121 (1996) MATHEMATICA BOHEMICA No. 1, 83–87

ON THEOREMS OF PU & PU AND GRANDE

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(Received October 4, 1994)

Abstract. Given a finite family of cliquish functions, $\mathfrak{A}$, we can find a Lebesgue function α such that $f + \alpha$ is Darboux and quasi-continuous for every $f \in \mathfrak{A}$. This theorem is a generalization both of the theorem by H. W. Pu & H. H. Pu and of the theorem by Z. Grande.

Keywords: quasi-continuous function, cliquish function, Lebesgue function

MSC 1991: 26A15, 54C08

In 1987 H. W. Pu and H. H. Pu [3] proved the following theorem.

Theorem 1. *Let $\mathfrak{A}$ be a finite family of Baire one functions. Then there is a Baire one function f such that $f + g$ is a Darboux function for each $g \in \mathfrak{A}$.*

This theorem was generalized in 1992 by Z. Grande [1].

Theorem 2. *Let $f_1, \dots, f_k$ be cliquish functions. There is a Baire one function f such that $f \neq 0$ only on a null set and all sums $f + f_i$, $i \in \{1, \dots, k\}$, are Darboux functions.*

In this paper I prove that given a finite family of cliquish functions, $\mathfrak{A}$, we can find a Lebesgue function α such that $f + \alpha$ is Darboux and quasi-continuous for every $f \in \mathfrak{A}$. Clearly we cannot require that $\alpha \neq 0$ only on a null set.

First we need some notation. The real line $(-\infty, \infty)$ is denoted by $\mathbb{R}$ and the set of positive integers by $\mathbb{N}$. The word *function* means a mapping from $\mathbb{R}$ into $\mathbb{R}$. The words *measure*, *summable* etc. refer to the Lebesgue measure and integral in $\mathbb{R}$. The Euclidean metric in $\mathbb{R}$ will be denoted by ϱ. For every set $A \subset \mathbb{R}$ let $\operatorname{cl} A$ be its

Supported by a KBN Research Grant 2 1144 91 01, 1992–94.

closure and $|A|$ its outer Lebesgue measure. A symbol like $\int_A f$ will always mean the Lebesgue integral.

Let f be a function and let $A \subset \mathbb{R}$ be non-empty. We will write $\sup(f, A)$ for $\sup\{f(x)\colon x \in A\}$ and we denote $\inf(f, A) = -\sup(-f, A)$. The *oscillation of f on A* will be denoted by $\omega(f, A)$, i.e., $\omega(f, A) = \sup(f, A) - \inf(f, A)$. Similarly, the *oscillation of f at a point $x \in \mathbb{R}$* will be denoted by $\omega(f, x)$, i.e., $\omega(f, x) = \lim_{r \to 0^+} \omega\big(f, [x-r, x+r]\big)$. The set of points of continuity of f will be denoted by $C(f)$.

We say that a function f is *quasi-continuous* (*cliquish*) at a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$ and each open set $U \ni x$ we can find a non-empty open set $V \subset U$ such that $\omega\big(f, \{x\} \cup V\big) < \varepsilon$ $\big(\omega(f, V) < \varepsilon$, respectively$\big)$. We say that f is *quasi-continuous* (*cliquish*) if it is quasi-continuous (cliquish) at each point $x \in \mathbb{R}$. Cliquish functions are also known as *pointwise discontinuous*.

We will use the following well-known (and easy to prove) facts.

- A function f is quasi-continuous iff for each $x \in \mathbb{R}$ there exists a sequence $x_1, x_2, \ldots \in C(f)$ such that $x_n \to x$ and $f(x_n) \to f(x)$.
- A function f is cliquish iff $C(f)$ is residual. In particular, every Baire one function is cliquish.

We say that $x \in \mathbb{R}$ is a *Lebesgue point* of a function α if α is locally summable at x and $\lim_{r \to 0} \int_x^{x+r} |\alpha - \alpha(x)|/r = 0$. We say that α is a *Lebesgue function* if each $x \in \mathbb{R}$ is a Lebesgue point of α.

The proof of the next lemma is straightforward. $\big($Cf. also Lemma 3.3 of [2].$\big)$

Lemma 3. *Let I be a compact interval, let functions $g_1, \ldots, g_k$ be cliquish, $K \subset \mathbb{R}$ nowhere dense, $L \geqslant \eta \geqslant \sup\big\{\omega(g_i, I)\colon i \in \{1, \ldots, k\}\big\}$, and $\varepsilon > 0$. Then there is a nowhere dense perfect set $F \subset I \cap \bigcap_{i=1}^{k} C(g_i) \setminus \operatorname{cl} K$ and a continuous function α such that $|\alpha| < L + \eta$, $\alpha = 0$ on $\operatorname{cl} K \cup (\mathbb{R} \setminus I)$, $\int_I |\alpha| < \varepsilon$, and $(g_i + \alpha)(F) \supset \big[\inf(g_i, I) - L, \sup(g_i, I) + L\big]$ for $i \in \{1, \ldots, k\}$.* □

Theorem 4. *Let $f_1, \ldots, f_k$ be cliquish functions and $\eta > 0$. There is a Lebesgue function α such that $f_i + \alpha$ is Darboux and quasi-continuous for each $i \in \{1, \ldots, k\}$, $C(\alpha) \supset \bigcap_{i=1}^{k} C(f_i)$ and $|\alpha| < \sup\big\{\omega(f_i, x)\colon i \in \{1, \ldots, k\}, x \in \mathbb{R}\big\} + \eta$.*

P r o o f . Denote $C = \bigcap_{i=1}^{k} C(f_i)$. Set $\eta_0 = \sup\big\{\omega(f_i, x)\colon i \in \{1, \ldots, k\}, x \in \mathbb{R}\big\}$, $\alpha_0 = 0$ and $B_0 = F_0 = \emptyset$. We will proceed by induction. Fix an $n \in \mathbb{N}$.

Put $\eta_n = \eta/2^{n+1}$ and $B_n = \big\{x \in \mathbb{R}\colon \omega(f_i, x) \geqslant \eta_n$ for some $i \in \{1, \ldots, k\}\big\}$. (Clearly we may assume that $\eta_1 < \eta_0$.) Find a family of non-overlapping compact intervals $\mathcal{I}_n = \big\{I_{n,m}\colon m \in \mathbb{N}\big\}$ such that $\bigcup \mathcal{I}_n = \mathbb{R} \setminus B_n$ and each $x \notin B_n$ belongs

to the interior of the union of some two elements of $\mathcal{I}_n$. Since each $I_{n,m}$ is compact and $\omega(f_i,x)<\eta_n$ for each $x\in I_{n,m}$ and $i\in\{1,\dots,k\}$, and α_{n-1} is continuous out of B_{n-1}, so we may assume that $\max\bigl\{\omega(f_i+\alpha_{n-1},I_{n,m})\colon i\in\{1,\dots,k\}\bigr\}<\eta_n$; we moreover assume that $|I_{n,m}|<\varrho(I_{n,m},B_n)$.

Fix an $m\in\mathbb{N}$. If $\eta_{n-1}<+\infty$, then set $L_{n,m}=\eta_{n-1}$. Otherwise set $L_{n,m}=2\max\bigl\{\sup(|f_i+\alpha_{n-1}|,I_{n,m})\colon i\in\{1,\dots,k\}\bigr\}+m$. (This case is possible only if $n=1$.) Use Lemma 3 to find a nowhere dense perfect set $F_{n,m}\subset I_{n,m}\cap C\setminus F_{n-1}$ and a continuous function $\alpha_{n,m}$ such that

$$|\alpha_{n,m}|<L_{n,m}+\eta_n \quad \text{on } I_{n,m}, \tag{1}$$

$$\alpha_{n,m}=0 \quad \text{on } F_{n-1}\cup(\mathbb{R}\setminus I_{n,m}), \tag{2}$$

$$\int_{I_{n,m}}|\alpha_{n,m}|<2^{-m}|I_{n,m}|, \tag{3}$$

and

$$\begin{aligned}(f_i+\alpha_{n-1}+\alpha_{n,m})(F_{n,m})\\ \supset\bigl[\inf(f_i+\alpha_{n-1},I_{n,m})-L_{n,m},\sup(f_i+\alpha_{n-1},I_{n,m})+L_{n,m}\bigr]\end{aligned} \tag{4}$$

for $i\in\{1,\dots,k\}$.

Define $F_n=F_{n-1}\cup\bigcup\limits_{m=1}^{\infty}F_{n,m}$ and $\alpha_n=\alpha_{n-1}+\sum\limits_{m=1}^{\infty}\alpha_{n,m}$. It is easy to show that each $x\in B_n$ is a Lebesgue point of α_n. Since α_n is continuous on $\mathbb{R}\setminus B_n$, so α_n is a Lebesgue function. Note that $\alpha_n=0$ on $B_n\cup F_{n-1}$.

By (1), the sequence (α_n) is uniformly convergent, so its sum, which we denote by α, is a Lebesgue function. By the construction, $C(\alpha)\supset\bigcap\limits_{n=1}^{\infty}(\mathbb{R}\setminus B_n)=C$, while $|\alpha|<\eta_0+2\sum\limits_{n=1}^{\infty}\eta_n=\eta_0+\eta$.

Suppose that $f_i+\alpha$ is not Darboux for some $i\in\{1,\dots,k\}$. Let $a,b,y\in\mathbb{R}$ be such that $(f_i+\alpha)(a)<y<(f_i+\alpha)(b)$ but $(f_i+\alpha)(x)=y$ for no x between a and b. Assume that, e.g., $a<b$. (The other case is analogous.) Set $x_0=\sup\bigl\{x\in[a,b]\colon (f_i+\alpha)(t)<y \text{ for each } t\in[a,x]\cap C\bigr\}$. By the definition, either $(f_i+\alpha)(x_0)<y$ and there is a sequence (t_l) of elements of C such that $t_l\searrow x_0$ and $(f_i+\alpha)(t_l)>y$ for each $l\in\mathbb{N}$, or $(f_i+\alpha)(x_0)>y$ and there is a sequence (t_l) of elements of C such that $t_l\nearrow x_0$ and $(f_i+\alpha)(t_l)<y$ for each $l\in\mathbb{N}$. We will consider the first case only. (The other case is analogous.)

One can easily see that $x_0\notin C(f_i)$. So $x_0\in B_n\setminus B_{n-1}$ for some $n\in\mathbb{N}$.

If $\eta_{n-1}=\infty$, then take an $m>|y|$ with $I_{n,m}\subset[a,b]$. Then by (4), there is an $x\in F_{n,m}\subset[a,b]$ with $(f_i+\alpha_n)(x)=y$. However, by (2), $\alpha_l(x)=0$ for each $l>n$, so $(f_i+\alpha)(x)=y$, contradicting our assumption.

If $\eta_{n-1} < \infty$, then there is a $\delta \in (0, b-x_0)$ with $\omega\big(f_i+\alpha_{n-1}, [x_0, x_0+\delta]\big) < \eta_{n-1}$. Let $t \in (x_0, x_0+\delta/4) \cap C$ be such that $(f_i+\alpha)(t) > y$. We will show that

$$(f_i+\alpha_{k-1})(t) > y \quad \text{for each} \quad k \geqslant n. \tag{5}$$

Indeed, suppose that this condition fails. Since by (1), $(f_i+\alpha_k)(t) > y$ for each sufficiently large k, there is a $k \geqslant n$ with

$$(f_i+\alpha_{k-1})(t) \leqslant y < (f_i+\alpha_k)(t). \tag{6}$$

Let $m \in \mathbb{N}$ be such that $t \in I_{k,m}$. Then $y > \inf(f_i+\alpha_{k-1}, I_{k,m}) - L_{k,m}$, so by (5), there is a $z \in F_{k,m}$ with

$$(f_i+\alpha_k)(z) = \sup(f_i+\alpha_{k-1}, I_{k,m}) + L_{k,m} < y. \tag{7}$$

Let $z \in I_{k+1,p}$. Since (1) and (6) yield

$$\begin{aligned}\sup(f_i+\alpha_k, I_{k+1,p}) + \eta_k &\geqslant (f_i+\alpha_k)(z) + \eta_k = \sup(f_i+\alpha_{k-1}, I_{k,m}) + L_{k,m} + \eta_k \\ &\geqslant (f_i+\alpha_{k-1})(t) + L_{k,m} + \eta_k \geqslant (f_i+\alpha_k)(t) > c,\end{aligned}$$

so (7) and (4) imply that there is an $x \in F_{k+1,p} \subset [x_0, b]$ with $(f_i+\alpha_{k+1})(x) = y$. It follows that $(f_i+\alpha)(x) = y$, contradicting our assumption.

The condition (5) implies, in particular, that $(f_i+\alpha_{n-1})(t) > y$. Let $m \in \mathbb{N}$ be such that $t \in I_{n,m}$. Using $t \in (x_0, x_0+\delta/4) \cap C$, we obtain

$$\begin{aligned}\sup(f_i+\alpha_{n-1}, I_{n,m}) + \eta_{n-1} &> (f_i+\alpha_{n-1})(t) > y > (f_i+\alpha_{n-1})(x_0) \\ &> (f_i+\alpha_{n-1})(t) - \eta_{n-1} \\ &\geqslant \inf(f_i+\alpha_{n-1}, I_{n,m}) - \eta_{n-1}.\end{aligned}$$

Hence by (4), there is an $x \in F_{n,m}$ with $(f_i+\alpha_n)(x) = y$. So $x \in [a,b]$ and $(f_i+\alpha)(x) = y$, contradicting our assumption.

We have shown that $f_i+\alpha$ is Darboux. Now we will prove that for each interval I we have

$$(f_i+\alpha)(I) = (f_i+\alpha)(I \cap C). \tag{8}$$

The inclusion "$\supset$" is evident. To prove the converse inclusion fix an interval I and take an $x_0 \in I \setminus C$. Arguing as above, we can find $n, m \in \mathbb{N}$ and $x \in F_{n,m} \subset I$ such that $(f_i+\alpha_n)(x) = (f_i+\alpha_n)(x_0)$. So (8) holds. This condition yields that $f_i+\alpha$ is quasi-continuous, which completes the proof. □

The next corollary is a generalization of Theorem 4.2 of [2].

Corollary 5. *Given a cliquish function f, we can find a quasi-continuous Lebesgue function α such that $f - \alpha$ is Darboux and quasi-continuous and $C(\alpha) \supset C(f)$. Moreover, we can require α to be bounded provided f is bounded.*

P r o o f . Use Theorem 4 for the family $\{-f, 0\}$. □

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