$ be the set of $(\gamma^1, \gamma^2) \in \hat{\mathcal{A}}$ such that

$$\gamma^1(0, \infty) \cap \gamma^2(0, \infty) = \emptyset.$$

We can define a measure $\hat{\mu}$ on $\hat{\mathcal{D}}$ by taking a weak limit of the measures μ^r as $r \to \infty$. It is easy to check that for each r, the measure on $\hat{\mathcal{D}}$ given by $\Theta_{-r}\Upsilon_r\hat{\mu}$ equals μ^∞.

The measure $\hat{\mu}$ is a measure on paths starting at 0 and "ending at infinity" conditioned never to intersect. We can use a conformal transformation to get a measure on a pair of paths starting at some fixed point and ending at some other fixed point. Suppose, for example, $f(z) = z/(z+1)$, so that $f(0) = 0$, $f(\infty) = 1$. If B_t is a Brownian motion starting at the origin in $\mathbf{C}$ and

$$\sigma_t = \int_0^t |f'(B_t)|^2 \, dt,$$

$$Y_t = f[B(\sigma^{-1}(t))],$$

then by conformal invariance Y_t is a Brownian motion. We can use the same time change to send the measure $\hat{\mu}$ to a measure $f(\hat{\mu})$ on Brownian paths starting at 0, ending at 1, and conditioned never to intersect. For unrestricted Brownian motions, there is a subtlety in performing this conformal transformation because recurrence implies with probability one

$$\int_0^\infty |f'(B_t)|^2 \, dt = \infty.$$

However, with probability one with respect to $\hat{\mu}$, if $(\gamma^1, \gamma^2) \in \hat{\mathcal{D}}$,

$$\int_0^\infty |f'(B_t)|^2 \, dt < \infty.$$

This essentially follows from (24) and (25).

7 Extensions

For ease we have considered the case of $k = 2$ independent Brownian motions. The results of this paper hold equally well for other numbers of Brownian motions. Since the proofs follows the same basic pattern we will only discuss the results. Throughout this section, constants may depend on k.

We will first consider $k = 1$. It is not obvious immediately, but the analogue of the intersection exponent for one Brownian motion is the disconnection exponent. Let B_t be a Brownian motion taking values in $\mathbf{C}$ starting at $|x| = 1$. For $s > 0$, let D_s be the event

$$D_s = \{B[0, T_{e^s}] \text{ does not disconnect 0 from infinity}\}.$$

The strong Markov property and Brownian scaling imply that

$$\mathbf{P}(D_{s+r}) \le \mathbf{P}(D_s)\mathbf{P}(D_r),$$

and hence there is an $\alpha > 0$ such that as $s \to \infty$,

$$\mathbf{P}(D_s) \approx e^{-2\alpha s},$$

and $\mathbf{P}(D_s) \ge e^{-2\alpha s}$. The number α is called the disconnection exponent for Brownian motion. It has been conjectured [12] from conformal field theory that $\alpha = 1/4$. It is known rigorously [5, 20] that

$$\frac{1}{4\pi} \le \alpha < .49.$$

The methods of this paper can be used to prove that

$$\mathbf{P}(D_s) \sim ce^{-2\alpha s}.$$

Also a limiting measure μ exists on $\mathcal{X}$ which lives on the set

$$\mathcal{Z} = \{\gamma \in \mathcal{X} : \gamma(0, b(\gamma)] \text{ does not disconnect } 0 \text{ from infinity}\}.$$

More precisely, $\mathcal{Z}$ is the set of curves $\gamma \in \mathcal{X}$ such that 0 is in the closure of the unbounded connected component of $\mathbf{C} \setminus \gamma[0, b(\gamma)]$. This measure is also the eigenstate of the Markov chain on $\mathcal{X}$ as described in the first section conditioned that the chain has stayed in $\mathcal{Z}$. There is also a measure $\hat{\mu}$ on $\hat{\mathcal{X}}$ that represents the measure on paths conditioned never to disconnect. If $z_1, z_2 \in \mathbf{C}$, we can conformally transform $\hat{\mu}$ by a linear fractional transformation to get a measure on Brownian paths B with the property that $B(0) = z_1, B(b) = z_2$ and $B(0, b)$ does not disconnect z_1 from z_2. Similar disconnection exponents can be defined by considering k independent Brownian motions and requiring that the union of the k paths not disconnect zero from infinity. The particular case $k = 2$ is interesting because the disconnection exponent is related to the Hausdorff dimension of the frontier or outer boundary of Brownian motion [18, 20].

Now assume $k \ge 2$ and we have $k + 1$ independent Brownian motions $B^1, \ldots, B^{k+1}$ starting at $x_1, \ldots, x_{k+1}$ with $|x_1| = |x_2| = \cdots = |x_{k+1}| = 1$. For $s > 0$, let D_s be the event

$$D_s = \{B^1[0, T^1_{e^s}] \cap (B^2[0, T^2_{e^s}] \cup \cdots \cup B^{k+1}[0, T^{k+1}_{e^s}]) = \emptyset\}. \tag{27}$$

Let

$$q_k(s) = \sup[\mathbf{P}(D_s)],$$

where the supremum is over all $x_1, \ldots, x_{k+1}$ on the unit circle. Again it is straightforward to show that there is a ζ_k such that

$$q_k(s) \approx e^{-2\zeta_k s}, \quad s \to \infty.$$

It is known that $\zeta_2 = 1$ [5]. No other values of ζ_k are known rigorously, and in fact, we know of no conjectures for $k > 2$. Using the methods in this paper, we can show that

$$q_k(s) \sim ce^{-2\zeta_k s}.$$

Let $\tilde{\mathcal{G}}_k = (\mathcal{X})^{k+1}$ and $\tilde{\mathcal{H}}_k$ the set of $(\gamma^1, \ldots, \gamma^{k+1}) \in \tilde{\mathcal{G}}_k$ such that

$$\gamma^1(0, b(\gamma^1)] \cap (\gamma^2(0, b(\gamma^2)] \cup \cdots \cup \gamma^{k+1}(0, b(\gamma^{k+1})]) = \emptyset.$$

Then there is a measure μ on $\tilde{\mathcal{H}}_k$ that represents the limiting measure of paths conditioned not to have intersected in the sense of (27), and similarly there is a measure $\hat{\mu}$ that represents the limiting measure of path conditioned never to intersect. Also by a linear fractional transformation we can define a measure on $k+1$ Brownian paths starting at z_1, ending at z_2, and conditioned so that the first path avoids the other k paths.

Another possible intersection rule is to forbid all intersections. Suppose $k \geq 2$ and let $B^1, \ldots, B^k$ be independent Brownian motions starting at $x_1, \ldots, x_k$ on the unit circle. For $s > 0$, let

$$D_s = \{B^j[0, T^j_{e^s}] \cap B^m[0, T^m_{e^s}] = \emptyset, \ 1 \leq j < m \leq k\}.$$

Let

$$q^k(s) = \sup[\mathbf{P}(D_s)],$$

where the supremum is over all $x_1, \ldots, x_k$ on the unit circle. There exists a λ_k such that

$$q^k(s) \approx e^{-\lambda_k s}, \quad s \to \infty,$$

and $q^k(s) \geq e^{-\lambda_k s}$. No value of λ_k is known rigorously, but conformal field theory [11, 12] predicts

$$\lambda_k = \frac{4k^2 - 1}{12}.$$

The methods of this paper allow us to prove

$$q^k(s) \sim c e^{-\lambda_k s}.$$

We can also define the limiting measure μ. If $k > 2$ the limiting measure lies on $\tilde{\mathcal{H}}^k$, the set of paths $(\gamma^1, \ldots, \gamma^k)$ in $\tilde{\mathcal{G}}_k$ such that

$$\gamma^j(0, b^j] \cap \gamma^m(0, b^m] = \emptyset, \ \ 1 \leq j < m \leq k$$

where $b^j = b(\gamma^j)$. The Markov chain conditioned to stay in $\tilde{\mathcal{H}}^k$ is not ergodic for $k > 2$. If we consider the cyclic order of the points $\gamma^1(b^1), \ldots, \gamma^k(b^k)$, we can see that the order cannot change if the paths are not allowed to cross. The space $\tilde{\mathcal{H}}^k$ splits into $(k-1)!$ equivalence classes all of which look the same. If we start Brownian motions on the unit circle and condition them not to intersect, they the order of the endpoints on the circle eventually stop changing. The measure on paths then converges to μ on $\tilde{\mathcal{H}}^k$ (normalized to be a probability measure). Equivalently we can think of μ as lying on the space of *unordered* k-tuples of curves.

To make things a little more complicated consider $k + m$ independent Brownian motions $B^1, \ldots, B^{k+m}$ starting on the unit circle and let

$$D_s = \{(B^1[0, T^1_{e^s}] \cup \cdots \cup B^k[0, T^k_{e^s}]) \cap (B^{k+1}[0, T^{k+1}_{e^s}] \cup \cdots \cup B^{k+m}[0, T^{k+m}_{e^s}]) = \emptyset\}.$$

Let

$$q_{k,m} = \sup[\mathbf{P}(D_s)], \tag{28}$$

where the supremum is over all starting points in the unit circle. Then there exists a $\zeta_{k,m}$ such that

$$q_{k,m}(s) \approx e^{-2\zeta_{k,m} s}, \quad s \to \infty,$$

and $q_{k,m}(s) \geq e^{-2\zeta_{k,m}s}$. Again the methods of this paper can be used to prove that

$$q_{k,m}(s) \sim ce^{-2\zeta_{k,m}s}.$$

When we try to define a limiting measure μ we run into the same problem as in the preceding paragraph. Certain crossing of points on the unit circle are forbidden, so μ must lie on a smaller set of paths. In this case we have the added difficulty that these different equivalent classes are not identical, and in fact we would expect them to have different eigenvalues associated with them. For example, if $k = m = 2$ then the points can be arranged x_1, x_2, x_3, x_4 or x_1, x_3, x_2, x_4. One would expect that paths of the first type would find it easier to survive. In fact we conjecture that the supremum in (28) is taken on by paths for which the points are "separated," i.e., for large s on the event D we can draw a straight line which separates $\{B^1(T^1_{e^s}), \ldots, B^k(T^k_{e^s})\}$ and $\{B^{k+1}(T^{k+1}_{e^s}), \ldots, B^{k+m}(T^{k+m}_{e^s})\}$. We do not have a proof of this fact. The methods of this paper do allow us to analyze each "equivalence class" and show that an appropriate limiting measure exists for each class. We could do similarly for even more complicated intersections rules, but we have discussed enough here.

8 Estimation of an Exponent

The methods of this paper say very little about how to compute the exponents. We will sketch the proof of one result in this chapter. Let the exponents $\zeta(k, m) = \zeta_{k,m}$ be defined as in the previous section. The case $k = l = 1, m = n = 2$ of the following proposition was proved in [7].

Proposition 8.1 *For any positive integers k, l, m, n,*

$$\zeta(k + l, m + n) > \zeta(k, m) + \zeta(l, n).$$

Let $B^1, \ldots, B^{k+l+m+n}$ be independent Brownian motions starting at $x_1, \ldots, x_{k+l+m+n}$ on the unit circle. For $s > 0$ define sets

$$V^1_s = B^1[0, T^1_{e^s}] \cup \cdots \cup B^k[0, T^k_{e^s}],$$
$$V^2_s = B^{k+1}[0, T^{k+1}_{e^s}] \cup \cdots \cup B^{k+l}[0, T^{k+l}_{e^s}],$$
$$V^3_s = B^{k+l+1}[0, T^{k+l+1}_{e^s}] \cup \cdots \cup B^{k+l+m}[0, T^{k+l+m}_{e^s}],$$
$$V^4_s = B^{k+l+m+1}[0, T^{k+l+m+1}_{e^s}] \cup \cdots \cup B^{k+l+m+n}[0, T^{k+l+m+n}_{e^s}],$$

and events

$$D^1_s = \{V^1_s \cap V^3_s = \emptyset\},$$
$$D^2_s = \{V^2_s \cap V^4_s = \emptyset\},$$
$$D^3_s = \{(V^1_s \cup V^2_s) \cap (V^3_s \cup V^4_s) = \emptyset\}.$$

Let $\alpha = 2\zeta(k, m), \beta = 2\zeta(m, n)$. We know that

$$\mathbf{P}(D^1_s) \leq ce^{-s\alpha}, \quad \mathbf{P}(D^2_s) \leq ce^{-s\beta}.$$

Also

$$D^3_s \subset D^1_s \cap D^2_s.$$

Since D_s^1 and D_s^2 are independent, it is immediate that $\zeta(k+l,m+n) \geq \zeta(k,m)+\zeta(l,n)$. The hard part of the proposition is to prove the strict inequality. Let

$$D_s^4 = \{V_s^1 \cap V_s^4 = \emptyset\}.$$

It suffices to prove that there is a $\lambda > 0$ such that

$$\mathbf{P}(D_{s-1}^4 \mid D_s^1 \cap D_s^2) \leq ce^{-\lambda s}, \tag{29}$$

for then

$$\begin{aligned} P(D_s^3) &\leq \mathbf{P}(D_s^1 \cap D_s^2 \cap D_{s-1}^4) \\ &\leq \mathbf{P}(D_s^1 \cap D_s^2)\mathbf{P}(D_{s-1}^4 \mid D_s^1 \cap D_s^2) \\ &\leq ce^{-s(\alpha+\beta+\lambda)}. \end{aligned}$$

The proof of the proposition is based on two observations. Fix starting points $x_1, \ldots, x_{k+l+m+n}$ on the unit circle. Let $\mathcal{G}_r^1 = (\mathcal{X}_{-r})^{k+m}, \mathcal{G}_r^2 = (\mathcal{X}_{-r})^{l+n}$. Let $\mathcal{H}_r^1$ be the set of $(\gamma^1, \ldots, \gamma^{k+m}) \in \mathcal{G}_r^1$ such that

$$(\gamma^1[0,b_{\gamma^1}] \cup \cdots \cup \gamma^k[0,b_{\gamma^k}]) \cap (\gamma^{k+1}[0,b_{\gamma^{k+1}}] \cup \cdots \cup \gamma^{k+m}[0,b_{\gamma^{k+m}}]) = \emptyset,$$

and define $\mathcal{H}_r^2$ similarly. We make the natural identification of paths which start at the circle of radius 1 and end at radius e^r with paths which start at the circle of radius e^{-r} and end at radius 1. Let P_s^1, P_s^2 be the measures on $\mathcal{H}_s^1$ and $\mathcal{H}_s^2$, respectively, given by Wiener measure conditioned on the events $\mathcal{H}_s^1$, $\mathcal{H}_s^2$ respectively, and let Q_s^j $(j=1,2)$ denote the measure on $\mathcal{H}_{s-1}^j$ obtained by projecting P_s^2 by Υ_{s-1}. Let $\hat{Q}_s^j$ denote the probability measure on $\mathcal{G}_{s-1}^j$ whose density with respect to Wiener measure W_{s-1}^j is a constant times $I(\mathcal{H}_{s-1}^j)\psi$ where I denotes indicator function and ψ is as defined in the previous section. By an analogue of Corollary 5.3, there exist constants c_1, c_2 such that for every $A \subset \mathcal{H}_{s-1}^j$,

$$c_1 Q_s^j(A) \leq \hat{Q}_s^j(A) \leq c_2 Q_s^j(A).$$

Hence it suffices to prove (29) for the measures $\hat{Q}_s^1, \hat{Q}_s^2$, i.e., it suffices to prove that if $\mathcal{H}_{s-1}^3 = \mathcal{H}_{s-1}^1 \times \mathcal{H}_{s-1}^2$; $\hat{Q}_s^3 = \hat{Q}_s^1 \times \hat{Q}_s^2$; and $\mathcal{H}_{s-1}^4$ is the subset of $((\gamma^1, \ldots, \gamma^{k+m}), (\gamma^{k+m+1}, \ldots, \gamma^{k+m+l+n})) \in \mathcal{H}_{s-1}^3$ with

$$\gamma^1[0, b(\gamma^1)] \cap \gamma^1[0, b(\gamma^{k+l+m+n})] = \emptyset,$$

then

$$\hat{Q}_s^3(\mathcal{H}_{s-1}^4) \leq ce^{-\lambda s}. \tag{30}$$

To prove this, we only need the analogue of Lemma 3.13. This lemma states that under the measure $\hat{Q}$, if γ^1 and $\gamma^{k+l+m+n}$ have not intersected up to time e^r there is a positive probability (independent of the path up to time e^r) such that the paths will intersect before time e^{r+2}. Iterating this fact gives (30). We omit the details.

References

[1] Ahlfors, L. (1973) . *Conformal Invariants.* McGraw-Hill.

[2] Aizenman, M. (1985). The intersection of Brownian paths as a case study of the renormalization group method for quantum field theory. Commun. Math. Phys. **97**, 91-110.

[3] Burdzy, K. (1989). Cut points on Brownian paths. Ann. Probab. **17** 1012–1036.

[4] Burdzy, K. and Lawler, G. (1990). Non-intersection exponents for random walk and Brownian motion. Part I: Existence and an invariance principle. Probab. Th. and Rel. Fields **84** 393-410.

[5] Burdzy, K. and Lawler, G. (1990). Non-intersection exponents for random walk and Brownian motion. Part II: Estimates and applications to a random fractal. Ann. Probab. **18** 981–1009.

[6] Burdzy, K., Lawler, G. and Polaski, T. (1989). On the critical exponent for random walk intersections. J. Stat. Phys. **56** 1–12.

[7] Burdzy, K. and Werner, W. (1994). No triple point of planar Brownian motion is accessible, preprint.

[8] Cardy, J. (1987). Conformal invariance, in *Phase Transitions*, Vol. 11, C. Domb and J. Lebowitz, ed., Academic Press.

[9] Cranston, M. and Mountford, T. (1991). An extension of a result of Burdzy and Lawler. Probab. Th. and Rel. Fields **89**, 487-502.

[10] Doob, J. L. (1984). *Classical Potential Theory and Its Probabilistic Counterparts*, Springer-Verlag.

[11] Duplantier, B. and Kwon, K.-H. (1988). Conformal invariance and intersections of random walks. Phys. Rev. Lett. **61** 2514–2517.

[12] Duplantier, B., Lawler, G., Le Gall, J.-F., and Lyons, T. (1993). The geometry of the Brownian curve. Bull. Sci. math. 2^{e} série, **117**, 91-106.

[13] Durrett, R. (1984). *Brownian Motion, Martingales, and Analysis.* Wadsworth.

[14] Dvoretzky, A., Erdős, P. and Kakutani, S. (1950). Double points of paths of Brownian motions in n-space. Acta. Sci. Math. Szeged **12** 75-81.

[15] Ityzkon, C. and Drouffe, J.-M. (1989). *Statistical Field Theory*, Vol. 2, Cambridge University Press.

[16] Lawler, G. (1991). *Intersections of Random Walks.* Birkhäuser-Boston.

[17] Lawler, G. (1996). Hausdorff dimension of cut points for Brownian motion, Electronic Journal of Probability **1**, paper no. 2.

[18] Lawler, G. (1995). The dimension of the frontier of planar Brownian motion, Duke University preprint 95-05.

[19] Li, B. and Sokal, A. (1990). High-precision Monte Carlo test of the conformal-invariance predictions for two-dimensional mutually avoiding walks. J. Stat. Phys. **61** 723-748.

[20] Werner, W. (1995). An upper bound to the disconnection exponent for two-dimensional Brownian motion, to appear in Bernoulli.