# ON THE RATE OF STRONG SUMMABILITY OF DOUBLE FOURIER SERIES

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*Abstract.* Estimates of the strong means of Marcinkiewicz type with the Cesàro means of negative order in one of the variables instead of square partial sums are obtained by characteristics constructed on the basis of moduli of continuity.

*Keywords*: double Fourier series, strong approximation

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## 1. Preliminaries

Let $L^{qq}$ $(1 \leqslant q \leqslant \infty)$ $[L^{\infty\infty} = C]$ be the class of all $2\pi$-periodic real-valued functions of two variables, integrable in the Lebesgue sense with $q$-th power (continuous) in the square $Q = [-\pi, \pi; -\pi, \pi]$. Let us define the norm of $f \in L^{qq}$ as

$$\|f\|_{L^{qq}} = \|f(x,y)\|_{L^{qq}} = \begin{cases} \left( \displaystyle\iint\limits_{Q} |f(x,y)|^q \,\mathrm{d}x \,\mathrm{d}y \right)^{1/q} & \text{when } 1 \leqslant q < \infty, \\ \sup\limits_{(x,y)\in Q} |f(x,y)| & \text{when } q = \infty. \end{cases}$$

Consider the trigonometric double Fourier series

$$S[f](x,y) = \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} A_{n\nu}(x,y;f)$$

of $f \in L^{qq}$ with partial sums $S_{jk}(x,y;f)$. Denote by $\sigma_{jk}^{(\gamma,\delta)}(x,y) = \sigma_{jk}^{(\gamma,\delta)}(x,y;f)$ the Cesàro means $(C,\gamma,\delta)$ of $S[f]$.

Then

$$\sigma_{jk}^{(\gamma,\delta)}(x,y;f)=\frac{1}{A_j^\gamma A_k^\delta}\sum_{\mu=0}^{j}\sum_{\nu=0}^{k}A_{j-\mu}^{\gamma}\,A_{k-\nu}^{\delta}\,A_{\mu\nu}(x,y;f),$$

where $j,k=0,1,2,\ldots,\ \gamma,\delta>-1$ and $A_i^k=\binom{i+k}{i}$.

Double Fourier series of a complex-valued function $f\in L^{qq}$ is defined as

$$S[f](x,y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}c_{mn}\mathrm{e}^{-\mathrm{i}(mx+ny)},$$

where

$$c_{mn}=(2\pi)^{-2}\iint\limits_{Q}f(x,y)\mathrm{e}^{-\mathrm{i}(mx+ny)}\,\mathrm{d}x\,\mathrm{d}y.$$

If $f(x,y)$ is real-valued, the Fourier coefficients $c_{mn}$ fulfil the obvious additional conditions. Hence, in our arrangement of $S[f](x,y)$, the expressions $A_{\mu,\nu}(x,y;f)$ are of the form

$$\begin{aligned}A_{\mu,\nu}(x,y;f)&=\alpha_{\mu\nu}\cos\mu x\cos\nu y+\beta_{\mu\nu}\sin\mu x\cos\nu y\\&\quad+\gamma_{\mu\nu}\cos\mu x\sin\nu y+\delta_{\mu\nu}\sin\mu x\sin\nu y,\end{aligned}$$

where $\alpha_{\mu\nu}$, $\beta_{\mu\nu}$, $\gamma_{\mu\nu}$, $\delta_{\mu\nu}$ are specific linear combinations of $c_{mn}$.

For the one dimensional case we will also use the notations $L^q$, $\|\times\|_{L^q}$, $S[f]$, $(C,\delta)$, $\sigma_k^\delta(x;f)$ which can be understood analogously to the above.

Let us introduce the integral moduli of continuity of $f\in L^{qq}$

$$\begin{aligned}\omega_1(\varrho)_{L^{qq}}&=\omega_1(\varrho,f)_{L^{qq}}=\sup_{h\in<0,\varrho>}\big\|f(x+h,y)-f(x,y)\big\|_{L^{qq}},\\\omega_2(\varrho)_{L^{qq}}&=\omega_2(\varrho,f)_{L^{qq}}=\sup_{h\in<0,\varrho>}\big\|f(x,y+h)-f(x,y)\big\|_{L^{qq}},\end{aligned}$$

and the characteristics

$$\varphi_{i,n}=\varphi_{i,n}(q)=\begin{cases}\omega_i\big(\frac{1}{n+1}\big)_{L^{qq}} & \text{when } 1<q<\infty,\\[2ex] \frac{1}{n+1}\sum\limits_{\nu=0}^{n}\omega_i\big(\frac{1}{\nu+1}\big)_{L^{qq}} & \text{when } q=1 \text{ or } q=\infty,\end{cases}$$

for $i=1,\,2,$

$$\Phi_{m,n}=\Phi_{m,n}(q)=\varphi_{1,m}(q)+\varphi_{2,n}(q),$$

$$\psi_{i,n}=\psi_{i,n}(p,q)=\begin{cases}\omega_i\big(\frac{1}{n+1}\big)_{L^{qq}} & \text{when } q\leqslant\max(p,2),\\[2ex] (n+1)^{(1/p)-(1/q)}\ \omega_i\big(\frac{1}{n+1}\big)_{L^{qq}} & \text{when } \max(p,2)<q<\infty,\end{cases}$$

for $i=1,\,2,$

$$\Psi_{m,n}=\Psi_{m,n}(p,q)=\psi_{1,m}(p,q)+\psi_{2,n}(p,q),$$

$$X_n = X_n(p,q) = \begin{cases} 1 & \text{when } q \leqslant \max(p,2), \\ (n+1)^{1/p-1/q} & \text{when } \max(p,2) < q < \infty; \end{cases}$$

$$L_n = L_n(\gamma) = L_n(\gamma,p,q) = \begin{cases} 1 & \text{when } \gamma > 0 \text{ or } \gamma = 0,\ p = 2,\ q \neq 1,\ \infty, \\ \log(n+2) & \text{when } \gamma = 0 \text{ and if } p = 2 \text{ then } q = 1,\ \infty. \end{cases}$$

We shall deal with a regular summability method, determined by an infinite functional sequence $\{\alpha_k(r)/A(r)\}$, $\alpha_k(r) \geqslant 0$ and $A(r) = \sum_{k=0}^{\infty} \alpha_k(r)$ for $0 < r < 1$, in such a way that the condition $w_k \to w$ when $k \to \infty$ implies

$$\frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) w_k \to w \quad \text{when} \quad r \to 1,\ 0 < r < 1.$$

The aim of the present paper is to estimate the following strong deviations of Marcinkiewicz type ($1 \leqslant q \leqslant \infty$):

$$P^p(r;f)_{L^{qq}} = \left( \frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) \left\| \sigma_{kk}^{(\gamma,\delta-1)} - f \right\|_{L^{qq}}^p \right)^{1/p} \qquad (q \neq \infty),$$

$$Q^p(r;f)_{L^{qq}} = \left\| \left( \frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) \left| \sigma_{kk}^{(\gamma,\delta-1)}(x,y) - f(x,y) \right|^p \right)^{1/p} \right\|_{L^{qq}}$$

$$R^{p,s}(r;f)_{L^{qq}} = \left( \frac{1}{A(r)} \sum_{l=0}^{\infty} \left\| \left( \sum_{k=2^l-1}^{2^{l+1}-2} \alpha_k(r) \left| \sigma_{kk}^{(\gamma,\delta-1)}(x,y) - f(x,y) \right|^p \right)^{1/p} \right\|_{L^{qq}}^s \right)^{1/s}$$

for some $p$, $s > 0$, $\gamma \geqslant 0$, $\delta > \frac{1}{2}$ and all $r \in (0,1)$ by means of the characteristics defined before.

Our theorems, presented in Section 2, correspond to the results announced in [3] and [4] for functions of one variable and generalize the results of [5].

The proofs of the theorems are based on a few lemmas which unfortunately cannot be deduced from the one dimensional results by iteration.

For convenience, let $N = \left[\frac{1}{1-r}\right]$, $p' = \min(p,q)$, $q' = \max(p,q)$, $\alpha_l = 2^l - 1$, $\beta_l = 2^{l+1} - 2$, $\beta'_l = \min\left(2^{l+1} - 2, N\right)$, and let $C_j$ $(j = 1, 2, 3, \ldots)$ be suitable positive constants independent of $f$ and $r$. Let $\tilde{\varepsilon}$ denote the real number conjugate to $\varepsilon$, i.e. $1/\varepsilon + 1/\tilde{\varepsilon} = 1$ $(\varepsilon, \tilde{\varepsilon} > 1)$.

## 2. Statement of results

Considering a function $f \in L^{qq}$ and numbers $r$ running through the interval $(0,1)$, we formulate the following seven approximation theorems:

**Theorem 1.** *Suppose that*

$$\sum_{l=0}^{\infty}\Bigg(\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_k^{\lambda}(r)\,(L_k\varphi_{i,k})^{\lambda p}}{(k+1)^{1-\lambda}}\Bigg)^{1/\lambda} \leqslant C_1 A(r)(1-r)\sum_{k=0}^{\infty} r^k\,(L_k\varphi_{i,k})^p \quad (i=1,2) \tag{1}$$

*for some finite $\lambda > 1, \delta$ and $p$ satisfying the condition $(1-\delta)p < 1-1/\lambda$. Then*

(i) *if $(1-\delta)\,q < 1-1/\lambda$, we have*

$$P^p(r;f)_{L^{qq}} \leqslant C_2\Bigg((1-r)\sum_{k=0}^{\infty} r^k\big(\Phi_{k,k}\,L_k\big)^p\Bigg)^{1/p}; \tag{2}$$

(ii) *if the estimate (1) holds with $\psi_{i,k}$ instead of $\varphi_{i,k}$ then*

$$P^p(r;f)_{L^{qq}} \leqslant C_3\Bigg((1-r)\sum_{k=0}^{\infty} r^k\big(\Psi_{k,k}\,L_k\big)^p\Bigg)^{1/p}. \tag{3}$$

**Theorem 2.** *Suppose that*

$$\sum_{l=0}^{j}\Bigg(\sum_{k=\alpha_l}^{\beta_l'}\frac{\alpha_k^{\lambda}(r)\,\varphi_{i,k}^{p\lambda}}{(k+1)^{1-\lambda}}\Bigg)^{1/\lambda} \leqslant C_4 A(r)(1-r)\sum_{k=0}^{N}\varphi_{i,k}^p \quad (i=1,2) \tag{4}$$

*where $2^j \leqslant N+1 < 2^{j+1}$ and that*

$$\Bigg((1-r)\sum_{k=N+1}^{\infty}\alpha_k^{\lambda}(r)\,r^{k(1-\lambda)}\Bigg)^{1/\lambda} \leqslant C_5 A(r)\,(1-r) \tag{5}$$

*with $\lambda > 1$ and $p$ such that $(1-\delta)p < 1-1/\lambda$. Then*

(i) *we have*

$$P^p(r;f)_{L^{qq}} \leqslant C_6\,L_{N-2}\big(\gamma, p\tilde{\lambda}, q\big)\,X_N\big(\tilde{\lambda}p, q\big)\Bigg((1-r)\sum_{k=0}^{N}\Phi_{k,k}^N\Bigg)^{1/p}; \tag{6}$$

(ii) *if $(1-\delta)q \leqslant 1-1/\lambda$, we have*

$$P^p(r,f)_{L^{qq}} \leqslant C_7 L_{N-2}\Bigg((1-r)\sum_{k=0}^{N}\Phi_{k,k}^p\Bigg)^{1/p}; \tag{7}$$

(iii) *if the estimate (4) holds with $\psi_{i,k}$ instead of $\varphi_{i,k}$, we have*

$$P^p(r;f)_{L^{qq}} \leqslant C_8 L_{N-2}\Bigg((1-r)\sum_{k=0}^{N}\Psi_{k,k}^p\big(\tilde{\lambda}p, q\big) + \Psi_{N,N}^p\big(\tilde{\lambda}p, q\big)\Bigg)^{1/p}. \tag{8}$$

**Theorem 3.** *Suppose that*

$$\left((1-r)\sum_{k=0}^{N}\alpha_k^{\lambda}(r)(k+1)^{-p\delta\lambda}\right)^{1/\lambda} \leqslant C_9 A(r)(1-r)^{\delta p+1}, \tag{9}$$

$$\sum_{k=0}^{N}\alpha_k(r)\varphi_{i,k}^{p} \leqslant C_{10}A(r)(1-r)\sum_{k=0}^{N}\varphi_{i,k}^{p} \qquad (i=1,2), \tag{10}$$

*and that the condition* (5) *holds with* $\lambda > 1$, $\delta > \frac{1}{2}$ *and* $p > 0$ *such that* $(1-\delta)\,p < 1 - 1/\lambda$. *Then*

(i) *condition* (6) *holds;*

(ii) *if* $(1-\delta)\,q < 1-1/\lambda$, *we have* (7).

The relation between $p$ and $q$ shows that sometimes it is more natural to consider the expression $Q^p(r;f)_{L^{qq}}$, the more so that it is possible to obtain an estimate for a larger range of $1 \leqslant q \leqslant \infty$.

**Theorem 4.** *Let the conditions* (4) *and* (5) *or* (5), (9) *and* (10) *hold with* $\lambda$, $\delta$, $p$ *as before. Then*

$$Q^p(r;f)_{L^{qq}} \leqslant C_{11}\max\bigl(1,(1-r)^{(1/p\tilde{\lambda})-(1/q)}\bigr)L_{N-2}\left((1-r)\sum_{k=0}^{N}\Phi_{k,k}^{p}\right)^{1/p}. \tag{11}$$

**Theorem 5.** *Let* $f \in L^{q'q'}$ *and let the conditions* (1) *or* (4) *and* (5) *or* (5), (9) *and* (10) *hold with* $\varphi_{i,k} = \varphi_{i,k}(q')$ *and* $\lambda, \delta, p$ *as before. Then*

$$Q^p(r;f)_{L^{q'q'}} \leqslant C_{12}L_{N-2}\left((1-r)\sum_{k=0}^{N}\Phi_{k,k}^{p}(q')\right)^{1/p}. \tag{12}$$

**Theorem 6.** *Let us assume (for* $i = 1, 2$*)*

$$\sum_{l=0}^{\infty}\left(\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_k^{\lambda}(r)\,(L_k\varphi_{i,k})}{(k+1)^{1-\lambda}}\right)^{s/\lambda p'} \leqslant C_{13}A(r)\,(1-r)\sum_{l=0}^{\infty}\left(\sum_{k=\alpha_l}^{\beta_l}r^k\,(L_k\varphi_{i,k})^{p'}\right)^{s/p'} \tag{13}$$

*for some finite* $\lambda > 1$, $\delta$ *and* $p'$ *such that* $(1-\delta)\,p' < 1-1/\lambda$. *Then*

$$R^{p',s}(r;f)_{L^{qq}} \leqslant C_{14}\left((1-r)\sum_{l=0}^{\infty}\left(\sum_{k=\alpha_l}^{\beta_l}r^k L_k^{p'}\,\Phi_{k,k}^{p'}\right)^{s/p'}\right)^{1/s}. \tag{14}$$

**Theorem 7.** *If the condition* (13) *holds with* $p' = p$, $\varphi_{i,k} = \varphi_{i,k}(q')$ *and with* $\lambda$, $\delta$, $p$ *as in Theorem* 1, *then*

$$R^{p,s}(r;f)_{L^{q'q'}} \leqslant C_{15}\left((1-r)\sum_{l=0}^{\infty}\left(\sum_{k=\alpha_l}^{\beta_l} r^k\big(L_k\Phi_{k,k}(q')\big)^p\right)^{s/p}\right)^{1/s}. \tag{15}$$

Let us postpone the proofs of our theorems to the next section. Here we will present a few lemmas, which will be needed in these proofs.

## 3. Fundamental lemmas

First we formulate an evident

**Lemma 1.** *Suppose that* $g_k \in L^{qq}$ $(k = 0, 1, \ldots)$ *and* $p > 0$. *Then*

$$\left(\sum_{k=0}^{n}\|g_k\|_{L^{qq}}^p\right)^{1/p} \leqslant \max\big(1, (n+1)^{1/p-1/q}\big)\left\|\left(\sum_{k=0}^{n}|g_k(x,y)|^p\right)^{1/p}\right\|_{L^{qq}}, \tag{16}$$

$$\left\|\left(\sum_{k=0}^{n}|g_k(x,y)|^p\right)^{1/p}\right\|_{L^{qq}} \leqslant \max\big(1, (n+1)^{1/q-1/p}\big)\left(\sum_{k=0}^{n}\|g_k\|_{L^{qq}}^p\right)^{1/p}, \tag{17}$$

$$\left\|\left(\sum_{k=0}^{\infty}r^{kp}|g_k(x,y)|^p\right)^{1/p}\right\|_{L^{qq}} \leqslant \max\big(1, (1-r^q)^{1/p-1/q}\big)\left(\sum_{k=0}^{\infty}r^{k\min(p,q)}\|g_k\|_{L^{qq}}^p\right)^{1/p} \tag{18}$$

*for* $n = 0, 1, 2, \ldots$ *and* $r \in (0,1)$ *(cf.* [6] p. 285*).*

**Lemma 2.** *Let* $u_k$ *and* $v_k$ *be the Poisson and the Poisson conjugate integrals of* $f_k \in L^q$ *and let*

$$F_k(z) = F(z;f_k) = u_k(r,x) + \mathrm{i}v_k(r,x) = \frac{a_0(f_k)}{2} + \sum_{\nu=1}^{\infty}\big(a_\nu(f_k) - \mathrm{i}b_\nu(f_k)\big)z^\nu, \tag{19}$$

*where* $z = r\mathrm{e}^{\mathrm{i}x}$, $0 < r < 1$ *and* $a_\nu(f_k)$, $b_\nu(f_k)$ *are the Fourier coefficients of* $f_k$. *Then for* $r \in (0,1)$ *and* $p \geqslant 1$ *we have*

$$\left\|\left(\frac{1}{n+1}\sum_{k=0}^{n}|F_k'(r\mathrm{e}^{\mathrm{i}x})|^p\right)^{1/p}\right\|_{L^q} \leqslant C_{16}\,\frac{1}{1-r}\left\|\left(\frac{1}{n+1}\sum_{k=0}^{n}|f_k(x)|^p\right)^{1/p}\right\|_{L^q}, \tag{20}$$

*but if* $f_k \in L^{qq}$ *and* $f_k^{(1)}(x) = f_k(x,y)$, *then*

$$(20') \qquad \Bigl\| \Bigl( \tfrac{1}{n+1} \sum_{k=0}^{n} \bigl| F'(r\mathrm{e}^{\mathrm{i}x};\, f_k^{(1)}) \bigr|^p \Bigr)^{1/p} \Bigr\|_{L^{qq}} \leqslant C_{17}\, \tfrac{1}{1-r} \Bigl\| \Bigl( \tfrac{1}{n+1} \sum_{k=0}^{n} \bigl| f_k(x,y) \bigr|^p \Bigr)^{1/p} \Bigr\|_{L^{qq}},$$

*where* $n = 0, 1, 2, \ldots$ *and* $F_k'(z) = \dfrac{\mathrm{d}F_k(z)}{\mathrm{d}z}$.

P r o o f. We will prove inequality (20) only, because the proof of the other, (20′), runs similarly.

Analogously to the proof of Lemma 1 in [3] we estimate every term on the right-hand side of the inequality

$$\begin{aligned} &\Bigl\| \Bigl( \frac{1}{n+1} \sum_{k=0}^{n} \Bigl| \tfrac{1}{r} \tfrac{\partial F_k(z)}{\partial x} \Bigr|^p \Bigr)^{1/p} \Bigr\|_{L^q} \\ &\qquad \leqslant \frac{1}{r} \Biggl( \Bigl\| \Bigl( \frac{1}{n+1} \sum_{k=0}^{n} \Bigl| \tfrac{\partial u_k(r,x)}{\partial x} \Bigr|^p \Bigr)^{1/p} \Bigr\|_{L^q} + \Bigl\| \Bigl( \frac{1}{n+1} \sum_{k=0}^{n} \Bigl| \tfrac{\partial v_k(r,x)}{\partial x} \Bigr|^p \Bigr)^{1/p} \Bigr\|_{L^q} \Biggr). \end{aligned}$$

On the basis of the Tôyama inequality ([6] p. 285) we have

$$\begin{aligned} &\Bigl\| \frac{1}{r} \Bigl( \frac{1}{n+1} \sum_{k=0}^{n} \Bigl| \tfrac{\partial u_k(r,x)}{\partial x} \Bigr|^p \Bigr)^{1/p} \Bigr\|_{L^q} \\ &\quad = \Bigl\| \frac{1}{r} \Bigl( \frac{1}{n+1} \sum_{k=0}^{n} \Bigl| \frac{1-r^2}{\pi} \int_{-\pi}^{\pi} \bigl( f_k(x) - f_k(x+t) \bigr) \frac{r \sin t \,\mathrm{d}t}{(1 - 2r\cos t + r^2)^2} \Bigr|^p \Bigr)^{1/p} \Bigr\|_{L^q} \\ &\quad \leqslant \Bigl\| \frac{1-r^2}{r\pi} \int_{-\pi}^{\pi} \Bigl( \frac{1}{n+1} \sum_{k=0}^{n} \bigl| f_k(x) - f_k(x+t) \bigr|^p \Bigr)^{1/p} \frac{|t| \,\mathrm{d}t}{(1 - 2r\cos t + r^2)^2} \Bigr\|_{L^q} \\ &\quad \leqslant \frac{1-r^2}{r\pi} \int_{-\pi}^{\pi} \Bigl\| \Bigl( \frac{1}{n+1} \sum_{k=0}^{n} \bigl| f_k(x) - f_k(x+t) \bigr|^p \Bigr)^{1/p} \Bigr\|_{L^q} \frac{|t| \,\mathrm{d}t}{(1 - 2r\cos t + r^2)^2} = I. \end{aligned}$$

If $0 < r \leqslant \frac{1}{2}$, then

$$I \leqslant \frac{1-r^2}{r\pi} \int_{-\pi}^{\pi} W_n^p(t)_{L^q} \frac{|t| \,\mathrm{d}t}{(1 - 2r\cos t + r^2)^2}$$

where

$$W_n^p(\varrho)_{L^q} = \sup_{|h| \leqslant \varrho} \Bigl\| \Bigl( \frac{1}{n+1} \sum_{k=0}^{n} \bigl| f_k(x) - f_k(x+h) \bigr|^p \Bigr)^{1/p} \Bigr\|_{L^q}.$$

In view of the relation

$$W_n^p(\delta\varrho)_{L^q} \leqslant (\delta+1)\, W_n^p(\varrho)_{L^q} \qquad (\delta>0) \tag{21}$$

we obtain

$$\begin{aligned} I &\leqslant C_{18}\, W_n^p(1-r)_{L^q} \int\limits_0^{\pi} \Big(\frac{t}{1-r}+1\Big)\, t\,\mathrm{d}t \leqslant C_{19}\, \frac{1}{1-r}\, W_n^p(1-r)_{L^q} \\ &\leqslant \frac{2\,C_{19}}{1-r} \left\| \Big(\frac{1}{n+1}\sum_{k=0}^{n} \big|f_k(x)\big|^p\Big)^{1/p} \right\|_{L^q}. \end{aligned}$$

In the case $\frac{1}{2}\leqslant r<1$, the substitution $y=\dfrac{2\sqrt{r}t}{\pi\,(1-r)}$ leads to

$$\begin{aligned} I &\leqslant \frac{\pi\,(1+r)}{2r(1-r)} \int\limits_0^{\frac{2\sqrt{r}}{1-r}} \frac{y}{(1+y^2)^2}\, W_n^p\Big(\frac{\pi y(1-r)}{2\sqrt{r}}\Big)_{L^q}\,\mathrm{d}y \leqslant C_{20}\, \frac{1}{1-r}\, W_n^p(1-r)_{L^q} \\ &\leqslant \frac{2\,C_{20}}{1-r} \left\| \Big(\frac{1}{n+1}\sum_{k=0}^{n} \big|f_k(x)\big|^p\Big)^{1/p} \right\|_{L^q} \end{aligned}$$

(cf. [2] p. 892).

The estimate of the second term is based on Lemma 2.31 from [7], p. 258, in which the modulus of the derivative $\Phi'(z)$ of a function $\Phi(z)$ regular in a disc is estimated by the Poisson integral of the modulus of $\operatorname{Re}\Phi$ (in our case $\Phi(z)=\frac{\partial}{\partial x}F_k(\varrho \mathrm{e}^{\mathrm{i}x})$ and the disc is $|z|<1-\frac{1-\varrho}{2}$). We obtain

$$\begin{aligned} &\left\| \frac{1}{r}\Big(\frac{1}{n+1}\sum_{k=0}^{n} \Big|\frac{\partial v_k(r,x)}{\partial x}\Big|^p\Big)^{1/p} \right\|_{L^q} \\ &\qquad \leqslant \left\| \frac{1}{r}\left(\frac{1}{n+1}\sum_{k=0}^{n} \left|\int\limits_0^r \frac{\partial}{\partial\varrho}\Big(\frac{\partial}{\partial x}F_k\big(\varrho \mathrm{e}^{\mathrm{i}x}\big)\Big)\,\mathrm{d}\varrho\right|^p\right)^{1/p} \right\|_{L^q} \\ &\qquad \leqslant \frac{1}{r}\int\limits_0^r \left\| \Big(\frac{1}{n+1}\sum_{k=0}^{n} \Big|\big(\tfrac{\partial}{\partial x}F_k(\varrho \mathrm{e}^{\mathrm{i}x})\big)'\Big|^p\Big)^{1/p} \right\|_{L^q} \mathrm{d}\varrho \\ &\qquad \leqslant \frac{1}{r}\int\limits_0^r \left\| \left(\frac{1}{n+1}\sum_{k=0}^{n} \left|\frac{4}{2\pi\,(1-\varrho)}\int\limits_{-\pi}^{\pi} \Big|\tfrac{\partial}{\partial x}u_k\big(1-\tfrac{1-\varrho}{2},\ t-x\big)\Big| \right.\right.\right. \\ &\qquad\quad \times \left.\left.\left. \frac{\big(1-\frac{1-\varrho}{2}\big)^2-\big(1-(1-\varrho)\big)^2}{2\big((1-\frac{1-\varrho}{2})^2+2\varrho\,(1-\frac{1-\varrho}{2})\cos t+\varrho^2\big)}\,\mathrm{d}t\right|^p\right)^{1/p} \right\|_{L^q} \mathrm{d}\varrho \end{aligned}$$

$$\leqslant \frac{1}{r}\int_0^r \frac{4}{2\pi(1-\varrho)} \int_{-\pi}^{\pi} \left\| \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \tfrac{\partial}{\partial x}\, u_k\big(\tfrac{1+\varrho}{2},\ t-x\big)\right|^p \right)^{1/p} \right\|_{L^q}$$

$$\times \frac{\big(\frac{1+\varrho}{2}\big)^2-\varrho^2}{2\big((\frac{1+\varrho}{2})^2-\varrho(1+\varrho)\cos t+\varrho^2\big)}\,\mathrm{d}t\,\mathrm{d}\varrho$$

$$= \frac{1}{r}\int_0^r \frac{4}{2\pi(1-\varrho)}\,\frac{1+\varrho}{2} \int_{-\pi}^{\pi} \left\| \frac{2}{1+\varrho}\left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \tfrac{\partial}{\partial x}\, u_k\big(\tfrac{1+\varrho}{2}, x\big)\right|^p \right)^{1/p} \right\|_{L^q}$$

$$\times \frac{\big(\frac{1+\varrho}{2}\big)^2-\varrho^2}{2\big(\frac{1+\varrho}{2}\big)^2-\varrho(1+\varrho)\cos t+\varrho^2\big)}\,\mathrm{d}t\,\mathrm{d}\varrho.$$

By the previous computations we have

$$\left\| \frac{2}{1+\varrho}\left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \tfrac{\partial}{\partial x}\, u_k\big(\tfrac{1+\varrho}{2}, x\big)\right|^p \right)^{1/p} \right\|_{L^q} \leqslant (C_{19}+C_{20})\,\frac{2}{1-\varrho}\, W_n^p\big(\tfrac{1-\varrho}{2}\big)$$

and so we finally obtain

$$\left\| \frac{1}{r}\left( \frac{1}{n+1} \sum_{k=0}^{n} \left| \frac{\partial v_k(r,x)}{\partial x}\right|^p \right)^{1/p} \right\|_{L^q} \leqslant \frac{1}{r}(C_{19}+C_{20}) \int_0^r \frac{4}{1-\varrho}\,\frac{1}{1-\varrho}\, W_n^p\big(\tfrac{1-\varrho}{2}\big)\,\mathrm{d}\varrho$$

$$\leqslant \frac{1}{r}\,C_{21}\,\frac{r}{1-r}\,W_n^p(1-r) \leqslant \frac{2}{1-r}\,C_{21} \left\| \left( \frac{1}{n+1} \sum_{k=0}^{n} \big|f_k(x)\big|^p \right)^{1/p} \right\|_{L^q}.$$

Collecting the results, we get (20). □

**Lemma 3.** *Suppose that $f_k \in L^q$ or $f_k \in L^{qq}$, $1 \leqslant q \leqslant \infty$ and $k = 0,1,2,\ldots$ If $\delta \geqslant 0$ and $p \geqslant 1$, then*

$$\left\| \left( \frac{1}{n+1} \sum_{\nu=n}^{2n} \left|\sigma_\nu^\delta(x; f_\nu)\right|^p \right)^{1/p} \right\|_{L^q} \leqslant C_{22} \left\| \left( \frac{L_n^p(\delta)}{n+1} \sum_{k=n}^{2n} \big|f_k(x)\big|^p \right)^{1/p} \right\|_{L^q} \tag{22}$$

*or*

$$\left\| \left( \frac{1}{n+1} \sum_{\nu=n}^{2n} \left|\sigma_\nu^\delta\big(x; f_\nu^{(1)}(x)\big)\right|^p \right)^{1/p} \right\|_{L^{qq}} \leqslant C_{23} \left\| \left( \frac{L_n^p(\delta)}{n+1} \sum_{k=n}^{2n} \big|f_k(x,y)\big|^p \right)^{1/p} \right\|_{L^{qq}} \tag{22'}$$

*for $n = 0,1,2,\ldots$*

P r o o f. In the proof of (22) we apply the integral representation of the Cesàro means $(C,\delta)$ and the Minkowski inequality. Then

$$\Bigg\|\Bigg(\frac{1}{n+1}\sum_{\nu=n}^{2n}\Big|\frac{1}{\pi}\int_{-\pi}^{\pi} f_\nu(x+t)\,K_\nu^\delta(t)\,\mathrm{d}t\Big|^p\Bigg)^{1/p}\Bigg\|_{L^q}$$

$$\leqslant \frac{1}{\pi}\int_{-\pi}^{\pi}\Bigg\|\Bigg(\frac{1}{n+1}\sum_{\nu=n}^{2n}\big|f_\nu(x+t)\big|^p\,\big|K_\nu^\delta(t)\big|^p\Bigg)^{1/p}\Bigg\|_{L^q}\,\mathrm{d}t$$

$$\leqslant \Bigg\|\Bigg(\frac{1}{n+1}\sum_{k=n}^{2n}\big|f_k(x)\big|^p\Bigg)^{1/p}\Bigg\|_{L^q}\,\frac{1}{\pi}\int_{-\pi}^{\pi}\max_{n\leqslant\nu\leqslant 2n}\big|K_\nu^\delta(t)\big|\,\mathrm{d}t,$$

where

$$K_k^0(t)=\frac{\sin\big(k+\frac{1}{2}\big)t}{2\sin\frac{t}{2}},$$

$$K_k^\delta(t)=\frac{1}{A_k^\delta}\sum_{\nu=0}^{k}A_{k-\nu}^{\delta-1}\,K_k^0(t)$$

for $\delta>-1$ and $k=0,1,2,\dots$

Since the kernel $K_\nu^\delta$ is quasi-positive for $\delta>0$ ([7] pp. 151, 157), we have

$$\frac{1}{\pi}\int_{\pi}^{\pi}\max_{n\leqslant\nu\leqslant 2n}\big|K_\nu^\delta(t)\big|\,\mathrm{d}t\leqslant\begin{cases}C_{22} & \text{for } \delta>0\\ C_{22}\log(n+2) & \text{for } \delta=0.\end{cases}$$

Thus, for $\delta>0$, (22) is proved.

Our estimates can be essentially improved only for $1<q<\infty$, $\delta=0$ and $p=2$. In this case the inequality (see [8] formula 2.16)

$$\text{(23)}\qquad \int_{-\pi}^{\pi}\Bigg(\sum_{\nu=0}^{n}\big|S_{k_\nu}(x;f_\nu)\big|^2\Bigg)^{\frac{q}{2}}\mathrm{d}x\leqslant C_{24}\int_{-\pi}^{\pi}\Bigg(\sum_{\nu=0}^{n}\big|f_\nu(x)\big|^2\Bigg)^{\frac{q}{2}}\mathrm{d}x\qquad(1<q<\infty),$$

where $k_\nu$ is any natural-valued function of $\nu$, for $k_\nu=\nu$ gives (22). In a similar way we can obtain (22′) and thus the proof of Lemma 3 is complete. □

**Lemma 4.** *If $\delta>\frac{1}{2}$ and $p>0$ such that $(1-\delta)\,p<1$, then, for any $q$, $s$ such that either $s=q$ or*

$$s\geqslant\begin{cases}\frac{p}{p-1} & \textit{when } p\geqslant 2,\\ 2 & \textit{when } 0<p\leqslant 2,\end{cases}$$

$$\text{(24)} \quad \left\| \left( \tfrac{1}{m+1} \sum_{k=0}^{m} \left( \tfrac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_\nu^{\delta-1}(x; f_k) \right|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q} \leqslant C_{25} \left\| \left( \tfrac{1}{m+1} \sum_{k=0}^{m} \left| f_k(x) \right|^s \right)^{1/s} \right\|_{L^q},$$

$$\text{(24')} \quad \left\| \left( \tfrac{1}{m+1} \sum_{k=0}^{m} \left( \tfrac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_\nu^{\delta-1}\bigl(x; f_k^{(1)}(x)\bigr) \right|^p \right)^{s/p} \right)^{1/s} \right\|_{L^{qq}} \leqslant C_{26} \left\| \left( \tfrac{1}{m+1} \sum_{k=0}^{m} \left| f_k(x,y) \right|^s \right)^{1/s} \right\|_{L^{qq}},$$

*where* $m, n = 0, 1, 2, \ldots$

P r o o f. We will prove only (24), since (24′) can be obtained analogously: using estimate (20′) instead of (20).

It can be easily observed that

$$\begin{aligned}
&\left\| \left( \frac{1}{m+1} \sum_{k=0}^{m} \left( \frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_\nu^{\delta-1}(x; f_k) \right|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q} \\
&\qquad \leqslant C_{27} \left( \left\| \left( \frac{1}{m+1} \sum_{k=0}^{m} \left( \frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_\nu^{\delta}(x; f_k) \right|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q} \right. \\
&\qquad\quad \left. + \left\| \left( \frac{1}{m+1} \sum_{k=0}^{m} \left( \frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_\nu^{\delta-1}(x; f_k) - \sigma_\nu^{\delta}(x; f_k) \right|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q} \right) \\
&\qquad = C_{27}\bigl(A^I + B^I\bigr).
\end{aligned}$$

In order to estimate $A^I$ we shall reason as in the proof of Lemma 3. Namely, for $\delta > \frac{1}{2}$, $p,\ s \geqslant 1$,

$$\begin{aligned}
A^I &\leqslant \left\| \left( \frac{1}{m+1} \sum_{k=0}^{m} \left( \frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f_k(x+t)\, K_\nu^\delta(t)\, \mathrm{d}t \right|^p \right)^{s/p} \right)^{1/s} \right\|_{L^q} \\
&\leqslant \frac{1}{\pi} \int_{-\pi}^{\pi} \left\| \left( \frac{1}{m+1} \sum_{k=0}^{m} \left| f_k(x+t) \right|^s \right)^{1/s} \right\|_{L^q} \left( \frac{1}{n+1} \sum_{\nu=n}^{2n} \left| K_\nu^\delta(t) \right|^p \right)^{1/p} \mathrm{d}t \\
&\leqslant \left\| \left( \frac{1}{m+1} \sum_{k=0}^{n} \left| f_k(x) \right|^s \right)^{1/s} \right\|_{L^q} \frac{1}{\pi} \int_{-\pi}^{\pi} \max_{n \leqslant \nu \leqslant 2n} \left| K_\nu^\delta(t) \right| \mathrm{d}t \\
&\leqslant C_{22} \left\| \left( \frac{1}{m+1} \sum_{k=0}^{m} \left| f_k(x) \right|^s \right)^{1/s} \right\|_{L^q}.
\end{aligned}$$

Without loss of generality we can assume $p \geqslant 1$ because $A^I$ is a non decreasing function of $p$ and the right-hand side of (24) is independent of $p$.

Now, let us consider the expression

$$B(r)=\left\|\left(\frac{1}{m+1}\sum_{k=0}^{m}\left(\sum_{\nu=n}^{2n}\nu^{\delta p}r^{\nu p}\left|\sigma_{\nu}^{\delta-1}(x;f_k)-\sigma_{\nu}^{\delta}(x;f_k)\right|^{p}\right)^{s/p}\right)^{1/s}\right\|_{L^q}.$$

Let $p\geqslant 2$ and $\widetilde{p}=\dfrac{p}{p-2}$. Then $\widetilde{p}\,\delta>1$, $s\geqslant\widetilde{p}$. Using the inequality

$$\left(\sum_{k=0}^{m}\left\|g_k\right\|_{L^t}^{s}\right)^{1/s}\leqslant\max\bigl(1,(m+1)^{1/s-1/t}\bigr)\left\|\left(\sum_{k=0}^{m}\left|g_k(x)\right|^{s}\right)^{1/s}\right\|_{L^t}$$

for $g_k\in L^t$, $s>0$, $t\in\langle 1,+\infty)$, $m=0,1,2,\ldots$ ([6], p. 285) and proceeding analogously as in [4] (see the proof of Lemma 3) we obtain

$$\begin{aligned}B(r)&\leqslant r\,C_{28}\left\|\left(\frac{1}{m+1}\sum_{k=0}^{m}\left(\int_{-\pi}^{\pi}\frac{\left|F_k'\bigl(r\mathrm{e}^{\mathrm{i}(x+\varphi)}\bigr)\right|^{\widetilde{p}}}{\left|1-r\mathrm{e}^{\mathrm{i}\varphi}\right|^{\delta\widetilde{p}}}\,\mathrm{d}\varphi\right)^{s/\widetilde{p}}\right)^{1/s}\right\|_{L^q}\\&\leqslant r\,C_{28}\left\|\left(\int_{-\pi}^{\pi}\frac{\left(\frac{1}{m+1}\sum_{k=0}^{m}\left|F_k'\bigl(r\mathrm{e}^{\mathrm{i}(x+\varphi)}\bigr)\right|^{s}\right)^{\widetilde{p}/s}}{\left|1-r\mathrm{e}^{\mathrm{i}\varphi}\right|^{\delta\widetilde{p}}}\,\mathrm{d}\varphi\right)^{1/\widetilde{p}}\right\|_{L^q}\qquad(s\neq q).\end{aligned}$$

Applying the Minkowski inequality for $q>\widetilde{p}$ and the mean value theorem for integrals for the other $q$, we get

$$B(r)\leqslant r\,C_{28}\left\|\left(\frac{1}{m+1}\sum_{k=0}^{m}\left|F_k'\bigl(r\mathrm{e}^{\mathrm{i}x}\bigr)\right|^{s}\right)^{1/s}\right\|_{L^q}\;\left\|\bigl(1-r\mathrm{e}^{\mathrm{i}x}\bigr)^{-\delta}\right\|_{L^{\widetilde{p}}}.$$

Lemma 2 leads to

$$B(r)\leqslant C_{16}\,C_{28}\,C_{29}\,(1-r)^{(1/\widetilde{p})-\delta-1}\left\|\left(\frac{1}{m+1}\sum_{k=0}^{m}\left|f_k(x)\right|^{s}\right)^{1/s}\right\|_{L^q},$$

and if $r=1-\frac{1}{n+1}$ $(n\geqslant 1)$, then

$$B^{I}\leqslant\mathrm{e}\left(\tfrac{2}{n+1}\right)^{\delta}(n+1)^{-1/p}\,B\bigl(1-\tfrac{1}{n+1}\bigr)<\mathrm{e}\,2^{\delta}C_{16}\,C_{28}\,C_{29}\left\|\left(\frac{1}{m+1}\sum_{k=0}^{m}\left|f_k(x)\right|^{s}\right)^{1/s}\right\|_{L^q}.$$

In the case $p \in (0,2)$, we can deduce from the inequality

$$(25)\qquad \left(\frac{1}{\sum\limits_{\nu=0}^{\infty}\zeta_\nu}\sum_{\nu=0}^{\infty}\zeta_\nu\,\xi_\nu^{\gamma_1}\right)^{1/\gamma_1} \leqslant \left(\frac{1}{\sum\limits_{\nu=0}^{\infty}\zeta_\nu}\sum_{\nu=0}^{\infty}\zeta_\nu\,\xi_\nu^{\gamma_2}\right)^{1/\gamma_2} \qquad \begin{array}{c}(0<\gamma_1\leqslant\gamma_2),\\ (\zeta_\nu,\ \xi_\nu>0)\end{array}$$

that the expression $B^I$ does not exceed

$$\mathrm{e}\,2^\delta C_{16}\,C_{28}\,C_{29}\left\|\left(\frac{1}{m+1}\sum_{k=0}^{m}\left|f_k(x)\right|^s\right)^{1/s}\right\|_{L^q} \quad \text{when} \quad s\neq q.$$

The mean value theorem implies that, for $s=q<\widetilde{p}$,

$$\begin{aligned}
B(r) &\leqslant r\,C_{28}\left(\frac{1}{m+1}\sum_{k=0}^{m}\left\|\left(\int_{-\pi}^{\pi}\frac{\left|F_k'\big(r\mathrm{e}^{\mathrm{i}(x+\varphi)}\big)\right|^{\widetilde{p}}}{|1-r\mathrm{e}^{\mathrm{i}\varphi}|^{\delta\widetilde{p}}}\,\mathrm{d}\varphi\right)^{1/\widetilde{p}}\right\|_{L^q}^s\right)^{1/s}\\
&= r\,C_{28}\left(\frac{1}{m+1}\sum_{k=0}^{m}\left\|F_k'\big(r\mathrm{e}^{\mathrm{i}(x+\varphi)}\big)\right\|_{L^q}^s\right)^{1/s}\left\|\big(1-r\mathrm{e}^{\mathrm{i}x}\big)^{-\delta}\right\|_{L^{\widetilde{p}}}\\
&= r\,C_{28}\left(\frac{1}{m+1}\sum_{k=0}^{m}\left\|F_k'\big(r\mathrm{e}^{\mathrm{i}x}\big)\right\|_{L^q}^s\right)^{1/s}\left\|\big(1-r\mathrm{e}^{\mathrm{i}x}\big)^{-\delta}\right\|_{L^{\widetilde{p}}}\\
&= r\,C_{28}\left\|\left(\frac{1}{m+1}\sum_{k=0}^{m}\left|F_k'\big(r\mathrm{e}^{\mathrm{i}x}\big)\right|^s\right)^{1/s}\right\|_{L^q}\left\|\big(1-r\mathrm{e}^{\mathrm{i}x}\big)^{-\delta}\right\|_{L^{\widetilde{p}}}.
\end{aligned}$$

Collecting the results we obtain (24). □

**Lemma 5.** *For any $\gamma\geqslant 0$ and $\delta>\frac{1}{2}$, $p>0$ such that $(1-\delta)\,p<1$ we have*

$$(26)\qquad \left\|\left(\frac{1}{n+1}\sum_{\nu=n}^{2n}\left|\sigma_{\nu\nu}^{(\gamma,\delta)}(x,y)-\sigma_{\nu\nu}^{(\gamma,\delta-1)}(x,y)\right|^p\right)^{1/p}\right\|_{L^{qq}} \leqslant C_{30}\,L_n\,\varphi_{2,n},$$

$$(27)\qquad \left\|\left(\frac{1}{n+1}\sum_{\nu=n}^{2n}\left|\sigma_{\nu\nu}^{(\delta,\gamma)}(x,y)-\sigma_{\nu\nu}^{(\delta-1,\gamma)}(x,y)\right|^p\right)^{1/p}\right\|_{L^{qq}} \leqslant C_{31}\,L_n\,\varphi_{1,n}$$

*when $n=0,1,2,\ldots$*

P r o o f. First we will prove (26) and therefore we will denote by $G$ the expression on the left-hand side of (26). Then

$$G=\left(\int_{-\pi}^{\pi}\left\|\left(\frac{1}{n+1}\sum_{\nu=n}^{2n}\left|\sigma_\nu^\gamma\big(x;\big[\sigma_\nu^\delta\big(y;f^{(2)}(y)\big)-\sigma_\nu^{\delta-1}\big(y;f^{(2)}(y)\big)\big]\big)\right|^p\right)^{1/p}\right\|_{L^q}^q\mathrm{d}y\right)^{1/q}.$$

By virtue of Lemma 3 we obtain

$$G \leqslant \begin{cases} C_7\, H & \text{when } \gamma > 0 \text{ or } \gamma = 0,\ p = 2,\ 1 < q < \infty, \\ C_7 \log(n+2)\, H & \text{when } \gamma = 0,\ p \neq 2 \text{ or } \gamma = 0,\ p = 2,\ q = 1,\ \infty. \end{cases}$$

where

$$H = \left\| \left( \frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_\nu^\delta\big(y; f^{(2)}(y)\big) - \sigma_\nu^{\delta-1}\big(y; f^{(2)}(y)\big) \right|^p \right)^{1/p} \right\|_{L^{qq}}.$$

From the proof of Lemma 2 and by Theorem 2.30 (II) of [7], p. 258, we can obtain the following generalization of the inequality (20):

$$\left\| \left( \tfrac{1}{n+1} \sum_{k=0}^{n} \left| F_k'\big(r\mathrm{e}^{\mathrm{i}x}\big) \right|^p \right)^{1/p} \right\|_{L^q} \leqslant \begin{cases} C_{32}\, \frac{W_n^p(1-r)_{L^q}}{1-r} & \text{when } 1 < q < \infty, \\ C_{33} \left( \frac{W_n^p(1-r)_{L^q}}{1-r} + \int_{1-r}^{1} \frac{W_n^p(t)_{L^q}}{t^2}\, \mathrm{d}t \right) & \text{when } q = 1,\ \infty. \end{cases}$$

Hence, analogously to the proof of Lemma 1 from [3], we can easily get

$$\left\| \left( \frac{1}{n+1} \sum_{k=0}^{n} \left| F_k'\big(r\mathrm{e}^{\mathrm{i}x}\big) \right|^p \right)^{1/p} \right\|_{L^q} \leqslant \begin{cases} C_{32}\, W_m^q\big(\tfrac{1}{n+1}\big)_{L^q} & \text{when } 1 < q < \infty, \\ C_{33} \int_{1-r}^{1} \frac{1}{t^2} W_n^p(t)_{L^q}\, \mathrm{d}t & \text{when } q = 1,\ \infty. \end{cases} \tag{28}$$

Applying (28) in the proof of Lemma 4, we have

$$\begin{aligned} &\left\| \left( \frac{1}{m+1} \sum_{k=0}^{m} \left( \frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_\nu^\delta(x; f_k) - \sigma_\nu^{\delta-1}(x; f_k) \right|^p \right)^{q/p} \right)^{1/q} \right\|_{L^q} \\ &\qquad \leqslant \begin{cases} C_{34}\, W_m^q\big(\tfrac{1}{n+1}\big)_{L^q} & \text{when } 1 < q < \infty, \\ C_{35} \left( \frac{1}{n+1} \sum_{\nu=0}^{n} W_m^q\big(\tfrac{1}{\nu+1}\big)_{L^q} \right) & \text{when } q = 1,\ \infty. \end{cases} \end{aligned} \tag{29}$$

Because the inequality (29) is homogeneous with respect to the sums $\sum_{k=0}^{m}$, it can be easily observed that the sums can be replaced by integrals and thus we get

$$\begin{aligned} &\left\| \left( \frac{1}{n+1} \sum_{\nu=n}^{2n} \left| \sigma_\nu^\delta\big(y; f^{(2)}(y)\big) - \sigma_\nu^{\delta-1}\big(y; f^{(2)}(y)\big) \right|^p \right)^{1/p} \right\|_{L^{qq}} \\ &\qquad \leqslant \begin{cases} C_{34}\ \omega_2\big(\tfrac{1}{n+1}\big)_{L^{qq}} & \text{when } 1 < q < \infty, \\ C_{35} \left( \frac{1}{n+1} \sum_{\nu=0}^{n} \omega_2\big(\tfrac{1}{\nu+1}\big)_{L^{qq}} \right) & \text{when } q = 1,\ \infty. \end{cases} \end{aligned} \tag{30}$$

Finally, the estimate of $G$ and (30) lead to (26). By symmetry (27) is also true and our proof is complete. □

**Lemma 6.** *Under the assumption $\gamma \geqslant 0$, $\delta > \frac{1}{2}$, $p > 0$, $(1-\delta)p < 1$ and $1 \leqslant q < \infty$, we have*

$$\left(\frac{1}{n+1}\sum_{\nu=n}^{2n}\left\|\sigma_{\nu\nu}^{(\gamma,\delta)} - \sigma_{\nu\nu}^{(\gamma,\delta-1)}\right\|_{L^{qq}}^{p}\right)^{1/p} \leqslant C_{30}\, L_n\, \psi_{2,n}, \tag{31}$$

$$\left(\frac{1}{n+1}\sum_{\nu=n}^{2n}\left\|\sigma_{\nu\nu}^{(\delta,\gamma)} - \sigma_{\nu\nu}^{(\delta-1,\gamma)}\right\|_{L^{qq}}^{p}\right)^{1/p} \leqslant C_{31}\, L_n\, \psi_{1,n} \tag{32}$$

*for $n = 0, 1, 2, \ldots$*

P r o o f. For the above expressions we apply inequalities (25) and (16) when $q \leqslant \max(p, 2)$ or only (16) otherwise and Lemma 5 (see Lemma 2 from [3]). Thus our inequalities follow. $\square$

**Lemma 7.** *If $\gamma \geqslant 0$ and $\delta > \frac{1}{2}$ and $p > 0$ are such that $(1-\delta)\,p < 1$, then*

$$\left\|\left((1-r)\sum_{\nu=N+1}^{\infty} r^{\nu}\left|\sigma_{\nu\nu}^{(\gamma,\delta)}(x,y) - \sigma_{\nu\nu}^{(\gamma,\delta-1)}(x,y)\right|^{p}\right)^{1/p}\right\|_{L^{qq}} \leqslant C_{36}\max\bigl(1, (1-r)^{1/p-1/q}\bigr) L_{N-2}\,\varphi_{2,N}, \tag{33}$$

$$\left\|\left((1-r)\sum_{\nu=N+1}^{\infty} r^{\nu}\left|\sigma_{\nu\nu}^{(\delta,\gamma)}(x,y) - \sigma_{\nu\nu}^{(\delta-1,\gamma)}(x,y)\right|^{p}\right)^{1/p}\right\|_{L^{qq}} \leqslant C_{37}\max\bigl(1, (1-r)^{1/p-1/q}\bigr) L_{N-2}\,\varphi_{1,N}, \tag{34}$$

*when $r \in (0,1)$.*

P r o o f. Denote by $G'$ the expression on the left-hand side of (33). Then Lemma 5 and inequality (18) imply

$$\begin{aligned} G' &= \left\|\left((1-r)\sum_{l=\log_2(N+2)}^{\infty}\sum_{\nu=\alpha_l}^{\beta_l} r^{\nu}\left|\sigma_{\nu\nu}^{(\gamma,\delta)}(x,y) - \sigma_{\nu\nu}^{(\gamma,\delta-1)}(x,y)\right|^{p}\right)^{1/p}\right\|_{L^{qq}} \\ &\leqslant \left((1-r)\sum_{l=\log_2(N+2)}^{\infty}\left\|\left(\sum_{\nu=\alpha_l}^{\beta_l} r^{(p'/p)\nu}\left|\sigma_{\nu\nu}^{(\gamma,\delta)}(x,y) - \sigma_{\nu\nu}^{(\gamma,\delta-1)}(x,y)\right|^{p}\right)^{1/p}\right\|_{L^{qq}}^{p}\right)^{1/p} \\ &\quad\times \max\bigl(1, (1-r^{q/p})^{1/p-1/q}\bigr) \end{aligned}$$

$$\leqslant \max\bigl(1,(1-r^{q/p})^{1/p-1/q}\bigr)$$

$$\times\left((1-r)\sum_{l=\log_2(N+2)}^{\infty}2^l\,r^{(p'/p)\alpha_l}\left\|\left(\frac{1}{2^l}\sum_{k=\alpha_l}^{\beta_l}\left|\sigma_{\nu\nu}^{(\gamma,\delta)}(x,y)-\sigma_{\nu\nu}^{(\gamma,\delta-1)}(x,y)\right|^p\right)^{1/p}\right\|_{L^{qq}}^p\right)^{1/p}$$

$$\leqslant \max\bigl(1,(1-r^{q/p})^{1/p-1/q}\bigr)\left((1-r)\sum_{l=\log_2(N+2)}^{\infty}2^l\,r^{(p'/p)\alpha_l}\left(C_{30}\,L_{\alpha_l}\,\varphi_{2,\alpha_l}\right)^p\right)^{1/p}.$$

It is clear that

$$\varphi_{i,k}(q)\leqslant\varphi_{i,k'}(q)\quad\text{for}\quad k\geqslant k',\ \ i=1,2,\ldots. \tag{35}$$

In the case $\gamma>0$ or $\gamma=0$, $p=2$, $1<q<\infty$, using the elementary inequality

$$1-r^{1/p}\leqslant 1-r<(p+1)\left(1-r^{1/p}\right)\qquad(p\geqslant 1) \tag{36}$$

and (35) we obtain

$$G'\leqslant C_{30}\left(\frac{p}{q}+1\right)^{1/q-1/p}\max\bigl(1,(1-r)^{1/p-1/q}\bigr)$$

$$\times\left((1-r)\sum_{l=\log_2(N+2)}^{\infty}2^l\,r^{(p'/p)\alpha_l}\right)^{1/p}\varphi_{2,N+1}$$

$$\leqslant 2^{1/p}\left(\frac{p}{p'}+1\right)^{1/p}\left(\frac{p}{q}+1\right)^{1/q-1/p}C_{30}\,\max\bigl(1,(1-r)^{1/p-1/q}\bigr)\,\varphi_{2,N+1}.$$

In the other case (35) and (36) lead to

$$G'\leqslant C_{30}\left(\frac{q}{p}+1\right)^{1/q-1/p}\max\bigl(1,(1-r)^{1/p-1/q}\bigr)\,\varphi_{2,N+1}$$

$$\times\left((1-r)\sum_{l=\log_2(N+2)}^{\infty}2^l\,r^{(p'/p)\alpha_l}\,\frac{\log^p(2^l+1)}{(2^l+1)^p}\left(2^l+1\right)^p\right)^{1/p}$$

$$\leqslant C_{30}\left(\frac{q}{p}+1\right)^{1/q-1/p}\max\bigl(1,(1-r)^{1/p-1/q}\bigr)\,\varphi_{2,N+1}$$

$$\times\left(\frac{\log^p N}{N^p}(1-r)\sum_{l=\log_2(N+2)}^{\infty}2^l\,r^{(p'/p)\alpha_l}\left(2^l+1\right)^p\right)^{1/p}$$

$$\leqslant C_{30}\left(\frac{q}{p}+1\right)^{1/q-1/p}\max\bigl(1,(1-r)^{1/p-1/q}\bigr)\,\log\frac{1}{1-r}\,\varphi_{2,N+1}$$

$$\times\left(\frac{1-r}{N^p}\sum_{k=[N/2]}^{\infty}k^p\,r^{(p'/p)k}\,2^{2p+1}\right)^{1/p}$$

$$\leqslant C_{38}\, C_{30}\, 2^{3+1/p} \left(\frac{p}{p'}+1\right)^{1+1/p} \max\bigl(1,(1-r)^{1/p-1/q}\bigr)\, \log \frac{1}{1-r}\, \varphi_{2,N+1}.$$

This implies our inequality and by symmetry (34) holds, too. □

**Lemma 8.** *Let $\gamma \geqslant 0$, $\delta > \frac{1}{2}$ and $p>0$. Then, with $r\in(0,1)$, we have*

$$\left((1-r)\sum_{\nu=N+1}^{\infty} r^{\nu} \left\| \sigma_{\nu\nu}^{(\gamma,\delta)} - \sigma_{\nu\nu}^{(\gamma,\delta-1)} \right\|_{L^{qq}}^{p}\right)^{1/p} \leqslant C_{39}\, L_{N-2}\, \psi_{2,N+1}, \tag{37}$$

$$\left((1-r)\sum_{\nu=N+1}^{\infty} r^{\nu} \left\| \sigma_{\nu\nu}^{(\delta,\gamma)} - \sigma_{\nu\nu}^{(\delta-1,\gamma)} \right\|_{L^{qq}}^{p}\right)^{1/p} \leqslant C_{40}\, L_{N-2}\, \psi_{1,N+1} \tag{38}$$

*for $1 \leqslant q < \infty$.*

Proof. The proof runs along the same lines as the proof of Lemma 7. The difference, for the case $q>\max(p,2)$, consists in the use of the inequality

$$\begin{aligned} &\left(\sum_{k=0}^{\infty} r^{k\max(p,q)} \|g_k\|_{L^{qq}}^{p}\right)^{1/p} \\ &\qquad \leqslant \max\bigl(1,(1-r^q)^{1/p-1/q}\bigr) \left\| \left(\sum_{k=0}^{\infty} r^{kp} |g_k(x,y)|^{p}\right)^{1/p} \right\|_{L^{qq}} \end{aligned} \tag{39}$$

instead of (18). As a consequence, the coefficient $\max\bigl(1,(1-r)^{1/p-1/q}\bigr)$ changes to $(1-r)^{1/q-1/p}$.

In the case $q \leqslant \max(p,2)$ we apply (25) and then (39) without any coefficient. □

**Lemma 9.** *Suppose that $\gamma \geqslant 0$, $\delta > \frac{1}{2}$ and $p>0$ are such that $(1-\delta)\,p<1$. Then, for $1\leqslant q<\infty$,*

$$\left(\frac{1}{(n+1)^{\delta p+1}} \sum_{\nu=0}^{n} (\nu+1)^{\delta p} \left\| \sigma_{\nu\nu}^{(\gamma,\delta)} - \sigma_{\nu\nu}^{(\gamma,\delta-1)} \right\|_{L^{qq}}^{p}\right)^{1/p} \leqslant C_{41}\, L_n\, \psi_{2,n}, \tag{40}$$

$$\left(\frac{1}{(n+1)^{\delta p+1}} \sum_{\nu=0}^{n} (\nu+1)^{\delta p} \left\| \sigma_{\nu\nu}^{(\delta,\gamma)} - \sigma_{\nu\nu}^{(\delta-1,\gamma)} \right\|_{L^{qq}}^{p}\right)^{1/p} \leqslant C_{42}\, L_n\, \psi_{1,n}, \tag{41}$$

*when $n = 0,1,2,\dots$*

P r o o f. Denote by $G''$ the left-hand side of inequality (40). In view of Lemma 6 we obtain

$$
\begin{aligned}
G'' &= \left(\frac{1}{(n+1)^{\delta p+1}} \sum_{l=0}^{\log_2[(n+2)/2]} \sum_{\nu=\alpha_l}^{\beta_l} (\nu+1)^{\delta p} \left\|\sigma_{\nu\nu}^{(\gamma,\delta)} - \sigma_{\nu\nu}^{(\gamma,\delta-1)}\right\|_{L^{qq}}^{p}\right)^{1/p} \\
&\leqslant \left(\frac{1}{(n+1)^{\delta p+1}} \sum_{l=0}^{\log_2[(n+2)/2]} \left(2^{l+1}-1\right)^{\delta p} 2^l\, C_{30}^p\, L_{\alpha_l}^p \psi_{2,\alpha_l}^p\right)^{1/p} \\
&\leqslant 2^{1+\delta}\, C_{30}\, L_n \psi_{2,n} \left(\frac{1}{(n+1)^{\delta p+1}} \sum_{l=0}^{\log_2[(n+2)/2]} 2^{l(p\delta+1)} \left(\frac{n+1}{2^l}\right)^{p}\right)^{1/p} \\
&= 2^{1+\delta}\, C_{30}\, L_n \psi_{2,n} \left(\frac{1}{(n+1)^{\delta p+1}} \frac{2^{(p\delta+1-p)\log_2[(n+2)/2]}-1}{2^{p\delta+1-p}-1}\right)^{1/p} \\
&\leqslant C_{43}\, L_n \psi_{2,n}
\end{aligned}
$$

for $p\,\delta+1-p>0$. Because the inequalities (41) and (40) are symmetric the desired results follow. $\square$

**Lemma 10.** *If $\gamma \geqslant 0$, $\delta > \frac{1}{2}$ and $p>0$ are such that $(1-\delta)p<1$, then*

$$
\left\|\left(\frac{1}{(n+1)^{\delta p+1}} \sum_{\nu=0}^{n} (\nu+1)^{\delta p} \left|\sigma_{\nu\nu}^{(\gamma,\delta)}(x,y) - \sigma_{\nu\nu}^{(\gamma,\delta-1)}(x,y)\right|^{p}\right)^{1/p}\right\|_{L^{qq}} \leqslant C_{44} \max\left(1,(n+1)^{1/q-1/p}\right) L_n \varphi_{2,n}, \tag{42}
$$

$$
\left\|\left(\frac{1}{(n+1)^{\delta p+1}} \sum_{\nu=0}^{n} (\nu+1)^{\delta p} \left|\sigma_{\nu\nu}^{(\delta,\gamma)}(x,y) - \sigma_{\nu\nu}^{(\delta-1,\gamma)}(x,y)\right|^{p}\right)^{1/p}\right\|_{L^{qq}} \leqslant C_{45} \max\left(1,(n+1)^{1/q-1/p}\right) L_n \varphi_{1,n} \tag{43}
$$

*for $n=0,1,2,\ldots$*

P r o o f. The proof runs analogously to the proof of Lemma 9, but we have to apply Lemma 5 and inequality (17) instead of Lemma 6. $\square$

## 4. Proofs of the theorems

Proof of Theorem 1. We start with the obvious inequality

$$\begin{aligned}\big(P^p(r;f)_{L^{qq}}\big)^p &\leqslant 2^p\bigg(\frac{1}{A(r)}\sum_{k=0}^{N}\alpha_k(r)\left(\big\|\sigma_{kk}^{(\gamma,\delta-1)}-\sigma_{kk}^{(\gamma,\delta)}\big\|_{L^{qq}}+\big\|\sigma_{kk}^{(\gamma,\delta)}-f\big\|_{L^{qq}}\right)\\ &\quad+\frac{1}{A(r)}\sum_{k=N+1}^{\infty}\alpha_k(r)\left(\big\|\sigma_{kk}^{(\gamma,\delta-1)}-\sigma_{kk}^{(\gamma,\delta)}\big\|_{L^{qq}}+\big\|\sigma_{kk}^{(\gamma,\delta)}-f\big\|_{L^{qq}}\right)\bigg)\\ &=2^p\big(A^{II}+B^{II}+C^{II}+D^{II}\big).\end{aligned}$$

As is well known,

$$\left\|\sigma_{kl}^{(\chi,\eta)}-f\right\|_{L^{qq}}\leqslant\begin{cases}C_{46}\,(\varphi_{1,k}+\varphi_{2,l})\\ \qquad\text{when } \chi,\ \eta>0,\ 1\leqslant q\leqslant\infty \text{ or } \chi,\ \eta\geqslant 0,\ 1<q<\infty,\\ C_{47}\big(\chi\log(l+2)+\eta\,\log(k+2)\big)\,(\varphi_{1,k}+\varphi_{2,l})\\ \qquad\text{when } \eta=0 \text{ or } \chi=0,\ q=1,\ \infty,\\ C_{48}\log(k+2)\,\log(l+2)\,(\varphi_{1,k}+\varphi_{2,l})\\ \qquad\text{when } \chi=\eta=0 \text{ and } q=1,\ \infty\end{cases}\tag{44}$$

for $k,l=0,1,2,\dots$ (see [7] pp. 178, 184).

Hence, in view of (25) and (1), we have

$$\begin{aligned}B^{II}+D^{II}&\leqslant C_{46}^p\frac{1}{A(r)}\sum_{k=0}^{\infty}\alpha_k(r)\,\Phi_{kk}^p\\ &\leqslant 2^{1/\lambda}\,C_{46}\frac{1}{A(r)}\sum_{l=0}^{\infty}\left(\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_k^{\lambda}(r)\,\Phi_{kk}^{\lambda_p}}{(k+1)^{1-\lambda}}\right)^{1/\lambda}\\ &\leqslant 2^{1/\lambda}\,C_{46}^p\,(1-r)\sum_{k=0}^{\infty}r^k\Phi_{kk}^p\end{aligned}$$

when $\gamma>0$ or $\gamma=0$ and $q\neq 1,\ \infty$.

If $\gamma=0$ and $q=1,\infty$, then

$$B^{II}+D^{II}\leqslant\delta^p 2^{1/\lambda}\,C_1C_{47}^p\,(1-r)\sum_{k=0}^{\infty}r^k\log^p(k+2)\,\Phi_{k,k}^p.$$

In case (i) the Hölder inequality and Lemma 6 lead to

$$\begin{aligned}
A^{II}+C^{II} &= \frac{1}{A(r)}\sum_{l=0}^{\infty}\left(\sum_{k=\alpha_l}^{\beta_l}\alpha_k(r)\left\|\sigma_{kk}^{(\gamma,\delta-1)}-\sigma_{kk}^{(\gamma,\delta)}\right\|_{L^{qq}}^{p}\right)\\
&\leqslant \frac{1}{A(r)}\sum_{l=0}^{\infty}\biggl(\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_k^{\lambda}(r)}{(k+1)^{1-\lambda}}\biggr)^{1/\lambda}\biggl(\frac{1}{2^l}\sum_{k=\alpha_l}^{\beta_l}\left\|\sigma_{kk}^{(\gamma,\delta-1)}-\sigma_{kk}^{(\gamma,\delta)}\right\|_{L^{qq}}^{p\tilde{\lambda}}\biggr)^{1/\tilde{\lambda}}\\
&\leqslant \frac{C_{30}^{p}}{A(r)}\sum_{l=0}^{\infty}\biggl(\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_k^{\lambda}(r)}{(k+1)^{1-\lambda}}\biggr)^{1/\lambda}L_{\alpha_l}^{p}\,\varphi_{2,\alpha_l}^{p}\\
&\leqslant \frac{2^pC_{30}^{p}}{A(r)}\sum_{l=0}^{\infty}\biggl(\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_k^{\lambda}(r)(L_k\varphi_{2,k})^{p\lambda}}{(k+1)^{1-\lambda}}\biggr)^{1/\lambda}.
\end{aligned}$$

Next, from (1),

$$\begin{aligned}
A^{II}+C^{II} &\leqslant 2^p\,C_{30}^{p}\,C_1(1-r)\sum_{k=0}^{\infty}r^k\,(L_k\varphi_{2,k})^p\\
&\leqslant 2^p\,C_{30}^{p}\,C_1\biggl((1-r)\sum_{k=0}^{\infty}r^k\,(L_k\Phi_{k,k})^p\biggr).
\end{aligned}$$

In case (ii) we proceed similarly as before taking $\psi_{2,k}$ and $\Psi_{k,k}$ instead of $\varphi_{2,k}$ and $\Phi_{k,k}$, respectively.

Finally, by the inequality $\Phi_{k,k}\leqslant\Psi_{k,k}$ $(k=0,1,2,\ldots)$, the desired assertion follows. □

Proof of Theorem 2. Under the above notation, the condition (4) leads to

$$B^{II}\leqslant C_{49}\,L_{N-2}^{p}\,(1-r)\sum_{k=0}^{N}\bigl(\Phi_{k,k}\bigr)^p$$

in all cases. However, in (i) we have

$$A^{II}\leqslant\begin{cases} C_{50}\,L_{N-2}^{p}\,(1-r)\sum\limits_{k=0}^{N}\bigl(\Phi_{k,k}\bigr)^p & \text{when } q\leqslant\max\bigl(\tilde{\lambda}p,2\bigr),\\ C_{51}(1-r)^{\tilde{\lambda}p/q}\sum\limits_{k=0}^{N}\Bigl(\omega_1\bigl(\tfrac{1}{k+1}\bigr)_{L^{qq}}+\omega_2\bigl(\tfrac{1}{k+1}\bigr)_{L^{qq}}\Bigr)^p & \text{when } q>\max\bigl(\tilde{\lambda}p,2\bigr),\end{cases}$$

while in case (ii)

$$A^{II}\leqslant C_{52}\,L_{N-2}^{p}\,(1-r)\sum_{k=0}^{N}\bigl(\Phi_{k,k}\bigr)^p,$$

and in case (iii)

$$A^{II}\leqslant C_{53}\,L_{N-2}^{p}\,(1-r)\sum_{k=0}^{N}\Psi_{k,k}^{p}\bigl(\tilde{\lambda}p,q\bigr).$$

From (44) and (5) it follows that in all cases

$$\begin{aligned}
D^{II} &\leqslant \frac{C_{46}^p + C_{47}^p}{A(r)} \sum_{k=N+1}^{\infty} \alpha_k(r)\, L_k^p\, \Phi_{k,k}^p \\
&\leqslant \frac{C_{46}^p + C_{47}^p}{A(r)} \Bigg( \sum_{k=N+1}^{\infty} \alpha_k^{\lambda}(r) r^{k(1-\lambda)} \Bigg)^{1/\lambda} \Bigg( \sum_{k=N+1}^{\infty} r^k\, (L_k \Phi_{k,k})^{p\tilde{\lambda}} \Bigg)^{1/\tilde{\lambda}} \\
&\leqslant C_{54} \Bigg( (1-r) \sum_{k=N+1}^{\infty} r^k\, (L_k \Phi_{k,k})^{p\tilde{\lambda}} \Bigg)^{1/\tilde{\lambda}} \\
&\leqslant C_{54}\, \Phi_{N,N}^p \Bigg( (1-r) \sum_{k=N+1}^{\infty} r^k L_k^{p\tilde{\lambda}} \Bigg)^{1/\tilde{\lambda}}.
\end{aligned}$$

If $L_k = 1$, then the expression in brackets is smaller than 1, but if $L_k = \log(k+2)$ then, similarly as in the proof of Lemma 7, we obtain

$$\begin{aligned}
D^{II} &\leqslant C_{54}\, \Phi_{N,N}^p \left( \left( \frac{\log N}{N} \right)^{p\tilde{\lambda}} (1-r) \sum_{k=N+1}^{\infty} r^k (k+2)^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \\
&\leqslant C_{55} \log^p \frac{1}{1-r}\, \Phi_{N,N}^p \left( (1-r)^{p\tilde{\lambda}+1} \frac{1}{(1-r)^{p\tilde{\lambda}+1}} \right) \\
&= C_{55} \log^p \frac{1}{1-r}\, \Phi_{N,N}^p.
\end{aligned}$$

To estimate $C^{II}$ we use the Hölder inequality, condition (5) and Lemma 8. Then

$$\begin{aligned}
C^{II} &\leqslant \frac{1}{(1-r)A(r)} \Bigg( (1-r) \sum_{k=N+1}^{\infty} \alpha_k^{\lambda}(r)\, r^{k(1-\lambda)} \Bigg)^{1/\lambda} \\
&\quad \times \Bigg( (1-r) \sum_{k=N+1}^{\infty} r^k \left\| \sigma_{kk}^{(\gamma,\delta-1)} - \sigma_{kk}^{(\gamma,\delta)} \right\|_{L^{qq}}^{p\tilde{\lambda}} \Bigg)^{1/\tilde{\lambda}} \\
&\leqslant C_{39}^p\, C_5 L_{N-2}^p \psi_{2,N+1}^p \big(p\tilde{\lambda}, q\big).
\end{aligned}$$

When $(1-\delta)\, q < 1 - 1/\lambda$, we obtain

$$C^{II} \leqslant C_{39}^p\, C_5\, L_{N-2}^p\, \varphi_{2,N+1}^p.$$

Because $\Phi_{k,k}$ is a non increasing function of $k$ and $\Phi_{k,k} \leqslant \Psi_{k,k}$ for $k = 0, 1, 2, \ldots$, then collecting the estimates, we obtain our result. $\square$

P r o o f  of Theorem 3. We use the same notation as in the proof of Theorem 1. Then, by (10) and (44), we get

$$\begin{aligned}
B^{II}+D^{II} &\leqslant \left(C_{46}^p+C_{47}^p\right)\frac{1}{A(r)}\left(\sum_{k=0}^{N}+\sum_{k=N+1}^{\infty}\right)\alpha_k(r)L_k^p\Phi_{k,k}^p \\
&\leqslant C_{10}\left(C_{46}^p+C_{47}^p\right)L_{N-2}^p\,(1-r)\sum_{k=0}^{N}\Phi_{k,k}^p+\left(C_{54}+C_{55}\right)L_{N-2}^p\Phi_{N,N}^p \\
&\leqslant \Big(C_{10}\left(C_{46}^p+C_{47}^p\right)+2\left(C_{54}+C_{55}\right)\Big)\,L_{N-2}^p(1-r)\sum_{k=0}^{N}\Phi_{k,k}^p.
\end{aligned}$$

To estimate $A^{II}$ and $C^{II}$ we use the Hölder inequality. By the condition (9) we obtain

$$\begin{aligned}
A^{II} &\leqslant \frac{1}{A(r)}\left(\sum_{k=0}^{N}\alpha_k^{\lambda}(r)(k+1)^{-\delta\lambda p}\right)^{1/\lambda}\left(\sum_{k=0}^{N}(k+1)^{\delta\tilde{\lambda}p}\left\|\sigma_{k,k}^{(\gamma,\delta-1)}-\sigma_{kk}^{(\gamma,\delta)}\right\|_{L^{qq}}^{p\tilde{\lambda}}\right)^{1/\tilde{\lambda}} \\
&\leqslant C_9\left((1-r)^{p\delta\tilde{\lambda}+1}\sum_{k=0}^{N}(k+1)^{\delta\tilde{\lambda}p}\left\|\sigma_{kk}^{(\gamma,\delta-1)}-\sigma_{kk}^{(\gamma,\delta)}\right\|_{L^{qq}}^{p\tilde{\lambda}}\right)^{1/\tilde{\lambda}}
\end{aligned}$$

and by the condition (5) we have

$$\begin{aligned}
C^{II} &\leqslant \frac{1}{A(r)}\left(\sum_{k=N+1}^{\infty}\alpha_k^{\lambda}(r)r^{k(1-\lambda)}\right)^{1/\lambda}\left(\sum_{k=N+1}^{\infty}r^k\left\|\sigma_{kk}^{(\gamma,\delta-1)}-\sigma_{kk}^{(\gamma,\delta)}\right\|^{p\tilde{\lambda}}\right)^{1/\tilde{\lambda}} \\
&\leqslant C_5\left((1-r)\sum_{k=N+1}^{\infty}r^k\left\|\sigma_{kk}^{(\gamma,\delta-1)}-\sigma_{kk}^{(\gamma,\delta)}\right\|_{L^{qq}}^{p\tilde{\lambda}}\right)^{1/p\tilde{\lambda}}.
\end{aligned}$$

Further, applying Lemmas 8 and 9 and collecting the results, we get our assertion. □

P r o o f  of Theorem 4. Clearly,

$$\begin{aligned}
Q^p(r;f)_{L^{qq}} \leqslant 4^{1/p}\Bigg(&\left\|\left(\frac{1}{A(r)}\sum_{k=0}^{N}\alpha_k(r)\left|\sigma_{kk}^{(\gamma,\delta-1)}(x,y)-\sigma_{kk}^{(\gamma,\delta)}(x,y)\right|^p\right)^{1/p}\right\|_{L^{qq}} \\
&+\left\|\left(\frac{1}{A(r)}\sum_{k=N+1}^{\infty}\alpha_k(r)\left|\sigma_{kk}^{(\gamma,\delta-1)}(x,y)-\sigma_{kk}^{(\gamma,\delta)}(x,y)\right|^p\right)^{1/p}\right\|_{L^{qq}}
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\left(\frac{1}{A(r)}\sum_{k=0}^{N}\alpha_k(r)\left|\sigma_{kk}^{(\gamma,\delta)}(x,y)-f(x,y)\right|^p\right)^{1/p}\right\|_{L^{qq}}\\
&+\left\|\left(\frac{1}{A(r)}\sum_{k=N+1}^{\infty}\alpha_k(r)\left|\sigma_{kk}^{(\gamma,\delta)}(x,y)-f(x,y)\right|^p\right)^{1/p}\right\|_{L^{qq}}\Bigg)\\
&=4^{1/p}\left(A^{III}+B^{III}+C^{III}+D^{III}\right).
\end{aligned}
$$

In view of (44), (18) and (5), in the case $L_n=1$ we have

$$
D^{III}\leqslant\begin{cases} C_{46}\Phi_{N,N} & \text{when } q\geqslant p,\\ C_{46}^{1/p}(1-r)^{1/p-1/q}\,\Phi_{N,N}\,2^{1-1/\lambda p}\Big(1+\frac{2(p/q-1)}{1-1/\lambda}\Big)^{1/q-1/p} & \text{when } q<p.\end{cases}
$$

In the case $L_n=\log(n+2)$, for $q\geqslant p$, we obtain

$$
\begin{aligned}
D^{III}&\leqslant C_{47}\,\Phi_{N,N}\Bigg(\frac{1}{A(r)}\sum_{k=N+1}^{\infty}\alpha_k(r)\,\log^p(k+2)\Bigg)^{1/p}\\
&\leqslant C_{47}\,C_5^{1/p}\,\Phi_{N,N}\Bigg((1-r)\sum_{k=N+1}^{\infty}r^k\log^{\tilde\lambda p}(k+2)\Bigg)^{1/p\tilde\lambda}\\
&\leqslant C_{47}\,C_5^{1/p}\log\big(\tfrac{1}{1-r}\big)\,\Phi_{N,N}\Bigg(\frac{1-r}{N^{\tilde\lambda p}}\sum_{k=N+1}^{\infty}(k+2)^{\tilde\lambda p}r^k\Bigg)^{1/p\tilde\lambda}\\
&\leqslant C_{56}\,C_{47}C_5^{\,1/p}\log\big(\tfrac{1}{1-r}\big)\,\Phi_{N,N},
\end{aligned}
$$

and for $q<p$,

$$
\begin{aligned}
D^{III}&\leqslant C_{47}\,\Phi_{N,N}(1-r_0)^{1/p-1/q}\Bigg(\frac{1}{A(r)}\sum_{k=N+1}^{\infty}r_0^{k\left(\frac{q}{p}-1\right)}\alpha_k(r)\,\log^p(k+2)\Bigg)^{1/p}\\
&\leqslant C_{47}\,C_5^{1/p}\,\Phi_{N,N}\,(1-r_0)^{1/p-1/q}\Bigg((1-r)\sum_{k=N+1}^{\infty}r^k r_0^{k\left(\frac{q}{p}-1\right)\tilde\lambda}\log^{\tilde\lambda p}(k+2)\Bigg)^{1/p\tilde\lambda}\\
&\leqslant C_5^{1/p}C_{47}(1-r_0)^{1/p-1/q}\log\big(\tfrac{1}{1-r}\big)\Phi_{N,N}\Bigg(\frac{1-r}{N^{\tilde\lambda p}}\sum_{k=N+1}^{\infty}(k+2)^{\tilde\lambda p}r^k r_0^{k\left(\frac{q}{p}-1\right)\tilde\lambda}\Bigg)^{1/p\tilde\lambda}.
\end{aligned}
$$

Taking $r_0=r^{1/\tilde\lambda}$, inequality (36) implies

$$
D^{III}\leqslant C_{57}\,C_{47}\,C_5^{1/p}(1-r)^{1/p\tilde\lambda-1/q}\,\log\big(\tfrac{1}{1-r}\big)\Phi_{N,N}.
$$

In view of condition (5) and Lemma 7 we have

$$\begin{aligned}
B^{III} &\leqslant C_5^{1/p} \left\| \left( (1-r) \sum_{k=N+1}^{\infty} r^k \left| \sigma_{kk}^{(\gamma,\delta-1)}(x,y) - \sigma_{kk}^{(\gamma,\delta)}(x,y) \right|^{p\tilde{\lambda}} \right)^{1/p\tilde{\lambda}} \right\|_{L^{qq}} \\
&\leqslant C_5^{1/p} C_{36} \max\bigl(1, (1-r)^{1/p\tilde{\lambda}-1/q}\bigr) L_{N-2} \Phi_{N,N}.
\end{aligned}$$

For the estimation of expressions $A^{III}$ and $C^{III}$ we first consider the case when the conditions (4) and (5) hold. Then, by (4), (17) and Lemma 5,

$$\begin{aligned}
A^{III} &\leqslant \left\| \left( \frac{1}{A(r)} \sum_{i=0}^{j} \left( \sum_{k=\alpha_l}^{\beta_l'} \frac{\alpha_k^{\lambda}(r)}{(k+1)^{1-\lambda}} \right) \left( \frac{1}{2^l} \sum_{k=\alpha_l}^{\beta_l'} \left| \sigma_{kk}^{(\gamma,\delta-1)}(x,y) - \sigma_{kk}^{(\gamma,\delta)}(x,y) \right|^{p\tilde{\lambda}} \right)^{1/\tilde{\lambda}} \right)^{1/p} \right\|_{L^{qq}} \\
&\leqslant C_{30} \max\bigl(1, (1+j)^{1/q-1/p\tilde{\lambda}}\bigr) \left( \frac{1}{A(r)} \sum_{l=0}^{j} \left( \sum_{k=\alpha_l}^{\beta_l'} \frac{\alpha_k^{\lambda}(r)}{(k+1)^{1-\lambda}} \right) L_{\alpha_l}^p \varphi_{2,\alpha_l}^p \right)^{1/p} \\
&\leqslant 2\, C_4^{1/p} C_{30} \max\bigl(1, (1-r)^{1/p\tilde{\lambda}-1/q}\bigr)\, L_{N-2} \left( (1-r) \sum_{k=0}^{N} \varphi_{2,k}^p \right)^{1/p}.
\end{aligned}$$

Applying (4), (17) and (44) we obtain

$$\begin{aligned}
C^{III} &\leqslant (C_{46}+C_{47}) \max\bigl(1, (1+N)^{1/q-1/p\tilde{\lambda}}\bigr) \left( \frac{1}{A(r)} \sum_{k=0}^{N} \alpha_k(r)\, (L_k \varphi_{2,k})^p \right)^{1/p} \\
&\leqslant (C_{46}+C_{47}) \max\bigl(1, (1-r)^{1/p\tilde{\lambda}-1/q}\bigr) L_N 2^{1/q} C_4^{1/p} \left( (1-r) \sum_{k=0}^{N} \varphi_{2,k}^p \right)^{1/p}.
\end{aligned}$$

In the other case, i. e. when the conditions (5), (9) and (10) hold, Lemma 10 yields

$$\begin{aligned}
A^{III} &\leqslant \left( \frac{1}{A(r)} \right)^{1/p} \left( \sum_{k=0}^{N} \alpha_k^{\lambda}(r) (k+1)^{-\delta\lambda p} \right)^{1/\lambda p} \\
&\quad \times \left\| \left( \sum_{k=0}^{N} (k+1)^{\delta\tilde{\lambda}p} \left| \sigma_{kk}^{(\gamma,\delta-1)}(x,y) - \sigma_{kk}^{(\gamma,\delta)}(x,y) \right|^{p\tilde{\lambda}} \right)^{1/p\tilde{\lambda}} \right\|_{L^{qq}} \\
&\leqslant C_9^{1/p} \left\| \left( (1-r)^{p\delta\tilde{\lambda}+1} \sum_{k=0}^{N} (k+1)^{\delta\tilde{\lambda}p} \left| \sigma_{kk}^{(\gamma,\delta-1)}(x,y) - \sigma_{kk}^{(\gamma,\delta)}(x,y) \right|^{p\tilde{\lambda}} \right)^{1/p\tilde{\lambda}} \right\|_{L^{qq}} \\
&\leqslant C_9^{1/p} C_{44}\, 2^{\delta+1/p\tilde{\lambda}} \max\bigl(1, (1-r)^{1/p\tilde{\lambda}-1/q}\bigr)\, L_N\, \varphi_{2,N} \\
&\leqslant C_9^{1/p} C_{44}\, 2^{\delta+1/p\tilde{\lambda}} \max\bigl(1, (1-r)^{1/p\tilde{\lambda}-1/q}\bigr)\, L_N \left( (1-r) \sum_{k=0}^{N} \varphi_{2,k}^p \right)^{1/p}.
\end{aligned}$$

In view of (44) we conclude

$$C^{III} \leqslant (C_{46}+C_{47}) \max\bigl(1, (1+N)^{1/q-1/p}\bigr) \biggl(\frac{1}{A(r)} \sum_{k=0}^{N} \alpha_k(r)\, (L_k \varphi_{2,k})^p\biggr)^{1/p}$$
$$\leqslant (C_{46}+C_{47})\, C_{25}^{1/p} \max\bigl(1, (1-r)^{1/p\tilde{\lambda}-1/q}\bigr)\, L_N \biggl((1-r) \sum_{k=0}^{N} \varphi_{2,k}^p\biggr)^{1/p} 2^{1/q}.$$

Putting together our results we complete our proof. □

P r o o f of Theorem 5. In the cases considered in Theorem 4 the proof proceeds similarly to the preceding ones and the factor

$$\max\bigl(1, (1-r)^{1/p-1/q}\bigr) = \max\bigl(1, (1-r)^{1/p-1/q'}\bigr)$$

is equal to 1. Therefore, we concentrate our attention to the case when only the condition (1) holds. Then, by (44) and the Minkowski inequality, we obtain

$$\biggl\| \biggl(\frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) \Bigl|\sigma_{kk}^{(\gamma,\delta)}(x,y) - f(x,y)\Bigr|^p\biggr)^{1/p} \biggr\|_{L^{q'q'}}$$
$$\leqslant \bigl(C_{46}+C_{47}\bigr) \biggl(\frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) \bigl(L_k \Phi_{k,k}(q')\bigr)^p\biggr)^{1/p}$$
$$\leqslant \bigl(C_{46}+C_{47}\bigr)\, 2^{1/\lambda} \left(\frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^{\lambda}(r) \bigl(L_k \Phi_{k,k}(q')\bigr)^{p\lambda}}{(k+1)^{1-\lambda}}\right)^{1/\lambda}\right)^{1/p}$$
$$\leqslant 2^{1/\lambda} \bigl(C_{46}+C_{47}\bigr)\, C_1^{1/p} C_{58}\, L_{N-2} \biggl((1-r) \sum_{k=0}^{N} \Phi_{kk}^p(q')\biggr)^{1/p},$$

and by Lemma 5,

$$\biggl\| \biggl(\frac{1}{A(r)} \sum_{k=0}^{\infty} \alpha_k(r) \Bigl|\sigma_{kk}^{(\gamma,\delta-1)}(x,y) - \sigma_{kk}^{(\gamma,\delta)}(x,y)\Bigr|^p\biggr)^{1/p} \biggr\|_{L^{q'q'}}$$
$$\leqslant \biggl(\frac{1}{A(r)} \sum_{l=0}^{\infty} \sum_{k=\alpha_l}^{\beta_l} \tfrac{\alpha_k^{\lambda}(r)}{(k+1)^{1-\lambda}}\biggr)^{1/\lambda} \biggl\| \biggl(\frac{1}{2^l} \sum_{k=\alpha_l}^{\beta_l} \Bigl|\sigma_{kk}^{(\gamma,\delta-1)}(x,y) - \sigma_{k,k}^{(\gamma,\delta)}(x,y)\Bigr|^{p\tilde{\lambda}}\biggr)^{1/\tilde{\lambda}p} \biggr\|_{L^{q'q'}}^p$$
$$\leqslant 2\, C_{30} \left(\frac{1}{A(r)} \sum_{l=0}^{\infty} \left(\sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^{\lambda}(r) \bigl(L_k \varphi_{2,k}(q')\bigr)^{p\lambda}}{(k+1)^{1-\lambda}}\right)^{1/\lambda}\right)^{1/p}$$

$$\leqslant 2\, C_{30}\, C_1^{1/p} \left( (1-r) \sum_{k=0}^{\infty} r^k \left( L_k \varphi_{2,k}(q') \right)^p \right)^{1/p}$$

$$\leqslant 2\, C_{58}\, C_{30}\, C_1^{1/p} L_{N-2} \left( (1-r) \sum_{k=0}^{N} \Phi_{kk}^p(q') \right)^{1/p} .$$

The proof of Theorem 5 is thus completed. $\square$

P r o o f of Theorem 6. We easily get the estimate

$$\left( R^{p',s}(r;f)_{L^{qq}} \right)^s$$

$$\leqslant 2^{1+s/p'} \left( \frac{1}{A(r)} \sum_{l=0}^{\infty} \left\| \left( \sum_{k=\alpha_l}^{\beta_l} \alpha_k(r) \left| \sigma_{kk}^{(\gamma,\delta-1)}(x,y) - \sigma_{k,k}^{(\gamma,\delta)}(x,y) \right|^{p'} \right)^{1/p'} \right\|_{L^{qq}}^s \right.$$

$$\left. + \frac{1}{A(r)} \sum_{l=0}^{\infty} \left\| \left( \sum_{k=\alpha_l}^{\beta_l} \alpha_k(r) \left| \sigma_{kk}^{(\gamma,\delta)}(x,y) - f(x,y) \right|^{p'} \right)^{1/p'} \right\|_{L^{qq}}^s \right)$$

$$= 2^{1+s/p'} \left( A^{IV} + B^{IV} \right).$$

By (13) and Lemma 5 we obtain

$$A^{IV} \leqslant \frac{1}{A(r)} \sum_{l=0}^{\infty} \left( \left( \sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^{\lambda}(r)}{(k+1)^{1-\lambda}} \right)^{s/\lambda p'} \left( C_{30} L_{\alpha_l} \varphi_{2,\alpha_l} \right)^s \right)$$

$$\leqslant C_{30}^s \frac{1}{A(r)} \sum_{l=0}^{\infty} \left( \sum_{k=\alpha_l}^{\beta_l} \frac{\alpha_k^{\lambda}(r) (L_k \varphi_{2,k})^{p'\lambda}}{(k+1)^{1-\lambda}} \right)^{s/\lambda p'}$$

$$\leqslant C_{30}^s\, C_{13} (1-r) \sum_{l=0}^{\infty} \left( \sum_{k=\alpha_l}^{\beta_l} r^k \left( L_k \varphi_{2,k} \right)^{p'} \right)^{s/p'}$$

$$\leqslant C_{30}^s\, C_{13} \left( (1-r) \sum_{l=0}^{\infty} \left( \sum_{k=\alpha_l}^{\beta_l} r^k \left( L_k \Phi_{k,k} \right)^{p'} \right)^{s/p'} \right).$$

From (17) and (44) we have

$$B^{IV} \leqslant \frac{1}{A(r)} \sum_{l=0}^{\infty} \left( \sum_{k=\alpha_l}^{\beta_l} \alpha_k(r) \left\| \sigma_{kk}^{(\gamma,\delta)}(x,y) - f(x,y) \right\|_{L^{qq}}^{p'} \right)^{s/p'}$$

$$\leqslant (C_{46}^s + C_{47}^s) \frac{1}{A(r)} \sum_{l=0}^{\infty} \left( \sum_{k=\alpha_l}^{\beta_l} \alpha_k(r) \left( L_k \Phi_{k,k} \right)^{p'} \right)^{s/p'}$$

$$\leqslant 2^{s/\lambda p'}\left(C_{46}^s + C_{47}^s\right)\frac{1}{A(r)}\sum_{l=0}^{\infty}\left(\sum_{k=\alpha_l}^{\beta_l}\frac{\alpha_k^\lambda(r)L_k^{\lambda p'}\Phi_{k,k}^{\lambda p'}}{(k+1)^{1-\lambda}}\right)^{s/\lambda p'}$$

$$\leqslant 2^{s/\lambda p'}\left(C_{46}^s + C_{47}^s\right)C_{13}\left((1-r)\sum_{l=0}^{\infty}\left(\sum_{k=\alpha_l}^{\beta_l} r^k\left(L_k\Phi_{k,k}\right)^{p'}\right)^{s/p'}\right).$$

Hence, the desired result is now evident. □

P r o o f of Theorem 7. The proof proceeds similarly to that of Theorem 6. We only apply the inequality (17) with exponents $p$ and $q'$ such that $q'/p \geqslant 1$. □

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