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ASSOCIATIVITY CONSTRAINTS, BRAIDINGS AND QUANTIZATIONS OF MODULES WITH GRADING AND ACTION

(submitted by V.V. Lychagin)

Abstract. We study quantizations, associativity constraints and braidings in the monoidal category of monoid graded modules over a commutative ring. All of them can be described in terms of the cohomology of the underlying (finite) monoid. The Fourier transform of finite groups gives a corresponding description in the monoidal category of modules with action by a group.

1. Introduction

We investigate the monoidal category of modules over a commutative ring R with grading by a monoid M and associativity constraints, quantizations, braidings of this category. We assume that the monoid is finite and commutative.

For this category the associativity constraints are 3-cocycles of M with coefficients in the group of invertible elements in R, $U(R)$. The 3-cocycle condition is equivalent to the Mac Lane coherence condition. Under action by natural isomorphisms of the tensor bifunctor the classes of associativity constraints are the 3^{rd} cohomology group of M with coefficients in the group of invertible elements in R, $U(R)$.

Similarly, for the category of M-graded modules quantizations up to natural isomorphisms of the identity functor are representations of the 2^{nd} cohomology group of M with coefficients in $U(R)$.

Braidings for this category are represented by symmetric 2-cochains of the monoid with coefficients in $U(R)$ satisfying conditions that are the bihomomorphism conditions when the associativity is trivial.

We consider the monoidal category of modules with an action by a finite abelian group G.

With the description of associativity constraints, quantizations and braidings for graded modules we obtain the corresponding associativity constraints, quantizations and braidings for modules with action by a finite abelian group by applying the Fourier transform of finite groups.

More precisely, for $R = \mathbb{C}$ the Fourier transform of finite groups

$$F : \mathbb{C}[G] \to \mathbb{C}\left(\hat{G}\right),$$

induces a functor isomorphism between the monoidal categories of $\hat{G}$-graded modules and G-modules. The isomorphism is that constructed by change of rings with the assumption that the ring homomorphism is an isomorphism.

2. Associativity constraints, braidings and quantizations of monoidal categories

2.1. **Associativity constraints.** Let C be a monoidal category with unit e.

An *associativity constraint* α [9] in a monoidal category C is a natural isomorphism

$$\begin{aligned} \alpha &: \otimes \circ (\mathbf{1} \times \otimes) \to \otimes \circ (\otimes \times \mathbf{1}), \\ \alpha &= \alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z, \end{aligned}$$

$X, Y, Z \in Ob(C)$, which satisfies the Mac Lane-coherence condition, namely the following diagram commutes,

$$\begin{tikzcd} X \otimes (Y \otimes (Z \otimes W)) \arrow[r, "\alpha"] \arrow[d, "1 \otimes \alpha"'] & (X \otimes Y) \otimes (Z \otimes W) \arrow[r, "\alpha"] & ((X \otimes Y) \otimes Z) \otimes W \arrow[d, "\alpha \otimes 1"] \\ X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "\alpha"] & & (X \otimes (Y \otimes Z)) \otimes W \end{tikzcd},$$

$X, Y, Z, W \in Ob(C)$. If the associativity constraint is trivial we say that the monoidal category is strict.

2.2. **Quantizations.** A quantization [6] of a monoidal category C is a natural isomorphism of the tensor bifunctor

$$\begin{aligned} q &: \otimes \to \otimes, \\ q_{X,Y} &: X \otimes Y \to X \otimes Y, \end{aligned}$$

$X, Y \in Ob(C)$, which preserves the unit and associativity so that the following diagram

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "1 \otimes q_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[d, "q_{X,Y} \otimes 1"'] \\ X \otimes (Y \otimes Z) \arrow[d, "q_{X,Y\otimes Z}"'] & (X \otimes Y) \otimes Z \arrow[d, "q_{X\otimes Y,Z}"'] \\ X \otimes (Y \otimes Z) \arrow[r, "\alpha"] & (X \otimes Y) \otimes Z \end{tikzcd} \tag{1}$$

commutes for all $X, Y, Z \in Ob(C)$. We call this the coherence condition for quantizations.

2.3. **Braidings.** A *braiding* [9] of a monoidal category C is a natural isomorphism

$$\begin{aligned} \sigma &: \otimes \to \otimes \circ \tau \\ \sigma &= \sigma_{X,Y} : X \otimes Y \to Y \otimes X, \end{aligned}$$

$X, Y \in Ob(C)$, which preserves the unit and associativity such that the following diagrams

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "1 \otimes \sigma"'] & (X \otimes Y) \otimes Z \arrow[d, "\sigma"] \\ X \otimes (Z \otimes Y) \arrow[d, "\alpha"'] & Z \otimes (X \otimes Y), \arrow[d, "\alpha"] \\ (X \otimes Z) \otimes Y \arrow[r, "\sigma \otimes 1"] & (Z \otimes X) \otimes Y \end{tikzcd} \quad \begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & (X \otimes Y) \otimes Z \arrow[d, "\sigma \otimes 1"] \\ (Y \otimes Z) \otimes X \arrow[d, "\alpha^{(-1)}"'] & (Y \otimes X) \otimes Z \arrow[d, "\alpha^{(-1)}"] \\ Y \otimes (Z \otimes X) \arrow[r, "1 \otimes \sigma"] & Y \otimes (X \otimes Z) \end{tikzcd} \tag{2}$$

commute. This is the coherence condition on braidings.

The braiding σ is a symmetry if

$$\sigma_{Y,X} \circ \sigma_{Y,Z} = Id, \tag{3}$$

and a monoidal category equipped with such is called symmetric. We shall mainly work with symmetries.

Remark 1. *When the monoidal category is symmetric, the associativity constraint can be considered as trivial, and if α is trivial then the category is symmetric.*

When the associativity constraint is trivial, the coherence condition gives what we call the bi(homo)morphism conditions for any braiding σ

$$(\sigma_{X,Z} \otimes 1) \circ (1 \otimes \sigma_{Y,Z}) \quad = \quad \sigma_{X \otimes Y, Z}, \tag{i}$$

$$(1 \otimes \sigma_{X,Z}) \circ (\sigma_{X,Y} \otimes 1) \quad = \quad \sigma_{X, Y \otimes Z}, \tag{ii}$$

$X, Y, Z \in Obj\,(C)$.

The trivial braiding is the twist, $\tau : X \otimes Y \to Y \otimes X$.

Any braiding composed with the twist, $\tau \circ \sigma$, is a quantizations since the coherence condition for quantizations then is satisfied.

Quantizations act on the set of braidings as follows

$$\sigma^q_{X,Y} = q^{-1}_{Y,X} \circ \sigma_{X,Y} \circ q_{X,Y}$$

and σ^q is also a braiding.

2.4. **Quantizations of functors.** Let $\Phi : C \to C'$ be a unit preserving functor of two monoidal categories. A quantization of Φ is

$$Q \quad : \quad \otimes \circ (\Phi \times \Phi) \to \Phi \circ \otimes,$$

$$Q_{X,Y} \quad : \quad \Phi\,(X) \otimes \Phi\,(Y) \to \Phi\,(X \otimes Y)\,,$$

$X, Y \in Ob\,(C)$, which preserves units and satisfies the coherence condition, i.e. the diagram

$$\begin{tikzcd}
\Phi(X) \otimes \Phi(Y) \otimes \Phi(Z) \arrow[r, "1_\Phi(X) \otimes Q_{Y,Z}"] \arrow[d, "Q_{X,Y} \otimes 1_\Phi(Z)"'] & \Phi(X) \otimes \Phi(Y \otimes Z) \arrow[d, "Q_{X,Y \otimes Z}"'] \\
\Phi(X \otimes Y) \otimes \Phi(Z) \arrow[r, "Q_{X \otimes Y,Z}"] & \Phi(X \otimes Y \otimes Z)
\end{tikzcd} \tag{4}$$

commutes.

A quantization of the identity functor $1 = id : C \to C$ is a quantization of the category.

If $\Phi : C \to C'$ and $\Psi : C' \to C''$ are functors of monoidal categories and Q^{Φ} and Q^{Ψ} are the corresponding quantizations, then

$$Q^{\Psi\circ\Phi}_{X,Y} \stackrel{def}{=} \Psi\left(Q^{\Phi}_{X,Y}\right) \circ Q^{\Psi}_{\Phi(X),\Phi(Y)}$$

is a quantization of $\Psi \circ \Phi$. See [6].

3. Associativity constraints, quantizations and braidings of graded modules

3.1. **The monoidal category of graded modules.** Let M be a finite commutative monoid. Let R be a commutative ring with unit.

Denote by $\mathrm{mod}_R(M)$ the strict monoidal category [9] of M-graded R-modules,

$$X = \oplus_{m\in M} X_m.$$

Denote the grading of a homogeneous element $x \in X$ either by $|x| \in M$, or write x_m, $m \in M$.

The arrows of $\mathrm{mod}_R(M)$ are the M-graded morphisms

$$f = \{f_m : X_m \to X_m\}_{m\in M},$$

i.e. morphisms that preserves gradings.

The tensor product $X \otimes_R X'$ of two objects in $\mathrm{mod}_R(M)$ is defined

$$(X \otimes_R X')_m = \oplus_{i+j=m}(X_i \otimes_R X'_j).$$

The ring R is a unit object as we define R to be indexed by $0 \in M$ and components indexed by $m \in M$, $m \neq 0$, are all zeros.

The direct sum of two objects X, X' in $\mathrm{mod}_R(M)$ is

$$X \oplus X' = \oplus_{m\in M}(X_m \oplus X'_m).$$

An object Y is a *M-graded submodule* of X if and only in Y_m is a submodule of X_m for all $m \in M$.

An object is a *M-graded factor module, X/Y*, if it has components

$$(X/Y)_m = X_m/Y_m$$

for all $m \in M$.

An algebra A in $\mathrm{mod}_R(M)$ is called a *M-graded R-algebra* and is equipped with multiplication

$$\mu : A \otimes A \to A$$

which maps $A_i \otimes A_j$ to A_{i+j}, $i, j \in M$.

An A-module E in $\mathrm{mod}_R(M)$ is called a *M-graded A-module* and is equipped with an action

$$\nu : A \otimes E \to E$$

which maps $A_i \otimes E_j$ to E_{i+j}, $i, j \in M$.

Remark 2. *A module E graded by a finite abelian group G can be considered as a bundle where E_g is the fiber at the point $g \in G$. The categorical constructions like associativity constraints, braidings and quantizations are sections of this bundle. (However, as we shall see, these are really just multiplications by constants depending on G).*

3.2. **Associativity constraints for M-graded modules.** For the monoidal category of M-graded R-modules we get an explicit description the associativity constraints in terms of the grading.

Let X, Y and Z be M-graded R-modules. Choose homogeneous elements $x \in X$, $y \in Y$ and $z \in Z$ with grading $|x|$, $|y|$ and $|z|$ respectively.

Let $R_{|x|}$ be the M-graded R-module such that

$$\begin{aligned} \left(R_{|x|}\right)_m &= R, \text{ if } m = |x|, \\ \left(R_{|x|}\right)_m &= 0, \text{ if } m \neq |x|. \end{aligned}$$

Construct a morphism

$$\phi_{|x|} : R_{|x|} \to X$$

such that

$$\phi_{|x|} : 1_{|x|} = (0, \ldots, 1, \ldots, 0) \mapsto (0, \ldots, x, \ldots, 0),$$

where 1 appears in the $|x|^{th}$ place. Similarly construct M-graded R-modules and morphisms

$$\begin{aligned} \phi_{|y|} &: R_{|y|} \to Y, \\ \phi_{|z|} &: R_{|z|} \to Z. \end{aligned}$$

By naturality of α the diagram

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\alpha_{X,Y,Z}"] & (X \otimes Y) \otimes Z \\ R_{|x|} \otimes (R_{|y|} \otimes R_{|z|}) \arrow[u, "\phi_{|x|} \otimes (\phi_{|y|} \otimes \phi_{|z|})"] \arrow[r, "\alpha_{R_{|x|},R_{|y|},R_{|z|}}"] & (R_{|x|} \otimes R_{|y|}) \otimes R_{|z|} \arrow[u, "(\phi_{|x|} \otimes \phi_{|y|}) \otimes \phi_{|z|}"'] \end{tikzcd} \tag{5}$$

commutes.

Since associativity constrains preserves units

$$\alpha_{R_{|x|},R_{|y|},R_{|z|}} : 1_{|x|} \otimes \left(1_{|y|} \otimes 1_{|z|}\right) \longmapsto \alpha\left(|x|, |y|, |z|\right) \left(1_{|x|} \otimes 1_{|y|}\right) \otimes 1_{|z|}$$

where $\alpha\left(|x|, |y|, |z|\right) \in U(R)$, $U(R)$ is the group of unit elements in R.

By commutativity of the diagram (5)

$$\alpha_{X,Y,Z} : x \otimes (y \otimes z) \longmapsto \alpha\left(|x|, |y|, |z|\right)(x \otimes y) \otimes z.$$

This is independent of the choice of x, y and z. We see that α only depends on the grading and can be represented as 3-cochain

$$\alpha : M \times M \times M \to U\left(R\right).$$

Clearly every associativity constraint α satisfies the normalization condition and is in $C^3\left(M, U\left(R\right)\right)$, the set of normalized 3-cochains of M with coefficients in $U\left(R\right)$, with trivial action of M.

It follows from the coherence condition that

$$\delta\left(\alpha\right) = \alpha\left(i,j,k\right)\alpha^{-1}\left(j,k,l\right)\alpha\left(i+j,k,l\right)\alpha^{-1}\left(i,j+k,l\right)\alpha\left(i,j,k+l\right) = 1,$$

$i, j, k, l \in M$, where δ is the coboundary operator, and each associativity constraint α is a 3-cocycle.

Proposition 1. *Any associativity constraint α in the monoidal category of M-graded R-modules is a normalized 3-cocycle of M with values in $\mathbf{U}\left(R\right)$ and of the form*

$$\alpha : x \otimes (y \otimes z) \longmapsto \alpha\left(|x|, |y|, |z|\right)(x \otimes y) \otimes z, \tag{6}$$

for homogeneous elements x, y, z.

Some of this result can be generalized.

Proposition 2. *Any unit preserving natural isomorphism γ from one n-functor F to another n-functor G in $\operatorname{mod}_R\left(M\right)$ can be represented as a normalized n-cochain of M, with trivial action of M, with coefficients in $U\left(R\right)$,*

$$\gamma : \underbrace{M \times \cdots \times M}_{n-times} \to U\left(R\right),$$

and is of the form

$$\gamma_{X_1,\ldots,X_n} : F\left(x_1, \ldots, x_n\right) \longmapsto \gamma\left(|x_1|, \ldots, |x_n|\right) G\left(x_1, \ldots, x_n\right)$$

for homogeneous $x_i \in X_i$, $X_i \in Obj\left(\operatorname{mod}_R\left(M\right)\right)$, $i = 1, \ldots, n$.

Let p be a natural isomorphism of the tensor bifunctor in a monoidal category C,

$$p : \otimes \to \otimes.$$

Define an action of p on any associativity constraint α by the commutative diagram

$$\begin{tikzcd}
X\otimes(Y\otimes Z) \arrow[r, "\alpha"] \arrow[d, "1\otimes p_{Y,Z}"'] & (X\otimes Y)\otimes Z \arrow[d, "p_{X,Y}\otimes 1"'] \\
X\otimes(Y\otimes Z) \arrow[d, "p_{X,Y\otimes Z}"'] & (X\otimes Y)\otimes Z, \arrow[d, "p_{X\otimes Y,Z}"'] \\
X\otimes(Y\otimes Z) \arrow[r, "p(\alpha)"] & (X\otimes Y)\otimes Z
\end{tikzcd} \tag{7}$$

where $X, Y, Y \in Ob(C)$.

If X, Y and Z are M-graded R-modules then

$$\begin{aligned} p(\alpha)(i,j,k) &= \alpha(i,j,k)\,p(j,k)\,p^{-1}(i+j,k)\,p(i,j+k)\,p^{-1}(i,j) \\ &= \alpha(i,j,k)\,\delta(p)(i,j,k), \end{aligned}$$

$i, j, k \in M$. Hence

$$p(\alpha) = \alpha \cdot \delta(p).$$

To sum up the results in this section, the following theorem give a complete description of associativity constraints in the category of M-graded R-modules up to the action above.

Theorem 3. *The orbits of all associativity constraints in the monoidal category* $\mathrm{mod}_R(M)$ *under the action of natural isomorphisms of the tensor bifunctor is in one to one correspondence with the* 3^{rd} *cohomology group* $H^3(M, \mathbf{U}(R))$.

3.3. Quantizations of the category of M-graded modules.

In $\mathrm{mod}_R(M)$ any quantization q is a normalized 2-cochain of M with coefficients in $U(R)$

$$q : M \times M \to U(R),$$

and for homogeneous elements x and y,

$$q : x \otimes y \longmapsto q(|x|, |y|)\, x \otimes y.$$

By the coherence condition of quantizations (1)

$$\delta(q) = q(i,j)\,q^{-1}(i,j+k)\,q(i+j,k)\,q^{-1}(j,k) = 1,$$

$i, j, k \in M$, hence any quantization q in $\mathrm{mod}_R(M)$ of is a 2-cocycle.

Remark 3. *By commutativity of the diagram (1), any quantization acts on the set of associativity constraints, defined as above by the diagram (7). This action is trivial as any quantization is a cocycle,* $\delta(q) = 1$.

Let λ be a unit preserving natural isomorphisms of the identity functor in a monoidal category C,

$$\lambda : Id \to Id.$$

Define an action of λ on the set of quantizations of a monoidal category by the commutativity of the diagram

$$\begin{tikzcd} X \otimes Y \arrow[r, "q_{X,Y}"] \arrow[d, "\lambda_X \otimes \lambda_Y"'] & X \otimes Y \arrow[d, "\lambda_{X \otimes Y}"] \\ X \otimes Y \arrow[r, "\lambda(q_{X,Y})"] & X \otimes Y \end{tikzcd} \tag{8}$$

where X, Y are objects of the category. It is easily checked that this is an action and that $\lambda(q)$ is a quantization.

If X and Y are M-graded R-modules we have the following representation of the action

$$\begin{aligned} \lambda(q)(i,j) &= q(i,j)\,\lambda(j)\,\lambda^{-1}(i+j)\,\lambda(i) \\ &= q(i,j)\,\delta(\lambda)(i,j), \end{aligned}$$

$i, j \in M$. We consider orbits of the quantization with respect to the action of natural isomorphisms of the identity functor.

The following theorem give a complete description of quantizations in the category of M-graded R-modules.

Theorem 4. *The orbits of all quantizations the monoidal category of* $\mathrm{mod}_R(M)$ *under the action of unit preserving natural isomorphisms of the identity functor is in one to one correspondence with the* 2^{nd} *cohomology group,* $H^2(M, \mathbf{U}(R))$.

3.4. **Braidings in the category of M-graded modules.** Any braiding σ can be represented by a normalized 2-cochain of M with coefficients in $U(R)$,

$$\sigma : M \times M \to U(R).$$

In addition, the coherence condition on braidings (2) gives certain properties of the 2-cochains summarized in the following theorem.

Theorem 5. *A braiding σ in the monoidal category* $\operatorname{mod}_R(M)$ *has the form*

$$\sigma : x \otimes y \longmapsto \sigma(|x|, |y|)\, y \otimes x$$

where $\sigma : M \times M \to \mathbf{U}(R)$ is a normalized 2-cochain satisfying

$$\begin{aligned} \sigma(i+j,k) &= \frac{\alpha(i,k,j)}{\alpha(i,j,k)\,\alpha(k,i,j)}\sigma(i,k)\,\sigma(j,k), \\ \sigma(i,j+k) &= \frac{\alpha(i,j,k)\,\alpha(j,k,i)}{\alpha(j,i,k)}\frac{\sigma(i,j)}{\sigma(k,i)}, \end{aligned} \tag{9}$$

and

$$\sigma(i,j)\,\sigma(j,i) = 1,$$

i.e., any braiding in $\operatorname{mod}_R(M)$ *is a symmetry. Furthermore, if the associativity constraint α is trivial we get bihomomorphism condition for σ,*

$$\begin{aligned} \sigma(i+j,k) &= \sigma(i,k)\,\sigma(j,k), \\ \sigma(i,j+k) &= \sigma(i,j)\,\sigma(i,k), \end{aligned} \tag{10}$$

$i,j,k \in M$.

4. Projections, the function algebra and the group algebra

4.1. **M-graded modules by projections.** First, more generally, let M be a finite commutative monoid.

There is a way to introduce M-graded modules equivalently to that of section 3.1.

Namely, let X be a R-module. Let

$$\{\pi_m : X \to X\}_{m \in M}$$

be a family of projectors with the properties

$$\begin{aligned} \pi_m \pi_n &= \pi_n \pi_m = \pi_{m+n}, \\ \sum_{m \in M} \pi_m &= Id. \end{aligned}$$

Then π_m projects X on X_m,

$$X_m = \operatorname{Im}(\pi_m).$$

Any such family of projectors determines a M-grading on X ensured by the properties of projections.

On the other hand, if X is a M-graded module then a family of projectors can be defined by

$$\pi_m(X) = X_m.$$

4.2. **M-graded algebras and algebra modules by projections.** Let A be a R-algebra with multiplication $\mu : A \otimes A \to A$. Then a family $\{\pi_m\}_{m \in M}$ of projectors introduces a grading on A, $A_m = \mathrm{Im}(\pi_m)$, and the product projects as follows

$$\pi_m(ab) = \sum_{i+j=m} \pi_i(a)\,\pi_j(b), \tag{11}$$

$a, b \in A$, i.e. the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu"] \arrow[d, "\Sigma_{i+j=m}\pi_i \otimes \pi_j"'] & A \arrow[d, "\pi_k"'] \\ (A \otimes A)_m \arrow[r, "\mu"] & A_m \end{tikzcd} \tag{12}$$

commutes.

Conversely, let A be a M-graded algebra with multiplication μ. Then a family of projectors is defined by

$$\pi_m(A) = A_m,$$

and (11) is satisfied.

Let E be an A-module with action $\nu : A \otimes E \to E$. Then, similarly, there is a grading introduced by projectors and

$$\pi_m(ax) = \sum_{i+j=m} \pi_i(a)\,\pi_j(x), \tag{13}$$

$a \in A$, $x \in E$.

Conversely, let E be a M-graded A-module. Then a family of projectors is defined by

$$\pi_m(E) = E_m,$$

and (13) is satisfied.

4.3. **M-graded operators.** Let X and Y be M-graded modules. By an operator $T_m : X \to Y$ of grade $m \in M$ we mean a R-linear operator such that

$$T_m(X_n) \subset Y_{n+m},$$

$n \in M$. The sum of such operators, $T = \oplus_{m \in M} T_m$, is called a M-graded operator. For M-graded operators the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi_n"'] & Y \arrow[d, "\pi_{m+n}"] \\ X_n \arrow[r, "T_m"] & Y_{m+n} \end{tikzcd}, \tag{14}$$

i.e.,

$$T \circ \pi_n = \pi_{m+n} \circ T. \tag{15}$$

Proposition 6. *Let M be a group and X and Y be M-graded modules. Then any operator $T : X \to Y$ is a M-graded operator.*

Proof. Let

$$T^m = T|_{X_m} : X_m \to Y = \sum_{n \in M} Y_n,$$

hence

$$T^m = \sum_{n \in M} T_n^m : X_m \to Y_n,$$

and write the operator as a matrix $T = \left[T_i^j\right]$. Let $\left[T_i^j\right]_m$ be the matrix where all entries except those of the form T_{n+m}^n, $n \in M$, are zero. Then $\left[T_i^j\right]_m$ is an operator of grade $m \in M$ and

$$T = \sum_{m \in M} \left[T_i^j\right]_m .$$

∎

4.4. **The bialgebra $R(M)$.** Consider the algebra $R(M)$ of R-valued functions on M with multiplication

$$\mu : R(M) \otimes R(M) \to R(M),$$

$$\mu (f \otimes g)(m) = f(m) g(m),$$

$m \in M$, and the multiplication of M, $m : M \otimes M \to M$, introduces a diagonal m^*,

$$m^* : R(M) \to R(M) \otimes R(M),$$

$$m^*(f)(m, n) = f(mn),$$

$m, n \in M$.

The unit is

$$\iota : R \to R(M), \quad e \longmapsto 1,$$

and counit,

$$\epsilon : R(M) \to R, \quad f \longmapsto f(e).$$

The functions $\{\theta_m\}_{m\in M}$,

$$\theta_m(n) = \begin{cases} 0 \text{ if } n \neq m \\ 1 \text{ if } n = m \end{cases}, \forall m, n \in M,$$

constitute a basis of $R(M)$.

The properties of $\{\theta_m\}_{m\in M}$ are

$$\begin{aligned} \mu(\theta_m \otimes \theta_n) &= \theta_m\theta_n = \theta_n\theta_m = \begin{cases} 0, m \neq n \\ \theta_m, m = n \end{cases}, \\ \sum_{m\in M} \theta_m &= 1, \\ m^*(\theta_m)(n, n') &= \theta_m(nn') = \begin{cases} 0, m \neq nn' \\ \theta_m, m = nn' \end{cases}, \end{aligned}$$

$m, n, n' \in M$.

Note the similarities with the properties of projections.

4.5. M**-graded modules are** $R(M)$**-modules.** For any $R(M)$-module X, define a M-grading on X,

$$\theta_m \cdot x \overset{def}{=} \pi_m(x) = x_m,$$

$m \in M$ and $x \in X$.

On the other hand, let X be a M-graded R-module. Define the action of any function $f \in R(M)$ by

$$f \cdot x = (\sum_{m\in M} f(m)\theta_m) \cdot x = \sum_{m\in M} f(m)\pi_m(x),$$

$x \in X$.

This gives the following proposition.

Proposition 7. *A R-module X is a $R(M)$-module if and only if X is a M-graded R-module.*

A (strict) monoidal structure is given on $R(M)$-modules as

$$f \cdot (x \otimes y) = m^*(f)(x \otimes y),$$

and there is an isomorphism of categories between the monoidal category of M-graded R-modules and the monoidal category of $R(M)$-modules.

4.6. **The bialgebra** $R[M]$**.** Denote by $R[M]$ the dual of the function algebra,

$$R(M)^* = Hom_R(R(M), R) = R[M],$$

which is the the group (or monoid) algebra of M, and the basis of $R[M]$ is consists of Dirac δ-functions,

$$\{\delta_m\}_{m\in M} = \{\theta_m^*\}_{m\in M},$$

such that

$$\begin{aligned} \langle \delta_m, \theta_m \rangle &= 1, \\ \langle \delta_m, \theta_n \rangle &= 0, \end{aligned}$$

$m, n \in M$, $m \neq n$.

The multiplication and comultiplication is

$$\begin{aligned} \delta_m \delta_n &= \delta_{mn}, \\ \Delta(\delta_m) &= \delta_m \otimes \delta_m, \end{aligned}$$

$m, n \in M,\ f, f' \in R(M)$.

The unit is

$$\epsilon^* : R \to R[M],$$

where

$$\langle \epsilon^*(e), f \rangle = \langle e, \epsilon(f) \rangle = \langle e, f(e) \rangle = f(e),$$

$f = \sum_{m\in M} f(m)\theta_m \in R(M)$, and the counit

$$\iota^* : R[M] \to R,$$

$$\langle \iota^*(s), e \rangle = \langle s, \iota(e) \rangle = \langle s, \theta_e \rangle = s(e),$$

$s = \sum_{m\in M} s(m)\delta_m \in R[M]$, $e \in M$.

We have the following well known fact.

Proposition 8. *M-action on a module X is equivalent to $R[M]$-module structure on X.*

5. Grading and action

In this section we will compare modules with an action by a finite abelian group G, i.e. modules with a cograding by G, and grading by the dual of G. By the Fourier transform of finite groups we will construct a functor isomorphism between the monoidal categories of graded modules and cograded modules.

5.1. **The Fourier transform for finite groups.** For any finite abelian group G and $R=\mathbb{C}$ the Fourier transform of finite groups provides us with an isomorphism from $\hat{G}$-grading to G-action.

Let

$$\hat{G}=Hom\left(G,T^{1}\right)$$

be the dual group of G, consisting of all group homomorphisms

$$\chi:G\rightarrow T^{1},$$

where $T^{1}=\{z\in\mathbb{C}\mid |z|=1\}$ is the 1-dimensional torus.

The Fourier transform F of finite abelian groups [13] is the map

$$F:\mathbb{C}\left[G\right]\rightarrow\mathbb{C}\left(\hat{G}\right),$$

given by

$$\begin{aligned} F\left(\delta_{g}\right)\left(\chi\right) &= \overline{\chi\left(g\right)} \\ &= \chi\left(g^{-1}\right), \end{aligned}$$

for $g\in G$ and $\chi\in\hat{G}$, and

$$F\left(s\right)\left(\chi\right)=\sum_{g\in G}s\left(g\right)\chi\left(g^{-1}\right),$$

for any element

$$s=\sum_{g\in G}s\left(g\right)\delta_{g}\in\mathbb{C}\left[G\right].$$

The inverse is the following

$$F^{-1}\left(\theta_{\chi}\right)=\frac{1}{|G|}\sum_{g\in G}\chi\left(g\right)\delta_{g}$$

for $\chi\in\hat{G}$, and

$$F^{-1}\left(f\right)=\frac{1}{|G|}\sum_{\substack{\chi\in\hat{G}\\ g\in G}}f\left(\chi\right)\chi\left(g\right)\delta_{g},$$

for any element

$$f=\sum_{\chi\in\hat{G}}f\left(\chi\right)\theta_{\chi}\in\mathbb{C}\left(\hat{G}\right).$$

Note that

$$F^{-1}\left(1\right)=F^{-1}\left(\sum_{\chi\in\hat{G}}\theta_{\chi}\right)=\frac{1}{|G|}\sum_{\substack{g\in G\\ \chi\in\hat{G}}}\chi\left(g\right)\delta_{g}=\delta_{e}.$$

We have the following well known fact.

Proposition 9. *The Fourier transform is a bialgebra isomorphism.*

5.2. **Change of rings.** Consider two rings A and B and a ring homomorphism $\phi : A \to B$. By Change of rings, see [1], we find a functor between the monoidal categories of A-modules and B-modules. Specifically, between $\mathbb{C}[G]$-modules and $\mathbb{C}\left(\hat{G}\right)$-modules.

Let mod_A and mod_B be the monoidal categories of A- and B-modules over R respectively.

A ring homomorphism $\phi : A \to B$ defines a unit preserving functor between the monoidal categories

$$\phi^! : \mathrm{mod}_B \to \mathrm{mod}_A,$$

by

$$\begin{aligned} \phi^!(X) &= X, \\ \phi^!(f : X \to X') &= f, \end{aligned}$$

where X and X' are B-modules and $f : X \to X'$ is an B-module homomorphism and the module structure is changed as follows.

Let X be a (left) B-module. Then, X is an A-module

$$ax = \phi(a)x,$$

$a \in A$, $x \in X$.

Let Y be a (left) A-module. The tensor product

$$Y_\phi = B \otimes_A Y,$$

is a B-module,

$$b'(b \otimes_A y) = (b'b) \otimes_A y,$$

$b, b' \in B$, $y \in Y$.

If ϕ is an isomorphism, the inverse of $\phi^!$ is $\left(\phi^!\right)^{-1} = \left(\phi^{-1}\right)^!$.

Theorem 10. *Let $\phi : A \to B$ be an isomorphism of rings. Then $\phi^!$ is an isomorphism of monoidal categories.*

Let $\Phi : C \to D$ be an isomorphism of monoidal categories. Let q^C be a quantization of C. Then

$$Q = q^D_{X,Y} : \Phi(X) \otimes_C \Phi(Y) \to \Phi(X \otimes_D Y),$$

$X, Y \in Obj\,(C)$, is the quantization of D via the functor Φ given by the commutativity of

$$\begin{tikzcd} \Phi(X)\otimes\Phi(Y) \arrow[r, "Q=q^D"] & \Phi(X\otimes Y) \\ X\otimes Y \arrow[u, "\Phi\otimes\Phi"] \arrow[r, "q^C"] & X\otimes Y \arrow[u, "\Phi"'] \end{tikzcd} ,$$

i.e.

$$Q = q^D = \Phi \circ q^C \circ (\Phi \otimes \Phi)^{-1}.$$

Note that Q is really the quantization of the functor Φ.

Similarly, associativity constraints and braidings of one of the categories corresponds to the ones in the other by the following diagrams. For associativity constraints,

$$\begin{tikzcd} \Phi(X)\otimes(\Phi(Y)\otimes\Phi(Z)) \arrow[r, "\alpha^D"] \arrow[d, "1\otimes Q"'] & (\Phi(X)\otimes\Phi(Y))\otimes\Phi(Z) \arrow[d, "Q\otimes 1"'] \\ \Phi(X)\otimes\Phi(Y\otimes Z) \arrow[d, "Q"'] & \Phi(X\otimes Y)\otimes\Phi(Z) \arrow[d, "Q"'] \\ \Phi(X\otimes(Y\otimes Z)) \arrow[r, "\Phi(\alpha^C)"] & \Phi((X\otimes Y)\otimes Z) \end{tikzcd} ,$$

and for braidings,

$$\begin{tikzcd} \Phi(X)\otimes\Phi(Y) \arrow[r, "\sigma^D"] \arrow[d, "Q"'] & \Phi(Y)\otimes\Phi(X) \arrow[d, "Q"'] \\ \Phi(X\otimes Y) \arrow[r, "\Phi(\sigma^C)"] & \Phi(Y\otimes X) \end{tikzcd} .$$

Hence the following theorem.

Theorem 11. *Let $\Phi : C \to D$ be an isomorphism of monoidal categories. Then q^C is a quantization of the monoidal category C if and only if*

$$q^D = \Phi \circ q^C \circ (\Phi \otimes \Phi)^{-1}$$

is a quantization of D. Moreover, given a quantization of either of the categories C and D, then α^C, σ^C are an associativity constraint and a braiding of C if and only if

$$\begin{aligned}\alpha^D &= \left(1\otimes q^D\right)^{-1}\circ\left(q^D\right)^{-1}\circ\Phi(\alpha^C)\circ q^D\circ\left(1\otimes q^D\right),\\ \sigma^D &= \left(q^D\right)^{-1}\circ\Phi(\sigma^C)\circ q^D,\end{aligned}$$

are an associativity constraint and a braiding of D.

Under the functor $\phi^!$ of Change of rings the objects are not changed, hence the picture is somewhat simplified as we shall see for G-modules and $\hat{G}$-graded modules.

5.3. **The Fourier functor between G-modules and $\hat{G}$-graded modules.** Let G be a finite abelian group.

By Theorem 13, the Fourier transform induces an isomorphism of categories $F^!$ between $\mathbb{C}[G]$-modules and $\mathbb{C}\left(\hat{G}\right)$-modules.

In the discussion above we have seen that $\mathbb{C}[G]$-module structure is equivalent to G-module structure and $\mathbb{C}\left(\hat{G}\right)$-module structure is equivalent to $\hat{G}$-grading.

Hence, as a special case of Change of rings, the *Fourier functor*

$$F^!:\mathrm{mod}_G\to\mathrm{mod}_{\mathbb{C}}\left(\hat{G}\right),$$

is a isomorphism of categories between the monoidal categories of G-modules, mod_G, and $\hat{G}$-graded modules, $\mathrm{mod}_{\mathbb{C}}\left(\hat{G}\right)$ where

$$\begin{aligned}F^!(X) &= X,\\ F^!(f) &= f,\end{aligned}$$

for a G-module X and a G-module homomorphism f.

Theorem 12. *Let X be a G-module. Then there is a $\hat{G}$-grading on X by the projection*

$$\pi_\chi(x)=F^{-1}(\chi)(x)=\frac{1}{|G|}\sum_{g\in G}\chi(g)\,g(x),$$

$x\in X$, $\chi\in\hat{G}$.

Let Y be a $\hat{G}$-graded R-module. There is a G-module structure on Y given by

$$gy=\sum_{\chi\in\hat{G}}F(\theta_g)(\chi)\,y_\chi=\sum_{\chi\in\hat{G}}\chi\left(g^{-1}\right)y_\chi,$$

$y=\sum_{\chi\in\hat{G}}y_\chi\in Y$, $g\in G$.

Proof. The projection π_χ is given by the commutativity of the diagram,

$$\begin{tikzcd} E \arrow[r, "\theta_\chi"] \arrow[d, "F^{-1}"'] & E \arrow[d, "F^{-1}"'] \\ E \arrow[r, "F^{-1}(\theta_\chi)"] & E \end{tikzcd},$$

i.e.

$$\begin{aligned} F^{-1} \circ \theta_\chi &= F^{-1}(\theta_\chi) \circ F^{-1}, \\ x_\chi &= \theta_\chi(x) \\ &= F^{-1}(\theta_\chi)(x) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi(g)\, g(x). \end{aligned}$$

and the action of g by the commutativity of

$$\begin{tikzcd} E \arrow[r, "\partial_g"] \arrow[d, "F"'] & E \arrow[d, "F"'] \\ E \arrow[r, "F(\partial_g)"] & E \end{tikzcd},$$

$$\begin{aligned} F \circ \partial_g &= F(\partial_g) \circ F, \\ g(x) &= \partial_g(x) = \sum_{\chi \in \hat{G}} F(\partial_g)(x_\chi) = \sum_{\chi \in \hat{G}} \chi\left(g^{-1}\right) x_\chi. \end{aligned}$$

∎

Remark 4. *Another way to see this isomorphism is the following. Let G be a finite abelian group and consider a representation of G on X,*

$$\rho : G \to GL(X).$$

Then ρ can be decomposed into one-dimensional irreducible representations and define projections by

$$\pi_\chi(X) = X_\chi$$

where each $\chi \in \hat{G}$ is the character or eigenvalue of a irreducible representation and X_χ is the subspace of X invariant with respect to the irreducible subrepresentation of ρ associated with the character χ.

X can then be expressed as a direct sum of irreducible G-subspaces

$$X=\sum_{\chi\in\hat{G}}X_{\chi}.$$

Remark 5. *For any group G, if an abelian group G_0 is contained in the set of automorphisms of a subcategory C_0 of the monoidal category* mod_G *of G-modules, then this subcategory C_0 is contained in the part of* mod_G *which is graded by the group G_0.*

6. Associativity constraints, quantizations and braidings for G-actions

We apply the Fourier transformation to the associativity constraints, quantizations and braidings for $\hat{G}$-graded modules to obtain the corresponding ones for G-modules.

6.1. **Associativity constraints of G-modules.** Any associativity constraint $\hat{\alpha}$ is a representation of a 3^{rd} cohomology group of $\hat{G}$ with coefficients in $U(\mathbb{C})$, and

$$\hat{\alpha}=\sum_{(\phi,\chi,\psi)\in\hat{G}\times\hat{G}\times\hat{G}}\hat{\alpha}(\phi,\chi,\psi)\,\theta_{(\phi,\chi,\psi)}=\sum_{\phi,\chi,\psi\in\hat{G}}\hat{\alpha}(\phi,\chi,\psi)\,\theta_{\phi}\otimes\theta_{\chi}\otimes\theta_{\psi}.$$

Theorem 13. *Any associativity constraint α in the monoidal category of G-modules is represented as follows*

$$\alpha=\frac{1}{|G|^3}\sum_{\substack{f,g,h\in G\\ \phi,\chi,\psi\in\hat{G}}}\hat{\alpha}(\phi,\chi,\psi)\,\phi(f)\,\chi(g)\,\psi(h)\,\delta_f\otimes\delta_g\otimes\delta_h,$$

where $\hat{\alpha}$ is an associativity constraint in the monoidal category of $\hat{G}$-graded modules.

Proof. Taking the inverse of the Fourier,

$$\begin{aligned}\alpha &= (F\otimes F\otimes F)^{-1}(\hat{\alpha})\\ &= (F\otimes F\otimes F)^{-1}\left(\sum_{\phi,\chi,\psi\in\hat{G}}\hat{\alpha}(\phi,\chi,\psi)\,\theta_{\phi}\otimes\theta_{\chi}\otimes\theta_{\psi}\right)\\ &= \frac{1}{|G|^3}\sum_{\substack{f,g,h\in G\\ \phi,\chi,\psi\in\hat{G}}}\hat{\alpha}(\phi,\chi,\psi)\,\phi(f)\,\chi(g)\,\psi(h)\,\delta_f\otimes\delta_g\otimes\delta_h.\end{aligned}$$

∎

We use the somewhat shorter notation

$$\alpha = \sum_{f,g,h\in G} A_{fgh}\delta_f \otimes \delta_g \otimes \delta_h.$$

6.2. **Quantizations of G-modules.** A quantization of a $\hat{G}$-graded module is represented by a 2-cocycle $\hat{q} : \hat{G} \times \hat{G} \to U(\mathbb{C})$, and

$$\hat{q} = \sum_{\phi,\chi\in\hat{G}} \hat{q}(\phi,\chi)\,\theta_\phi \otimes \theta_\chi.$$

Theorem 14. *Any quantization q in the monoidal category of G-modules is represented as follows*

$$q = \frac{1}{|G|^2} \sum_{\substack{g,h\in G\\ \phi,\chi\in\hat{G}}} \hat{q}(\phi,\chi)\,\phi(g)\,\chi(h)\,\delta_g \otimes \delta_h : X \otimes Y \to X \otimes Y,$$

where $\hat{q}$ is a quantization in the monoidal category of $\hat{G}$-graded modules.

Proof. Applying the inverse of the Fourier,

$$\begin{aligned} q &= (F \otimes F)^{-1}(\hat{q}) \\ &= (F \otimes F)^{-1}\left(\sum_{\phi,\chi\in\hat{G}} \hat{q}(\phi,\chi)\,\theta_\phi \otimes \theta_\chi\right) \\ &= \frac{1}{|G|^2} \sum_{\substack{g,h\in G\\ \phi,\chi\in\hat{G}}} \hat{q}(\phi,\chi)\,\phi(g)\,\chi(h)\,\delta_g \otimes \delta_h. \end{aligned}$$

∎

We use the notation

$$q = \sum_{g,h\in G} Q_{gh}\delta_g \otimes \delta_h.$$

6.3. **Braidings of G-modules.** A braiding of $\mathbb{C}\left(\hat{G}\right)$-modules is represented by 2-cochains $\hat{\sigma} : \hat{G} \times \hat{G} \to U(\mathbb{C})$, and

$$\hat{\sigma} = \tau \circ \sum_{\phi,\chi\in\hat{G}} \hat{\sigma}(\phi,\chi)\,\theta_\phi \otimes \theta_\chi.$$

Theorem 15. *Any braiding σ in the monoidal category of G-modules is represented as follows*

$$\sigma = \tau \circ \left(\frac{1}{|G|^2} \sum_{\substack{g,h\in G\\ \phi,\chi\in\hat{G}}} \hat{\sigma}(\phi,\chi)\,\phi(g)\,\chi(h)\,\delta_g \otimes \delta_h\right),$$

where $\hat{\sigma}$ is a braiding in the monoidal category of $\hat{G}$-graded modules.

Proof. By the Fourier isomorphism, a braiding σ in the monoidal category of G-modules is

$$\begin{aligned}
\sigma &= (F \otimes F)^{-1} (\hat{\sigma}) \\
&= \tau \circ (F \otimes F)^{-1} \left(\sum_{\phi,\chi \in \hat{G}} \hat{\sigma} (\phi, \chi) (\theta_\phi \otimes \theta_\chi) \right) \\
&= \tau \circ \left(\frac{1}{|G|^2} \sum_{\substack{g,h \in G \\ \phi,\chi \in \hat{G}}} \hat{\sigma} (\phi, \chi) \phi (g) \chi (h) \delta_g \otimes \delta_h \right).
\end{aligned}$$

∎

Use the notation

$$\sigma = \tau \circ \left(\sum_{g,h \in G} S_{g,h} \delta_g \otimes \delta_h \right).$$

Remark 6. *The braidings provides us with solutions of the Yang-Baxter equation for any finite group G.*

A quantization of a braiding is

$$\begin{aligned}
\sigma_q &= \tau \circ \left(\sum_{g,h \in G} S^q_{g,h} \delta_g \otimes \delta_h \right) \\
&= \tau \circ \left(\frac{1}{|G|^2} \sum_{\substack{g,h \in G \\ \phi,\chi \in \hat{G}}} \hat{q}^{-1} (\chi, \phi) \hat{\sigma} (\phi, \chi) \hat{q} (\phi, \chi) \phi (g) \chi (h) \delta_g \otimes \delta_h \right),
\end{aligned}$$

where $\hat{q}$ is a quantization in the monoidal category of $\hat{G}$-graded modules.

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