ON WEAKLY MEASURABLE STOCHASTIC PROCESSES AND ABSOLUTELY SUMMING OPERATORS

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(Received December 1, 2005)

Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. A characterization of absolutely summing operators by means of McShane integrable stochastic processes is considered.

Keywords: Pettis integral, McShane integral, amart, uniform amart, absolutely summing operators

MSC 2000: 60G48, 28B05

1. Introduction

A linear and continuous operator between Banach spaces is said to be absolutely summing if it maps unconditionally convergent series into absolutely convergent series. Moreover, it improves properties of stochastic processes. Indeed, N. Ghoussoub in [7] proved that an operator is absolutely summing if and only if it maps amarts (asymptotic martingales) into uniform amarts. In this paper we go a bit further studying the composition of stochastic processes consisting of weakly measurable functions with absolutely summing operators. In particular, we consider stochastic processes of McShane and Pettis integrable functions. Both these processes generalize the more familiar notion of Bochner stochastic processes. For functions taking values in an infinite dimensional Banach space the Bochner integral and the Pettis integral are the most known generalizations of the Lebesgue integral. The family of all McShane integrable functions is strictly contained between the family of all

Supported by MURST of Italy.

Bochner integrable functions and the family of all Pettis integrable functions. Even if the Bochner integral is the natural candidate to generalize the Lebesgue integral in Banach spaces, elementary classical examples show that it is highly restrictive. In fact it integrates few functions, for example the function

$$f\colon\ [0,1] \to \ell_\infty[0,1]$$

defined as

$$f(t) = \chi_{[0,t]},$$

is not strongly measurable. Even for strongly measurable functions the Bochner integral does not necessarily exist. Indeed, if E is an infinite dimensional Banach space, by the Dvoretzky-Rogers Theorem there exists an unconditionally convergent series $\sum\limits_{n=1}^{\infty} x_n$ that is not absolutely convergent. If $(A_n)_n$ is a countable measurable partition of a probability space (Ω, P) such that $P(A_n) > 0$ for each $n \in \mathbb{N}$, the function

$$X = \sum_{n=1}^{\infty} \frac{x_n}{P(A_n)} \chi_{A_n}$$

is strongly measurable but not Bochner integrable.

Actually little is known about adapted sequences of Pettis integrable functions (see for example [5], [9], [12] and [17]) or about adapted sequences of McShane integrable functions (see [11]). Indeed, the conditional expectation of Pettis integrable functions does not necessarily exist. Moreover, the class of all Pettis integrable functions, endowed with the Pettis norm, is complete if and only if the Banach space is of finite dimension.

In Proposition 1 we extend to amarts of McShane integrable functions a characterization known in the case of Bochner integrable stochastic processes. We use this result and a recent characterization of absolutely summing operators (see [10] and [14]) to characterize absolutely summing operators by means of amarts of McShane integrable functions (Theorem 1). In Examples 1 and 2 we give applications of this result.

2. Definitions and a preliminary result

Let E and F be two Banach spaces with a norm $\|\cdot\|_E$ and $\|\cdot\|_F$ respectively; E^* and $B(E^*)$ denote respectively the dual of E and its unit ball.

Throughout, $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_m \subset \mathcal{F}_n$ if $m < n$. A *stopping time* is a map $\tau\colon\ \Omega \to \mathbb{N}\cup\{\infty\}$ such that, for each $n \in \mathbb{N}$, $\{\tau \leqslant n\} := \{\omega \in \Omega\colon\ \tau(\omega) \leqslant n\} \in \mathcal{F}_n$. We denote by T

the collection of all *simple* stopping times (i.e. taking finitely many values and not taking the value ∞). Then T is a directed set filtering to the right.

Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{F}$, then a function $X\colon \Omega \to E$ is called *weakly $\mathcal{F}_0$-measurable* if the function fX is $\mathcal{F}_0$-measurable for every $f \in E^*$. A weakly $\mathcal{F}$-measurable function is called *weakly measurable.* A function $X\colon \Omega \to E$ is said to be *Pettis integrable* if fX is Lebesgue integrable on Ω for each $f \in E^*$ and there exists a set function $\nu\colon \mathcal{F} \to E$ such that

$$f\nu(A) = \int_A fX$$

for all $f \in E^*$ and $A \in \mathcal{F}$. In this case we write $\nu(A) = (P)\int_A X$ and we call $\nu(\Omega)$ the *Pettis integral* of X *over* Ω and ν is the *indefinite Pettis integral of* X.

The space of all Pettis integrable functions $X\colon \Omega \to E$ is denoted by $\mathcal{P}(E)$. The Pettis norm of a Pettis integrable functions is

$$|X|_P = \sup\left\{ \int_\Omega |fX| :\ f \in B(E^*)\right\}.$$

It is well known that

$$\sup\left\{\left\|(P)\int_A X\right\|:\ A \in \mathcal{F}\right\}$$

defines an equivalent norm in $\mathcal{P}(E)$.

It should be noted that, in general, if X is only Pettis integrable and not bounded enough, then even in the space $E = \ell_2(\mathbb{N})$, there is no *Pettis conditional expectation* of X with respect to a sub-σ-algebra of $\mathcal{F}$ (see [16], Example 6-4-1).

We say that $(X_n, \mathcal{F}_n)_n$ is a *stochastic process of Pettis integrable functions* if, for each $n \in \mathbb{N}$, $X_n\colon \Omega \to E$ is Pettis integrable and weakly $\mathcal{F}_n$-measurable.

Let $(\Omega, \mathcal{A}, \mathcal{F}, P)$ be a probability space which is Radon and outer regular, where $\mathcal{A}$ denotes the topology in Ω. A *McShane partition* of Ω is a set $\{(S_i, \omega_i), i = 1, 2\ldots\}$ where $(S_i)_i$ is a disjoint family of measurable sets of finite measure, $P\Big(\Omega \setminus \bigcup_{i=1}^{\infty} S_i\Big) = 0$ and $\omega_i \in \Omega$ for each $i = 1, 2, \ldots$. A *gauge* on Ω is a function $\Delta\colon \Omega \to \mathcal{A}$ such that $\omega \in \Delta(\omega)$ for each $\omega \in \Omega$. A McShane partition $\{(S_i, \omega_i), i = 1, 2, \ldots\}$ is *subordinate to a gauge* Δ if $S_i \subset \Delta(\omega_i)$ for $i = 1, 2, \ldots$.

A function $f\colon \Omega \to E$ is said to be *McShane integrable* on $\mathcal{G}$, when $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ (see [6] Definition 1A) (briefly McS-*integrable* on $\mathcal{G}$), with *McShane integral* $z \in E$, if for each $\varepsilon > 0$ there exists a gauge $\Delta\colon \Omega \to \mathcal{A}$ such that

$$\limsup_n \left\| \sum_{i=1}^{n} P(S_i) f(\omega_i) - z \right\| < \varepsilon$$

for each McShane partition $\{(S_i,\omega_i)\colon\ i=1,2,\ldots\}$ subordinate to Δ with $S_i\in\mathcal{G}$. In case $\mathcal{G}=\mathcal{F}$ we simply say that X is McS-integrable.

It is known that if $f\colon\ \Omega\to E$ is McS-integrable, then its indefinite Pettis integral is totally bounded (see [6], Corollary 3E), hence it is norm relatively compact. Denote by $\mathrm{McS}(E)$ the set of all McS-integrable functions $X\colon\ \Omega\to E$ endowed with the norm

$$|X|_{\mathrm{McS}}=\sup\left\{\int_\Omega |fX|:\ f\in B(E^*)\right\}.$$

We know that it is equivalent to the norm ([15])

$$\sup\left\{\left\|(\mathrm{McS})\int_A X\right\|:\ A\in\mathcal{F}\right\}.$$

If $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, if X is McS-integrable and if Y is McS-integrable on $\mathcal{G}$, then Y is called the *McShane conditional expectation* of X with respect to $\mathcal{G}$ if for every $A\in\mathcal{G}$,

$$(\mathrm{McS})\int_A Y=(\mathrm{McS})\int_A X.$$

We observe that the conditional expectation of a McS-integrable function does not always exist. Indeed, the same is true for strongly measurable Pettis integrable functions and then for McShane integrable functions (see [8], Theorem 17).

We say that $(X_n,\mathcal{F}_n)_n$ is a *stochastic process of* McS-*integrable functions*, if for each $n\in\mathbb{N}$, X_n is McS-integrable and weakly measurable with respect to $\mathcal{F}_n$. We recall that $(X_n,\mathcal{F}_n)_n$ is a *stochastic process* if, for each $n\in\mathbb{N}$, X_n is Bochner integrable and $\mathcal{F}_n$-measurable. For $\tau\in T$, let

$$\mathcal{F}_\tau=\{A\in\mathcal{F}\colon\ A\cap\{\tau=n\}\in\mathcal{F}_n,\ \text{for each } n\in\mathbb{N}\}$$

and

$$X_\tau=\sum_{n=\min\tau}^{\max\tau}X_n\chi_{\{\tau=n\}}.$$

Definition 1. A stochastic process $(X_n,\mathcal{F}_n)_n$ of McS-integrable functions is called an *amart* if the net $((\mathrm{McS})\int_\Omega X_\tau)_{\tau\in T}$ converges in E, that is there is $y\in E$ such that for each $\varepsilon>0$ there is $\sigma_0\in T$ such that if $\tau\in T$ and $\tau\geqslant\sigma_0$ then

$$\left\|(\mathrm{McS})\int_\Omega X_\tau-y\right\|_E<\varepsilon.$$

It is worth recalling at this point that the previous definition extends to stochastic processes of McS-integrable functions the more familiar notion of amart (see [1]) known for stochastic processes.

In the sequel we shall make use of the following facts and proposition.

For each $\tau \in T$ define $\mu_\tau(A) = (\mathrm{McS}) \int_A X_\tau$ for $A \in \mathcal{F}_\tau$.

The following proposition extends to amarts of McShane integrable functions a result known for Bochner integrable functions.

Proposition 1. *Let $(X_n, \mathcal{F}_n)_n$ be an amart. Then the family $(\mu_\tau(A))_\tau$ converges to a limit $\mu_\infty(A)$ for each $A \in \mathcal{F}_\infty = \bigcup_{\tau \in T} \mathcal{F}_\tau = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, and the convergence is uniform on $\mathcal{F}_\infty$ in the sense that for each $\varepsilon > 0$ there is $\sigma_0(\varepsilon) \in T$ such that if $\tau \in T$ and $\tau \geqslant \sigma_0$ then*

$$\|\mu_\tau(A) - \mu_\infty(A)\|_E < \varepsilon \quad \text{for all} \quad A \in \mathcal{F}_\tau.$$

Proof. Let $(X_n, \mathcal{F}_n)_n$ be an amart. Since the net $\big((\mathrm{McS}) \int X_\tau\big)_{\tau \in T}$ converges, for any fixed $\varepsilon > 0$ there is $\sigma_0(\varepsilon) \in T$ such that if $\tau \geqslant \sigma \geqslant \sigma_0(\varepsilon) \in T$ then

$$\left\|(\mathrm{McS}) \int X_\tau - (\mathrm{McS}) \int X_\sigma\right\|_E < \frac{\varepsilon}{2}. \tag{1}$$

Fix now $\tau \geqslant \sigma \geqslant \sigma_0(\varepsilon)$. Let $A \in \mathcal{F}_\sigma$. Choose a natural number $n \in \mathbb{N}$ such that $n \geqslant \tau \geqslant \sigma$ and define

$$\tau_1 = \begin{cases} \tau \text{ on } A, \\ n \text{ on } \Omega \setminus A, \end{cases} \qquad \sigma_1 = \begin{cases} \sigma \text{ on } A, \\ n \text{ on } \Omega \setminus A. \end{cases}$$

Then τ_1 and σ_1 are stopping times and $\tau_1 \geqslant \sigma_1 \geqslant \sigma_0$. Moreover,

$$\mu_\tau(A) - \mu_\sigma(A) = (\mathrm{McS}) \int_A X_\tau - (\mathrm{McS}) \int_A X_\sigma = (\mathrm{McS}) \int X_{\tau_1} - (\mathrm{McS}) \int X_{\sigma_1},$$

hence by (1) we get

$$\|\mu_\tau(A) - \mu_\sigma(A)\|_E < \frac{\varepsilon}{2} \tag{2}$$

for all $A \in \mathcal{F}_\sigma$. Let $A \in \mathcal{F}_\infty$, then $A \in \mathcal{F}_m$ for some $m \in \mathbb{N}$. Now $\mu_\tau(A)$ is defined for all $\tau \geqslant m$ and $(\mu_\tau(A))_{\tau \geqslant m}$ is Cauchy in E. Indeed, if $\sigma \geqslant \sigma_0(\varepsilon) \vee m$ and $\tau \geqslant \sigma_0(\varepsilon) \vee m$ by (2) we obtain

$$\|\mu_\tau(A) - \mu_\sigma(A)\|_E \leqslant \|\mu_\tau(A) - \mu_{\sigma_0(\varepsilon) \vee m} \sigma(A)\|_E + \|\mu_\sigma(A) - \mu_{\sigma_0(\varepsilon) \vee m}(A)\|_E < \varepsilon.$$

Hence the limit $\mu_\infty(A)$ exists in E for all $A \in \mathcal{F}_\infty$. Thus we can take the limit in (2) as required. $\square$

We recall that a stochastic process $(X_n, \mathcal{F}_n)_n$ is a *uniform amart* (see [1]) if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that if $\sigma \in T$ and $\sigma \geqslant n_0$ then

$$\mathrm{Var}(\mu_\sigma - \mu_\infty | \mathcal{F}_\sigma) \leqslant \varepsilon,$$

where the symbol $\mathrm{Var}(\mu)$ denotes the variation of the measure μ, that is

$$\mathrm{Var}(\mu) = \sup \sum_{i=1}^{n} \|\mu(E_i)\|,$$

where the supremum is taken over the finite partitions $\{E_1, \ldots, E_n\}$ of Ω.

3. Composition of absolutely summing operators with weakly measurable functions

In this section we consider the composition of operators with weakly measurable functions. As usual, bounded linear maps between Banach spaces are referred to as operators. The symbol $\mathcal{L}(E, F)$ denotes the space of operators from E to F. Let $u \in \mathcal{L}(E, F)$. Define U from $\mathrm{McS}(E)$ to $\mathrm{McS}(F)$ (or respectively from $\mathcal{B}(E)$ to $\mathcal{B}(F)$, where the symbols $\mathcal{B}(E)$ and $\mathcal{B}(F)$ denote respectively the family of all Bochner integrable functions taking values in E or F) by

$$(UX)(\omega) = u(X(\omega)).$$

Then u "lifts" to an operator $U \in \mathcal{L}(\mathrm{McS}(E), \mathrm{McS}(F))$ (or to an operator $U \in \mathcal{L}(\mathcal{B}(E), \mathcal{B}(F))$) (see [6] (or resp. [3])).

If $\mu\colon \Omega \to E$ is an E-valued additive set function defined on an algebra $\mathcal{G}$ of a subset of Ω, then $U\mu$ defined as $U\mu(A) = U(\mu(A))$ is an F-valued additive set function defined on the algebra $\mathcal{G}$. Recall that the *semivariation* of μ is defined as

$$|\mu| = \sup \{\mathrm{Var}\ f(\mu)\colon\ f \in B(E^*)\}.$$

It is well known that

$$|\mu| \leqslant 4 \sup \{\|\mu(A)\|\colon\ A \in \mathcal{G}\}.$$

An operator $u \in \mathcal{L}(E, F)$ is said to be *absolutely summing* if there is a constant $c \geqslant 0$ such that, for every choice of an integer n and vectors $\{x_i\}_{i=1}^n$ in E, we have

$$\sum_{i=1}^{n} \|u(x_i)\|_F \leqslant c \sup_{f \in B(E^*)} \sum_{i=1}^{n} |f(x_i)|. \tag{3}$$

The least c for which inequality (3) always holds is denoted by $\pi(u)$.

Proposition 2. *Let $u\colon E\to F$ be an absolutely summing operator. Then there is a constant C such that for every McShane integrable function $X\colon\ \Omega\to E$ it follows that*

$$\int \|U(X)\| \leqslant C|X|_{\mathrm{McS}}.$$

P r o o f. By [14] Theorem 3.13 the function $UX\colon\ \Omega\to F$ is Bochner integrable. We want to prove that the operator U from $\mathrm{McS}(E)$ to $\mathcal{B}(F)$ is continuous. Let $s=\sum\limits_{i=1}^{n} x_i\chi_{A_i}$ be a simple function, then

$$\begin{aligned}\|Us\|_{\mathcal{B}(F)} &= \int_\Omega \|(Us)(\omega)\|_F = \int_\Omega \|u(s(\omega))\|_F\\ &= \int_\Omega \Bigl\|u\Bigl(\sum_{i=1}^{n} x_i\chi_{A_i}(\omega)\Bigr)\Bigr\|_F = \int_\Omega \Bigl\|\sum_{i=1}^{n} u(x_i)\chi_{A_i}(\omega)\Bigr\|_F.\end{aligned}\tag{4}$$

Applying the disjointness of A_i's and the linearity of the integral we get

$$\begin{aligned}\int_\Omega \Bigl\|\sum_{i=1}^{n} u(x_i)\chi_{A_i}(\omega)\Bigr\|_F &= \sum_{i=1}^{n}\int_{A_i} \|u(x_i)\|_F\\ &= \sum_{i=1}^{n} \|u(x_i)\|_F P(A_i) = \sum_{i=1}^{n} \|u(P(A_i)x_i)\|_F\\ &\leqslant \pi(u)\sup\Bigl\{\sum_{i=1}^{n} |f(P(A_i)x_i)|\colon\ f\in B(E^*)\Bigr\} = \pi(u)|s|_{\mathrm{McS}(E)}\end{aligned}\tag{5}$$

where the last inequality follows from the definition of the McS-norm. Thus by (4) and (5), we get

$$\|Us\|_{\mathcal{B}(F)} \leqslant \pi(u)|s|_{\mathrm{McS}(E)}\tag{6}$$

for every simple function. If $X\in\mathrm{McS}(E)$, then its indefinite Pettis integral is relatively compact (see [6] Corollary 3E). Therefore simple functions are dense in $\mathrm{McS}(E)$ ([13] Theorem 9.1) with the $\mathrm{McS}(E)$-norm. Let (t_n) be a sequence of simple functions converging to X in the $\mathrm{McS}(E)$-norm. Then it is Cauchy in the $\mathrm{McS}(E)$-norm, moreover, there is a subsequence (s_n) of (t_n) such that for each $f\in E^*$

$$\lim_{n\to\infty} f(s_n(\omega)) = f(X(\omega))$$

for all $\omega\notin N_f$, $P(N_f)=0$. By (6) and the linearity of U we get

$$\|Us_n - Us_m\|_{\mathcal{B}(F)} \leqslant \pi(u)|s_n - s_m|_{\mathrm{McS}(X)}.$$

Therefore the sequence (Us_n) is Cauchy in $\mathcal{B}(F)$. Since $\mathcal{B}(F)$ is complete there is a function $Y \in \mathcal{B}(F)$ such that (Us_n) converges to Y in $\mathcal{B}(F)$. Without loss of generality we can assume that the convergence is also a.e. So there is a set N with $P(N) = 0$ such that for each $g \in F^*$ and $\omega \notin N$

$$\lim_{n\to\infty} g((Us_n)(\omega)) = g(Y(\omega)). \tag{7}$$

Let $u^*\colon F^* \to E^*$ be the adjoint operator of u. Since it is weak*-continuous we obtain

$$\begin{aligned} \lim_{n\to\infty} \langle (Us_n)(\omega), g\rangle &= \lim_{n\to\infty} \langle u(s_n)(\omega), g\rangle \\ &= \lim_{n\to\infty} \langle s_n(\omega), u^*g\rangle = \langle X(\omega), u^*g\rangle \\ &= \langle u(X(\omega)), g\rangle = \langle (UX)(\omega), g\rangle \end{aligned} \tag{8}$$

a.e. on Ω. From (7) and (8) we get that $g(Y(\omega)) = g((UX)(\omega))$ a.e. on Ω. Since the functions UX and Y are strongly measurable, $Y(\omega) = (UX)(\omega)$ a.e. So the sequence (Us_n) converges to (UX) a.e. and in $\mathcal{B}(F)$. Furthermore, we have

$$\|UX\|_{\mathcal{B}(F)} = \lim_{n\to\infty} \|Us_n\|_{\mathcal{B}(F)} \leqslant \pi(u) \lim_{n\to\infty} |s_n|_{\mathrm{McS}(E)} = \pi(u)|X|_{\mathrm{McS}(E)}.$$

Therefore U is continuous and there is a constant C such that

$$\int \|U(X)\| \leqslant C|X|_{\mathrm{McS}},$$

as required. □

4. Main result

Our main result is

Theorem 1. *Let $u \in \mathcal{L}(E, F)$. Then u is absolutely summing if and only if U maps every E-valued amart of McShane integrable functions to an F-valued uniform amart.*

Proof. Let $X\colon \Omega \to E$ be a McShane integrable function, then

$$\mu(A) = (\mathrm{McS}) \int_A X$$

is an E-valued additive set function. If u is absolutely summing then UX is Bochner integrable by Proposition 2 and there is a constant C such that

$$\text{(9)} \qquad \operatorname{Var}(U(\mu)) = \int \|U(X)\| \leqslant C|X|_{\mathrm{McS}} = C|\mu|.$$

Let $(X_n, \mathcal{F}_n)_n$ be an E-valued amart of McShane integrable functions and $\varepsilon > 0$. According to Proposition 1 there is $m_0 \in \mathbb{N}$ such that

$$\text{(10)} \qquad |\mu_\sigma - \mu_\infty|\mathcal{F}_\sigma| < \frac{\varepsilon}{C}$$

for all $\sigma \in T$, $\sigma \geqslant m_0$. By (9) and (10) we get

$$\operatorname{Var}(U(\mu_\sigma - \mu_\infty|\mathcal{F}_\sigma)) \leqslant C|\mu_\sigma - \mu_\infty|\mathcal{F}_\sigma| < \varepsilon.$$

Therefore $(UX_n)_n$ is a uniform amart as required.

To prove the converse assume that the operator U is not absolutely summing. Then there is a series $\sum\limits_{n=1}^{\infty} x_n$ in E which is unconditionally summable, but such that $\sum\limits_{n=1}^{\infty} \|Ux_n\| = \infty$. We can find an increasing sequence of integers $(n_k)_k$ such that

$$\sum_{n=n_k+1}^{n_{k+1}} \|Ux_n\| \geqslant 1.$$

Multiplying some x_n's by coefficients smaller than 1, we can assume, without loss of generality, that

$$\sum_{n=n_k+1}^{n_{k+1}} \|Ux_n\| = 1.$$

For every $k \in \mathbb{N}$, divide the interval $[0,1]$ into $(n_{k+1} - n_k)$ subintervals $A_{k,n}$ of length $\|Ux_n\|$. Let $(\Omega, \mathcal{F}, P)$ be the interval $[0,1]$ with the Lebesgue measure. We define a sequence of functions $X_k \colon \Omega \to E$ as

$$\text{(11)} \qquad X_k(\omega) = \sum_{n=n_k+1}^{n_{k+1}} \frac{x_n}{\|Ux_n\|} \chi_{A_{k,n}}.$$

As, for each k, X_k is a countably valued function, it is strongly measurable. Since

$$\sum_{n=n_k+1}^{n_{k+1}} \frac{x_n}{\|Ux_n\|} P(A_{k,n}) = \sum_{n=n_k+1}^{n_{k+1}} x_n$$

is unconditionally convergent, each X_k is McShane integrable (see [8], Theorem 15). For every $k \in \mathbb{N}$, $\mathcal{F}_k$ will be the σ-algebra $\sigma(X_1, X_2, \ldots, X_k)$. Then the sequence $(X_k)_k$ is adapted to the the family $(\mathcal{F}_k)_k$. To prove that $(X_k)_k$ is an amart, let $N \in \mathbb{N}$ and let $\sigma \geqslant N$ be a stopping time. For each $k \geqslant N$, let $B_{k,n} = A_{k,n} \cap \{\sigma = k\}$. Then it follows that

$$\text{(12)} \qquad (\mathrm{McS}) \int X_\sigma = \sum_{k \geqslant N} (\mathrm{McS}) \int_{\{\sigma = k\}} X_k = \sum_{k \geqslant N} \sum_{n=n_k+1}^{n_{k+1}} P(B_{k,n}) \frac{x_n}{\|Ux_n\|}.$$

Since $B_{k,n} \subseteq A_{k,n}$, $P(B_{k,n})/\|Ux_n\| = \alpha_{k,n} \leqslant 1$. Therefore, since $\sum\limits_{i=1}^{\infty} x_n$ is unconditionally summable, by (12) we get that

$$(\mathrm{McS}) \int X_\sigma = \sum_{k \geqslant N} \sum_{n=n_k+1}^{n_{k+1}} \alpha_{k,n} x_n$$

converges to zero. Since each X_n is strongly measurable, hence $(UX_n)_n$ is an F-valued strongly measurable amart which is not a uniform amart. Indeed, one should have $\lim\limits_{\sigma} \int \|UX_\sigma\| = 0$. By the real-valued amart convergence theorem (see [4] Theorem 1.2.5), $(\|UX_k\|)_k$ must converge to zero, but for each $k \geqslant N$ and for each $\omega \in \Omega$ we have $\|UX_k(\omega)\|_F = 1$. Therefore the assertion holds true. □

The previous theorem extends to amarts of McShane integrable functions a result due to Ghoussoub (see [7] Theorem 1) in the case of Bochner integrable amarts.

Corollary 1. *Let $u \in \mathcal{L}(E, F)$. Then u is absolutely summing if and only if U maps every E-valued amart of strongly measurable Pettis integrable functions to an F-valued uniform amart.*

Proof. By [6, Corollary 4C], the class of strongly mesurable Pettis integrable functions is included in that of McShane integrable ones. Therefore the assertion follows from Theorem 1. □

The following examples are applications of Theorem 1.

Example 1. Let $E = \ell_1$ and let $x_n = (x_1^n, x_2^n, \ldots)$ be such that $\sum\limits_{n=1}^{\infty} x_n$ is a series in ℓ_1 converging unconditionally but not absolutely. We can find an increasing sequence of integers $(n_k)_k$ such that

$$\sum_{n=n_k+1}^{n_{k+1}} \|x_n\| \geqslant 1.$$

Multiplying some x_n's by coefficients smaller than 1, we can assume, without loss of generality, that

$$\sum_{n=n_k+1}^{n_{k+1}} \|x_n\| = 1.$$

For every $k \in \mathbb{N}$, divide the interval $[0,1]$ into $(n_{k+1} - n_k)$ subintervals $A_{k,n}$ of length $\|x_n\|$. Let $(\Omega, \mathcal{F}, P)$ be the interval $[0,1]$ with the Lebesgue measure. We define a sequence of functions $X_k\colon \Omega \to \ell_1$ as

$$X_k(\omega) = \sum_{n=n_k+1}^{n_{k+1}} \frac{x_n}{\|x_n\|} \chi_{A_{k,n}}. \tag{13}$$

As, for each k, X_k is a countably valued function, it is strongly measurable. Since

$$\sum_{n=n_k+1}^{n_{k+1}} \frac{x_n}{\|x_n\|} P(A_{k,n}) = \sum_{n=n_k+1}^{n_{k+1}} x_n$$

is unconditionally convergent, each X_k is McShane integrable (see [8], Theorem 15). For every $k \in \mathbb{N}$, let $\mathcal{F}_k$ be the σ-algebra $\sigma(X_1, X_2, \dots, X_k)$. Then the sequence $(X_k)_k$ is adapted to the the family $(\mathcal{F}_k)_k$. To prove that $(X_k)_k$ is an amart, let $N \in \mathbb{N}$ and let $\sigma \geqslant N$ be a stopping time. For each $k \geqslant N$, let $B_{k,n} = A_{k,n} \cap \{\sigma = k\}$. Then it follows that

$$(\mathrm{McS}) \int X_\sigma = \sum_{k \geqslant N} (\mathrm{McS}) \int_{\{\sigma = k\}} X_k = \sum_{k \geqslant N} \sum_{n=n_k+1}^{n_{k+1}} P(B_{k,n}) \frac{x_n}{\|x_n\|}. \tag{14}$$

Since $B_{k,n} \subseteq A_{k,n}$, we have $P(B_{k,n})/\|x_n\| = \alpha_{k,n} \leqslant 1$. Therefore, since $\sum\limits_{n=1}^{\infty} x_n$ is unconditionally summable, by (14) we get that

$$(\mathrm{McS}) \int X_\sigma = \sum_{k \geqslant N} \sum_{n=n_k+1}^{n_{k+1}} \alpha_{k,n} x_n$$

converges to zero. Since $\sum\limits_{n=1}^{\infty} x_n$ is not absolutely convergent, f is not Bochner integrable (see [2] Theorem 2). If $i\colon \ell_1 \hookrightarrow \ell_2$ is the canonical immersion then i is an absolutely summing operator. Therefore applying Theorem 1 we get that the composition

$$i \circ X_k\colon [0,1] \to \ell_2$$

is a uniform amart.

E x a m p l e 2. Let $E = \ell_1$ and let $(e_n)_{n\in\mathbb{N}}$ be the canonical basis in ℓ_1. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of independent real integrable functions such that $Y_n(\Omega) = \{-1, +1\}$ with $P(Y_n = -1) = P(Y_n = 1) = \frac{1}{2}$. Define

$$X_n = \frac{1}{n}\sum_{i=1}^{n} Y_i e_i.$$

For each $f \in E^*$, by the strong law of large numbers, we get that $\lim\limits_{n\to\infty} f(X_n) = 0$ a.e. and thus also $\lim\limits_{\tau} f(X_\tau) = 0$ a.e. Moreover, $\|X_n(\omega)\| = 1$ for each $n \in \mathbb{N}$, therefore $\lim\limits_{\tau} \int_\Omega f(X_\tau) = 0$. Thus for every $f \in E^*$, $(f(X_n))_{n\in\mathbb{N}}$ is a real valued amart. By the Schur Theorem we get that $\lim\limits_{\tau} \int_\Omega X_\tau$ exists in ℓ_1, therefore $(X_n)_{n\in\mathbb{N}}$ is an ℓ_1-amart. As in the previous example if $i\colon \ell_1 \hookrightarrow \ell_2$ is the canonical immersion then i is an absolutely summing operator. Thus applying once again Theorem 1 we get that the composition

$$i \circ X_n\colon [0,1] \to \ell_2$$

is a uniform amart.

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