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**ON  $\phi$ -RECURRENT GENERALIZED  $(k, \mu)$ -CONTACT  
METRIC MANIFOLDS**

(submitted by M. A. Malakhaltsev)

**ABSTRACT.** The aim of the present paper is to introduce a type of contact metric manifolds called  *$\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifolds* and to study their geometric properties. The existence of such manifolds is ensured by a non-trivial example.

## 1. Introduction

In 1995 Blair, Koufogiorgos, and Papantoniou [5] introduced the notion of  $(k, \mu)$ -contact metric manifolds, where  $k$  and  $\mu$  are real constants, and a full classification of such manifolds was given by E. Boeckx [6]. Assuming  $k, \mu$  be smooth functions, T. Koufogiorgos and C. Tsihlias introduced the notion of generalized  $(k, \mu)$ -contact metric manifolds and gave several examples [8].

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [9] introduced the notion of local  $\phi$ -symmetry on a Sasakian manifold. Recently, Shaikh etl. [2] studied the locally  $\phi$ -symmetric  $(k, \mu)$ -contact metric manifolds and proved that such a manifold exists whereas a locally  $\phi$ -symmetric *generalized*

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$(k, \mu)$ -contact metric manifold does not exist. Extending the notion of local  $\phi$ -symmetry, in the present paper we introduce the notion of *locally  $\phi$ -recurrent  $(k, \mu)$ -contact metric manifolds* and *locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifolds*. In [8], the authors proved that the generalized  $(k, \mu)$ -contact metric manifolds exist only for dimension 3 and hence we confined ourselves to the study of 3-dimensional *generalized  $(k, \mu)$ -contact metric manifolds*. The  $(k, \mu)$ -contact metric manifold is of our special interest as it contains both the Sasakian and non-Sasakian cases. The paper is organized as follows. Section 2 is concerned with contact metric manifolds. Section 3 deals with  $(k, \mu)$ -contact metric manifolds and section 4 is the discussion of *generalized  $(k, \mu)$ -contact metric manifolds*. In section 5 we study *locally  $\phi$ -recurrent  $(k, \mu)$ -contact metric manifolds*. Section 6 is devoted to the study of *locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifolds* and proved that such a manifold is either flat or an  $\eta$ -Einstein manifold. Finally, we construct an example of a locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifold which is neither locally symmetric nor locally  $\phi$ -symmetric.

## 2. Contact metric manifolds

A contact manifold is a  $C^\infty$  manifold  $M^{2n+1}$  equipped with a global 1-form  $\eta$  such that  $\eta \wedge (d\eta)^n \neq 0$  everywhere on  $M^{2n+1}$ . Given a contact form  $\eta$  it is well known that there exists a unique vector field  $\xi$ , called the characteristic vector field of  $\eta$ , such that  $\eta(\xi)=1$  and  $d\eta(X, \xi) = 0$  for every vector field  $X$  on  $M^{2n+1}$ . A Riemannian metric is said to be associated metric if there exists a tensor field  $\phi$  of type (1,1) such that

$$d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad (2.1)$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2.3)$$

for all vector fields  $X, Y$  on  $M^{2n+1}$ . Then the structure  $(\phi, \xi, \eta, g)$  on  $M^{2n+1}$  is called a contact metric structure and the manifold  $M^{2n+1}$  equipped with such structure is called a contact metric manifold [3].

Given a contact metric manifold  $M^{2n+1}(\phi, \xi, \eta, g)$  we define a (1, 1) tensor field  $h$  by  $h = \frac{1}{2}\mathcal{L}_\xi\phi$ , where  $\mathcal{L}$  denotes the Lie differentiation. Then  $h$  is symmetric and satisfies  $h\phi = -\phi h$ . Thus, if  $\lambda$  is an eigenvalue of  $h$  with eigenvector  $X$ ,  $-\lambda$  is also an eigenvalue with eigenvector  $\phi X$ . Also we have  $Tr.h = Tr.\phi h = 0$  and  $h\xi = 0$ . Moreover, if  $\nabla$  denotes the Riemannian connection of  $g$ , then the following relation holds:

$$\nabla_X \xi = -\phi X - \phi hX. \quad (2.4)$$

The vector field  $\xi$  is a Killing vector with respect to  $g$  if and only if  $h = 0$ . A contact metric manifold  $M^{2n+1}(\phi, \xi, \eta, g)$  for which  $\xi$  is a Killing vector is said to be a  $K$ -contact manifold. A contact structure on  $M^{2n+1}$  gives rise to an almost complex structure on the product  $M^{2n+1} \times R$ . If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. Equivalently, a contact metric manifold is Sasakian if and only if the relation

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

holds for all  $X, Y$ , where  $R$  denotes the curvature tensor of the manifold. We shall now state a result which will be used later on.

**Lemma 2.1.** ([4]) *Let  $M^{2n+1}(\phi, \xi, \eta, g)$  be a contact metric manifold with  $R(X, Y)\xi = 0$  for all vector fields  $X, Y$  tangent to  $M$ . Then  $M$  is locally isometric to the Riemannian product  $E^{n+1}(0) \times S^n(4)$ .*

### 3. $(k, \mu)$ -Contact metric manifolds

For a contact metric manifold  $M^{2n+1}(\phi, \xi, \eta, g)$ , the  $(k, \mu)$ -nullity distribution is

$$p \rightarrow N_p(k, \mu) = [Z \in T_p M \quad : \quad R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\} \\ + \mu\{g(Y, Z)hX - g(X, Z)hY\}]$$

for any  $X, Y \in T_p M$ ,  $k, \mu$  are real numbers. Hence, if the characteristic vector field  $\xi$  belongs to the  $(k, \mu)$ -nullity distribution, then we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \quad (3.1)$$

Thus a contact metric manifold satisfying relation (3.1) is called a  $(k, \mu)$ -contact metric manifold [5]. In particular, if  $\mu = 0$ , then the notion of  $(k, \mu)$ -nullity distribution reduces to the notion of  $k$ -nullity distribution, introduced by S. Tanno [7]. A  $(k, \mu)$ -contact metric manifold is Sasakian if and only if  $k = 1$ . In a  $(k, \mu)$ -contact metric manifold the following relations hold ([1], [5]):

$$h^2 = (k - 1)\phi^2, \quad k \leq 1, \quad (3.2)$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \quad (3.3)$$

$$(\nabla_X h)(Y) = \{(1 - k)g(X, \phi Y) + g(X, h\phi Y)\}\xi \\ + \eta(Y)[h(\phi X + \phi hX)] - \mu\eta(X)\phi hY, \quad (3.4)$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX], \quad (3.5)$$

$$\begin{aligned}\eta(R(X, Y)Z) &= k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ &\quad + \mu[g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)],\end{aligned}\quad (3.6)$$

$$S(X, \xi) = 2nk\eta(X), \quad (3.7)$$

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi, \quad (3.8)$$

$$\begin{aligned}S(X, Y) &= [2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y) \\ &\quad + [2(1-n) + n(2k + \mu)]\eta(X)\eta(Y), \quad n \geq 1,\end{aligned}\quad (3.9)$$

$$r = 2n(2n-2 + k - n\mu), \quad (3.10)$$

$$S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 2(2n-2 + \mu)g(hX, Y), \quad (3.11)$$

where  $S$  is the Ricci tensor of type  $(0, 2)$ ,  $Q$  is the Ricci-operator, i.e.,  $g(QX, Y) = S(X, Y)$  and  $r$  is the scalar curvature of the manifold. From (2.4), it follows that

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y). \quad (3.12)$$

#### 4. Generalized $(k, \mu)$ -contact metric manifolds

A generalized  $(k, \mu)$ -contact metric manifold  $M^3(\phi, \xi, \eta, g)$  is a  $(k, \mu)$ -contact metric manifold in which  $k$  and  $\mu$  are smooth functions on  $M$ . In [8] the authors proved that a generalized  $(k, \mu)$ -contact metric manifold does not exist for dimension greater than three. Hence the generalized  $(k, \mu)$ -contact metric manifold exists for dimension three and several examples are given in [8]. In a generalized  $(k, \mu)$ -contact metric manifold  $M^3(\phi, \xi, \eta, g)$ , the relations (3.2), (3.5)-(3.11) hold ([2], [8]) and also the following relations hold :

$$\xi k = 0, \quad (4.1)$$

$$\xi r = 0, \quad (4.2)$$

$$h \operatorname{grad} \mu = \operatorname{grad} k. \quad (4.3)$$

**Definition 4.1.** A generalized  $(k, \mu)$ -contact metric manifold is said to be  $\eta$ -Einstein if its Ricci tensor  $S$  is of the form

$$S = \alpha g + \beta \eta \otimes \eta \quad (4.4)$$

where  $\alpha$  and  $\beta$  are smooth functions on the manifold.

### 5. Locally $\phi$ -recurrent $(k, \mu)$ -contact metric manifolds

**Definition 5.1.** ([9]) A  $(k, \mu)$ -contact metric manifold is said to be locally  $\phi$ -symmetric in the sense of Takahashi if the relation

$$\phi^2((\nabla_W R)(X, Y)Z) = 0 \quad (5.1)$$

holds for all vector fields  $X, Y, Z, W$  orthogonal to  $\xi$ .

**Definition 5.2.** A  $(k, \mu)$ -contact metric manifold  $M^{2n+1}(\phi, \xi, \eta, g)$  is said to be locally  $\phi$ -recurrent if and only if there exists a non-zero 1-form  $A$  such that

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z \quad (5.2)$$

for all vector fields  $X, Y, Z, W$  tangent to  $M$ , where  $A(X) = g(X, \rho)$ .

If the 1-form  $A$  vanishes identically and the vector fields  $X, Y, Z, W$  are orthogonal to  $\xi$ , then the manifold reduces to a locally  $\phi$ -symmetric manifold in the sense of Takahashi.

**Theorem 5.1.** Let  $M^{2n+1}(\phi, \xi, \eta, g)$  be a locally  $\phi$ -recurrent  $(k, \mu)$ -contact metric manifold. Then any one of the following holds:

- (i) The manifold is locally isometric to the Riemannian product  $E^{n+1}(0) \times S^n(4)$ , including a flat contact metric structure for  $n = 1$ .
- (ii) The manifold is locally  $\phi$ -symmetric in the sense of Takahashi.
- (iii) The characteristic vector field  $\xi$  and the associated vector field  $\rho$  of the 1-form of recurrence are codirectional.

**Proof.** In a locally  $\phi$ -recurrent  $(k, \mu)$ -contact metric manifold the relation (5.2) holds. Then, by virtue of (2.2), we obtain

$$-(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi = A(W)R(X, Y)Z. \quad (5.3)$$

Taking the inner product on both sides of (5.3) by any vector field  $U$ , we get

$$\begin{aligned} -g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) \\ = A(W)g(R(X, Y)Z, U). \end{aligned} \quad (5.4)$$

Let  $\{e_i : i = 1, 2, \dots, 2n+1\}$  be an orthonormal basis of the tangent space at any point of the manifold. Then setting  $X = U = e_i$  in (5.4) and taking summation over  $i$ ,  $1 \leq i \leq 2n+1$ , we get

$$-(\nabla_W S)(Y, Z) + \eta((\nabla_W R)(\xi, Y)Z) = A(W)S(Y, Z). \quad (5.5)$$

Plugging  $Z$  by  $\xi$  in (5.5) we obtain, by virtue of the skew-symmetry property of the curvature tensor, that

$$-(\nabla_W S)(Y, \xi) = A(W)S(Y, \xi). \quad (5.6)$$

In view of (3.7) and (3.12), (5.6) reduces to

$$2nkg(Y, \phi W + \phi hW) - S(Y, \phi W + \phi hW) = 2nkA(W)\eta(Y). \quad (5.7)$$

Setting  $Y = \xi$  in (5.7) we get by virtue of (2.2) and (3.7) that

$$kA(W) = 0,$$

which yields either  $k = 0$ , or  $A(W) = 0$  for all vector fields  $W$  tangent to  $M$ .

Again, changing  $W, X, Y$  cyclically in (5.3) and then adding the results, we obtain

$$\begin{aligned} & -[(\nabla_W R)(X, Y)Z + (\nabla_X R)(Y, W)Z + (\nabla_Y R)(W, X)Z] \\ & +[\eta((\nabla_W R)(X, Y)Z) + \eta((\nabla_X R)(Y, W)Z) + \eta((\nabla_Y R)(W, X)Z)]\xi \\ & = A(W)R(X, Y)Z + A(X)R(Y, W)Z + A(Y)R(W, X)Z, \end{aligned} \quad (5.8)$$

which, by virtue of Bianchi identity, yields

$$\begin{aligned} & A(W)\eta(R(X, Y)Z) + A(X)\eta(R(Y, W)Z) \\ & + A(Y)\eta(R(W, X)Z) = 0. \end{aligned} \quad (5.9)$$

In view of (3.6), (5.9) reduces to

$$\begin{aligned} & A(W) [k\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} + \mu\{g(hY, Z)\eta(X) \\ & - g(hX, Z)\eta(Y)\}] + A(X) [k\{g(W, Z)\eta(Y) - g(Y, Z)\eta(W)\} \\ & + \mu\{g(hW, Z)\eta(Y) - g(hY, Z)\eta(W)\}] + A(Y) [k\{g(X, Z)\eta(W) \\ & - g(W, Z)\eta(X)\} + \mu\{g(hX, Z)\eta(W) - g(hW, Z)\eta(X)\}] = 0. \end{aligned} \quad (5.10)$$

Setting  $Y = Z = e_i$  in (5.10) and taking summation over  $i$ ,  $1 \leq i \leq 2n+1$ , we get

$$\begin{aligned} & (2n-1)k[A(W)\eta(X) - A(X)\eta(W)] \\ & + \mu[A(hX)\eta(W) - A(hW)\eta(X)] = 0. \end{aligned} \quad (5.11)$$

Substituting  $X$  by  $\xi$  in (5.12), we have

$$(2n-1)k[A(W) - A(\xi)\eta(W)] - \mu A(hW) = 0. \quad (5.12)$$

If  $k = 0$ , then (5.13) yields either  $\mu = 0$ , or  $A(hW) = 0$ . Thus for  $k = 0 = \mu$ , (3.1) reduces to  $R(X, Y)\xi = 0$  for all  $X, Y$  and hence, by virtue of Lemma 2.1, the manifold under consideration is locally isometric to the Riemannian product  $E^{n+1}(0) \times S^n(4)$  and, for  $n = 1$ , the manifold is a flat contact metric manifold.

Again, for  $k = 0$  and  $A(hW) = 0$ , we have  $A(W) = A(\xi)\eta(W)$ , which can be written as  $A(W)\eta(\xi) = A(\xi)\eta(W)$ . This implies that the vector

field  $\xi$  and  $\rho$  associated to the 1-form  $A$  are codirectional. Finally, if  $A(W) = 0$ , (for  $k \neq 0$ ) for all  $W$ , then (5.2) reduces to (5.1) and hence the manifold under consideration is locally  $\phi$ -symmetric in the sense of Takahashi. This proves the theorem.

## 6. Locally $\phi$ -recurrent generalized $(k, \mu)$ -contact metric manifolds

**Definition 6.1.** A generalized  $(k, \mu)$ -contact metric manifold is said to be locally  $\phi$ -recurrent if and only if relation (5.2) holds.

In particular, if  $A$  vanishes, then a generalized  $(k, \mu)$ -contact metric manifold is said to be a locally  $\phi$ -symmetric manifold.

**Theorem 6.1.** A locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifold  $M^3(\phi, \xi, \eta, g)$  is either a flat contact metric manifold or an  $\eta$ -Einstein manifold.

**Proof.** Proceeding similarly to the proof of Theorem 5.1, we can easily show that in a generalized  $(k, \mu)$ -contact metric manifold the relation (5.6) holds, and hence in view of (3.7) and (3.12), we obtain

$$\begin{aligned} (\nabla_W S)(Y, \xi) \\ = 2(Wk)\eta(Y) - 2kg(Y, \phi W + \phi hW) + S(Y, \phi W + \phi hW). \end{aligned} \quad (6.1)$$

Using (6.1) in (5.6) we get

$$\begin{aligned} -2(Wk)\eta(Y) + 2kg(Y, \phi W + \phi hW) - S(Y, \phi W + \phi hW) \\ = 2kA(W)\eta(Y). \end{aligned} \quad (6.2)$$

Replacing  $Y$  by  $\phi Y$  in (6.2) we have

$$2kg(\phi Y, \phi W + \phi hW) - S(\phi Y, \phi W + \phi hW) = 0. \quad (6.3)$$

By virtue of (2.2) and (3.11), it follows from (6.3) that

$$\begin{aligned} S(W, Y) + S(Y, hW) &= -2[k - \mu]g(Y, W) \\ &\quad -2[k - \mu]g(Y, hW) \\ &\quad +[4k + 2(k - 1)\mu]\eta(W)\eta(Y). \end{aligned} \quad (6.4)$$

Again, Replacing  $W$  by  $hW$  in (6.4) and then using (2.2) and (3.2), we obtain

$$\begin{aligned} S(W, Y) + S(Y, hW) - kS(Y, W) &= -2[k - \mu]g(Y, hW) \\ &\quad +2(k - 1)[k - \mu]g(Y, W) \\ &\quad -2(k - 1)[2k - \mu]\eta(W)\eta(Y). \end{aligned} \quad (6.5)$$

Subtracting (6.5) from (6.4), it follows that either  $k = 0$ , or

$$S(W, Y) = -2kg(W, Y) + 4k\eta(W)\eta(Y). \quad (6.6)$$

The relation (6.6) implies that the manifold under consideration is an  $\eta$ -Einstein manifold.

If  $k = 0$ , then (6.4) reduces to

$$S(W, Y) + S(Y, hW) = 2\mu[g(Y, W) + g(Y, hW) - \eta(W)\eta(Y)]. \quad (6.7)$$

Again, if  $k = 0$ , then for  $n = 1$ , (3.9) takes the form

$$S(X, Y) = -\mu g(X, Y) + \mu g(hX, Y) + \mu\eta(X)\eta(Y), \quad (6.8)$$

which yields by setting  $Y = hY$  that

$$S(X, hY) = -\mu g(X, hY) + \mu g(X, Y) - \mu\eta(X)\eta(Y). \quad (6.9)$$

By virtue of (6.8) and (6.9), we obtain from (6.7) that either  $\mu = 0$ , or  $hY = -Y + \eta(Y)\xi$ . If  $k = 0 = \mu$ , then the manifold is a flat contact metric structure. Again, if  $k = 0$  and  $hY = -Y + \eta(Y)\xi$ , then (6.8) yields

$$S(W, Y) = -2\mu g(Y, W) + 2\mu\eta(W)\eta(Y),$$

which implies that the manifold is an  $\eta$ -Einstein manifold. This proves the theorem.

**Theorem 6.2.** *In a locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifold  $M^3(\phi, \xi, \eta, g)$ , the vector field associated to the 1-form of recurrence is given by*

$$\rho = -\frac{1}{k} \text{grad } k, \quad \text{provided that } k \neq 0.$$

**Proof.** In a locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifold relation (6.2) holds. Setting  $Y = \xi$  in (6.2), we obtain  $A(W) = -\frac{1}{k}(Wk)$  for  $k \neq 0$ , which implies that  $\rho = -\frac{1}{k} \text{grad } k$ . This proves the theorem.

**Theorem 6.3.** *In a locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifold  $M^3(\phi, \xi, \eta, g)$ , the characteristic vector field  $\xi$  and the vector field  $\rho$  associated to the 1-form  $A$  are orthogonal to each other provided that  $k \neq 0$ .*

**Proof.** In a locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifold relation (6.2) holds. Setting  $W = \xi$  in (6.2) and then using (4.1), we get  $A(\xi) = 0$ , i.e.,  $g(\xi, \rho) = 0$ , provided that  $k \neq 0$ . This proves the theorem.



We now give an example of a locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifold.

**Example.** We consider the 3-dimensional manifold  $M = \{(x, y, z) \in R^3 : x \neq 0\}$ , where  $(x, y, z)$  are the standard coordinates in  $R^3$ . Let  $\{E_1, E_2, E_3\}$  be linearly independent global frame on  $M$  given by

$$E_1 = \frac{2}{x} \frac{\partial}{\partial y}, \quad E_2 = 2 \frac{\partial}{\partial x} - \frac{4z}{x} \frac{\partial}{\partial y} + xy \frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}.$$

Let  $g$  be the Riemannian metric defined by

$$\begin{aligned} g(E_1, E_3) &= g(E_2, E_3) = g(E_1, E_2) = 0, \\ g(E_1, E_1) &= g(E_2, E_2) = g(E_3, E_3) = 1. \end{aligned}$$

Let  $\eta$  be the 1-form defined by  $\eta(U) = g(U, E_3)$  for any  $U \in \chi(M)$ . Let  $\phi$  be the  $(1, 1)$ -tensor field defined by  $\phi E_1 = E_2$ ,  $\phi E_2 = -E_1$ ,  $\phi E_3 = 0$ . Then using the linearity of  $\phi$  and  $g$ , we have  $\eta(E_3) = 1$ ,  $\phi^2 U = -U + \eta(U)E_3$  and  $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$  for any  $U, W \in \chi(M)$ . Moreover  $hE_1 = -E_1$ ,  $hE_2 = E_2$  and  $hE_3 = 0$ . Thus for  $E_3 = \xi$ ,  $(\phi, \xi, \eta, g)$  defines a contact metric structure on  $M$ .

Let  $\nabla$  be the Levi-Civita connection with respect to the Riemannian metric  $g$  and  $R$  be the curvature tensor of  $g$ . Then we have

$$[E_1, E_2] = 2E_3 + \frac{2}{x}E_1, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 2E_1.$$

Taking  $E_3 = \xi$  and using Koszul formula for the Riemannian metric  $g$ , we can easily calculate

$$\begin{aligned} \nabla_{E_1} E_3 &= 0, & \nabla_{E_2} E_3 &= 2E_1, & \nabla_{E_3} E_3 &= 0, \\ \nabla_{E_3} E_1 &= 0, & \nabla_{E_1} E_2 &= \frac{2}{x}E_1, & \nabla_{E_2} E_1 &= -2E_3, \\ \nabla_{E_2} E_2 &= 0, & \nabla_{E_3} E_2 &= 0, & \nabla_{E_1} E_1 &= -\frac{2}{x}E_2. \end{aligned}$$

From the above it can be easily seen that  $(\phi, \xi, \eta, g)$  is a generalized  $(k, \mu)$ -contact metric structure on  $M$ . Consequently  $M^3(\phi, \xi, \eta, g)$  is a generalized  $(k, \mu)$ -contact metric manifold with  $k = -\frac{2}{x}$  and  $\mu = -\frac{2}{x}$ .

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows :

$$R(E_2, E_3)E_2 = -\frac{4}{x}E_1, \quad R(E_2, E_3)E_1 = \frac{4}{x}E_2, \quad (6.10)$$

and the components which can be obtained from these by the symmetry properties. We shall now show that such a generalized  $(k, \mu)$ -contact

metric manifold is locally  $\phi$ -recurrent. Since  $\{E_1, E_2, E_3\}$  form a basis of  $M^3$ , any vector field  $X \in \chi(M)$  can be written as

$$X = a_1 E_1 + a_2 E_2 + a_3 E_3$$

where  $a_i \in R^+$  (the set of all positive real nonumbers),  $i = 1, 2, 3$ . Thus the covariant derivatives of the curvature tensor are given by

$$(\nabla_X R)(E_2, E_3)E_1 = -\frac{8a_2}{x^2}E_2, \quad (\nabla_X R)(E_2, E_3)E_2 = \frac{8a_2}{x^2}E_1.$$

This implies that

$$\begin{aligned} \phi^2((\nabla_X R)(E_2, E_3)E_1) &= \frac{8a_2}{x^2}E_2, \\ \phi^2((\nabla_X R)(E_2, E_3)E_2) &= -\frac{8a_2}{x^2}E_1. \end{aligned} \quad (6.11)$$

Let us now consider the non-vanishing 1-form

$$A(X) = \frac{2a_2}{a_3 x}, \quad (6.12)$$

at any point  $p \in M$ . From (6.10)–(6.12), it follows that

$$\phi^2((\nabla_X R)(E_2, E_3)E_1) = A(X)R(E_2, E_3)E_1$$

$$\text{and } \phi^2((\nabla_X R)(E_2, E_3)E_2) = A(X)R(E_2, E_3)E_2$$

This implies that the manifold under consideration is a locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifold, which is neither locally symmetric nor locally  $\phi$ -symmetric. This leads to the following:

**Theorem 6.4.** *There exists a 3-dimensional locally  $\phi$ -recurrent generalized  $(k, \mu)$ -contact metric manifold which is neither locally symmetric nor locally  $\phi$ -symmetric in the sense of Takahashi.*

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