MULTIPLIERS FOR GENERALIZED RIEMANN INTEGRALS IN THE REAL LINE

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Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. We use an elementary method to prove that each BV function is a multiplier for the C-integral.

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1. Introduction

It is well known that if f is Henstock-Kurzweil integrable on a compact interval $[a,b] \subset \mathbb{R}$ and g is of bounded variation there, then fg is Henstock-Kurzweil integrable on $[a,b]$ and the integration by parts formula holds; see, for example, [7, Theorem 12.21]. Here g is known as a multiplier for the Henstock-Kurzweil integral. In [2] Bongiorno used the above mentioned result to prove that each BV function is a multiplier for the C-integral. See [2, Theorem 4.2] for details. In this paper, we will use elementary properties of the C-integral to obtain a new proof of [2, Theorem 4.2]. As a result, we also obtain an alternative proof of the well-known results that each BV function is a multiplier for both the McShane and Henstock-Kurzweil integrals.

The set of all real numbers is denoted by $\mathbb{R}$. A set $Z \subset \mathbb{R}$ is said to be μ_1-*negligible* whenever $\mu_1(Z) = 0$, where μ_1 is the one-dimensional Lebesgue measure. Given two subsets X, Y of $\mathbb{R}$, we say that X and Y are non-overlapping if their intersection is μ_1-negligible. A function is always real-valued. When no confusion is possible we do not distinguish between a function defined on a set Z and its restriction to a set $W \subset Z$.

An interval in $\mathbb{R}$ is always a compact non-degenerate interval in $\mathbb{R}$. The family of all non-degenerate subintervals of $[a,b]$, where $-\infty < a < b < \infty$, is denoted by $\mathcal{I}_1$. For any given $I \in \mathcal{I}_1$, we write $\mu_1(I)$ as $|I|$.

A *partition* P is a finite collection $\{(I_1,\xi_1),\dots,(I_p,\xi_p)\}$, where $I_1,\dots,I_p$ are pairwise non-overlapping intervals in $\mathcal{I}_1$, and $\xi_i \in [a,b]$ for each $i = 1,\dots,p$. Given $Z \subseteq [a,b]$, a positive function δ on Z is called a *gauge* on Z. A partition $\{(I_1,\xi_1),\dots,(I_p,\xi_p)\}$ is said to be:

(i) a partition *of* Z if $\bigcup\limits_{i=1}^{p} I_i = Z$;

(ii) a subpartition *of* Z if $\bigcup\limits_{i=1}^{p} I_i \subseteq Z$;

(iii) δ-*fine* if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1,\dots,p$;

(iv) McShane if for each $i = 1,\dots,p$, ξ_i need not be in I_i.

Lemma 2.1 [8, Lemma 6.2.6]**.** *Given a gauge δ on $[a,b]$, δ-fine partitions of $[a,b]$ exist.*

Definition 2.2 ([3])**.** A function $f\colon [a,b] \longrightarrow \mathbb{R}$ is said to be C-*integrable on* $[a,b]$ if there exists $A \in \mathbb{R}$ with the following property: for each $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that

$$\left|\sum_{i=1}^{p} f(\xi_i)|I_i| - A\right| < \varepsilon$$

for each δ-fine McShane partition $\{(I_1,\xi_1),\dots,(I_p,\xi_p)\}$ of the interval $[a,b]$ such that $\sum\limits_{i=1}^{p} \operatorname{dist}(\xi_i, I_i) < 1/\varepsilon$. Here A is called the C-*integral of* f *over* $[a,b]$, and we write A as $\int_a^b f(x)\,\mathrm{d}x$ or $\int_{[a,b]} f(x)\,\mathrm{d}x$.

The C-integral is the minimal integral which includes Lebesgue integrable functions and derivatives. See [3, Main Theorem] for details. The following properties of the C-integral can be found in [1], [2], [3], [6].

Remark 2.3. (a) The C-integral is linear; the class of C-integrable functions on $[a,b]$ is a linear space.

(b) C-integrability on an interval I implies C-integrability on each subinterval of I.

Lemma 2.4 (Saks-Henstock). *Let f be C-integrable on $[a,b]$. Then for each $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that*

$$\sum_{i=1}^{p} \left| f(\xi_i)|I_i| - \int_{I_i} f(x)\,\mathrm{d}x \right| < \varepsilon \tag{1}$$

for each δ-fine McShane subpartition $\{(I_1,\xi_1),\dots,(I_p,\xi_p)\}$ of $[a,b]$ such that $\sum\limits_{i=1}^{p} \operatorname{dist}(\xi_i, I_i) < 1/\varepsilon$.

3. Multipliers for the C-integral

Let χ_X denote the characteristic function of a set X. The following lemma is an easy consequence of [9, 4.32 Theorem].

Lemma 3.1. *If $a \leqslant u < v \leqslant b$, $g \in BV[a,b]$ and $g(a) = 0$, then*

$$\int_a^b \chi_{[u,v]}(x) g(x)\,\mathrm{d}x = \int_a^b \left(\int_x^b \chi_{[u,v]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x).$$

As an easy application of Lemma 3.1, we have the following crucial theorem for this paper.

Theorem 3.2. *Let f be C-integrable on $[a,b]$. If $g \in BV[a,b]$ and $g(a) = 0$, then the inequality*

$$\begin{aligned} &\left| \sum_{i=1}^{p} \left\{ f(\xi_i) g(\xi_i)(v_i - u_i) - \int_a^b \left(\int_x^b f(t)\chi_{[u_i,v_i]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x) \right\} \right| \\ &\qquad \leqslant \sum_{i=1}^{p} |f(\xi_i)| \int_{u_i}^{v_i} |g(\xi_i) - g(t)|\,\mathrm{d}t \\ &\qquad\quad + \sup_{x\in[a,b]} \left| \int_x^b \sum_{i=1}^{p} \{ f(\xi_i)\chi_{[u_i,v_i]}(t) - f(t)\chi_{[u_i,v_i]}(t) \}\,\mathrm{d}t \right| \operatorname{Var}(g,[a,b]) \end{aligned}$$

holds for each subpartition $\{([u_1,v_1],\xi_1),\dots,([u_p,v_p],\xi_p)\}$ of $[a,b]$.

Proof. Let $\{([u_1,v_1],\xi_1),\dots,([u_p,v_p],\xi_p)\}$ be a subpartition of $[a,b]$. In view of Lemma 3.1, we see that

$$\left| \sum_{i=1}^{p} \left\{ f(\xi_i) g(\xi_i)(v_i - u_i) - \int_a^b \left(\int_x^b f(t)\chi_{[u_i,v_i]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x) \right\} \right|$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{p} |f(\xi_i)| \left| g(\xi_i)(v_i-u_i) - \int_{u_i}^{v_i} g(t)\,\mathrm{d}t \right| \\
&\quad + \left| \sum_{i=1}^{p} \left\{ f(\xi_i) \int_{u_i}^{v_i} g(t)\,\mathrm{d}t - \int_a^b \left(\int_x^b f(t)\chi_{[u_i,v_i]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x) \right\} \right| \\
&\leqslant \sum_{i=1}^{p} |f(\xi_i)| \left| g(\xi_i)(v_i-u_i) - \int_{u_i}^{v_i} g(t)\,\mathrm{d}t \right| \\
&\quad + \left| \sum_{i=1}^{p} \left\{ f(\xi_i) \int_a^b \left(\int_x^b \chi_{[u_i,v_i]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x) \right.\right. \\
&\quad \left.\left. - \int_a^b \left(\int_x^b f(t)\chi_{[u_i,v_i]}(t)\,\mathrm{d}t \right) \mathrm{d}g(x) \right\} \right| .
\end{aligned}
$$

Since

$$
\sum_{i=1}^{p} |f(\xi_i)| \left| g(\xi_i)(v_i-u_i) - \int_{u_i}^{v_i} g(t)\,\mathrm{d}t \right| \leqslant \sum_{i=1}^{p} |f(\xi_i)| \int_{u_i}^{v_i} |g(\xi_i) - g(t)|\,\mathrm{d}t
$$

and

$$
\begin{aligned}
&\left| \int_a^b \left(\int_x^b \sum_{i=1}^{p} \{ f(\xi_i)\chi_{[u_i,v_i]}(t) - f(t)\chi_{[u_i,v_i]}(t) \}\,\mathrm{d}t \right) \mathrm{d}g(x) \right| \\
&\qquad \leqslant \sup_{x\in[a,b]} \left| \int_x^b \sum_{i=1}^{p} \{ f(\xi_i)\chi_{[u_i,v_i]}(t) - f(t)\chi_{[u_i,v_i]}(t) \}\,\mathrm{d}t \right| \operatorname{Var}(g,[a,b]),
\end{aligned}
$$

the theorem is proved.

We can now give an elementary proof of the following result.

Theorem 3.3 [2, Theorem 4.2]**.** *Each BV function is a multiplier for the C-integral.*

Proof. We may assume that $g(a) = 0$ and $\operatorname{Var}(g,[a,b]) < 1$. According to the Saks-Henstock Lemma for the C-integral, given $\varepsilon > 0$ there exists a gauge δ_1 on $[a,b]$ such that

$$
\sum_{i=1}^{q} \left| f(\zeta_i)(t_i - s_i) - \int_{s_i}^{t_i} f(x)\,\mathrm{d}x \right| < \frac{\varepsilon}{3} \tag{2}
$$

for each δ_1-fine McShane subpartition $\{([s_1,t_1],\zeta_1),\ldots,([s_q,t_q],\zeta_q)\}$ of $[a,b]$ such that

$$
\sum_{i=1}^{q} \operatorname{dist}(\zeta_i,[s_i,t_i]) < \frac{3}{\varepsilon}.
$$

Observe that if $s < r < t$, then $(r,t] = (s,t] - (s,r]$. Then it follows from our choice of δ_1 that for each $x \in [a,b]$, the inequality

$$\Bigl|\sum_{i=1}^{q}\Bigl\{f(\zeta_i)\mu_1([x,b]\cap[s_i,t_i]) - \int_a^b f(t)\chi_{[x,b]\cap[s_i,t_i]}(t)\,\mathrm{d}t\Bigr\}\Bigr| < \frac{2\varepsilon}{3}$$

holds for each δ_1-fine McShane subpartition $\{([s_1,t_1],\zeta_1),\ldots,([s_q,t_q],\zeta_q)\}$ of $[a,b]$ such that

$$\sum_{i=1}^{q}\operatorname{dist}(\zeta_i,[s_i,t_i]) < \frac{3}{\varepsilon}.$$

As f is real-valued and g is of bounded variation on $[a,b]$, it is not difficult to select a gauge δ_2 on $[a,b]$ such that

$$\sum_{j=1}^{r}|f(z_j)|\int_{\alpha_i}^{\beta_i}|g(z_j)-g(t)|\,\mathrm{d}t < \frac{\varepsilon}{3}$$

for each δ_2-fine McShane subpartition $\{([\alpha_1,\beta_1],z_1),\ldots,([\alpha_r,\beta_r],z_r)\}$ of $[a,b]$.

Define a gauge δ on $[a,b]$ by $\delta(x) = \min\{\delta_1(x),\delta_2(x)\}$. For each δ-fine McShane partition $\{([u_1,v_1],\xi_1),\ldots,([u_p,v_p],\xi_p)\}$ of $[a,b]$ satisfying

$$\sum_{i=1}^{p}\operatorname{dist}(\xi_i,[u_i,v_i]) < \frac{1}{\varepsilon},$$

we infer from Theorem 3.2 and the above estimates that

$$\begin{aligned}
&\Bigl|\sum_{i=1}^{p} f(\xi_i)g(\xi_i)(v_i-u_i) - \int_a^b\Bigl(\int_x^b f(t)\,\mathrm{d}t\Bigr)\,\mathrm{d}g(x)\Bigr| \\
&\quad= \Bigl|\sum_{i=1}^{p}\Bigl\{f(\xi_i)g(\xi_i)(v_i-u_i) - \int_a^b\Bigl(\int_x^b f(t)\chi_{[u_i,v_i]}(t)\,\mathrm{d}t\Bigr)\,\mathrm{d}g(x)\Bigr\}\Bigr| \\
&\quad\leqslant \sum_{i=1}^{p}|f(\xi_i)|\int_{u_i}^{v_i}|g(\xi_i)-g(t)|\,\mathrm{d}t \\
&\qquad+ \sup_{x\in[a,b]}\Bigl|\int_x^b\sum_{i=1}^{p}\Bigl\{f(\xi_i)\chi_{[u_i,v_i]}(t) - f(t)\chi_{[u_i,v_i]}(t)\Bigr\}\,\mathrm{d}t\Bigr|\operatorname{Var}(g,[a,b]) < \varepsilon,
\end{aligned}$$

thereby completing the proof of the theorem.

By modifying the proof of the above theorem, we obtain the following well-known theorem.

Theorem 3.4. *Each BV function is a multiplier for each of the generalized Riemann integrals:*

(i) the McShane integral;
(ii) the classical Henstock-Kurzweil integral;
(iii) the $\widetilde{C}$-integral in [5];
(iv) the improper Lebesgue integral in [4].

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