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CONTINUOUS AND GENERALIZED SOLUTIONS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS

(submitted by A. M. Elizarov)

Abstract. The paper discusses continuous and generalized solutions of equations with partial derivatives having the operator coefficients which operate in Banach spaces. The operator with the elder derivative with respect to time is Fredholm. We apply Lyapunov–Schmidt's ideas and the generalized Jordan sets techniques to reduce partial differential-operator equations with the Fredholm operator in the main part to regular problems. In addition this technique has been exploited to prove the theorem of existence and uniqueness for a singular initial-value problem, as well as to construct the left and right regularizators of singular operators in Banach spaces and to construct fundamental operators in the theory of generalized solutions of singular equations.

1. Introduction

In 2002 N.Sidorov and M.Falaleev have described (see [14] chapter 6) applications of Lyapunov–Schmidt's ideas [17] to the theory of ordinary differential operator equations in Banach spaces with the irreversible operator in the main part (briefly, singular DOE). A number of initial-value and boundary-value problems, which model real dynamic processes of filtering, thermal convection, deformation of mechanical systems, electrical engineering (models of Barrenblatt–Zheltova, Kochina, Oskolkov, Hoff,

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V. Dolexal, M. Korpusov, N. Pletner, A. Svechnikov and others), can be reduced to such equations.

Singular differential operator equations have been investigated in the works by S. Krein, N. Sidorov, B. Loginov, I. Melnikhova, K. Akhmedov, A. Kozhanov, R. Schowalter, G. Sviridyuk, M. Falaleev and others. Extended bibliographies can be found in monographs by N. Sidorov [11], N. Sidorov, B. Loginov, A. Sinitsyn and M. Falaleev [14], R. Cassol and R. Schowalter [1], G. Sviridyuk and V. Fedorov [15].

The problem of applying Lyapunov-Schmidt's ideas to singular differential operator equations having Fredholm operators in the main part had been stated already by L. Lusternik in the course of work of his symposia held at Moscow State University in the mid of 1950s and has been solved by N. Sidorov (see [11], chapter 4). It appeared obvious that the analog of the classical branching equation for such equations (see [17]) is a system of differential equations of an infinite order. In view of substantial difficulties, which arise in the process of investigation of this system, the theory of singular DOE is presently far from being completed, moreover, there are few results for the nonlinear case. In the monograph [14] an explication of foundations of the general theory of singular differential operator equations is given. Authors have employed the apparatus of generalized Jordan chains (developed in [17]) and the fundamental operators of singular integro-differential expressions (constructed in [2]), the theory of generalized functions, the Nekrasov–Nazarov's method of undeterminate coefficients, which is combined with asymptotic methods of the theory of differential equations with singular points, topological methods and the techique of construction of the regularizator algorithm by N. Sidorov's [11], methods of semigroups and groups with kernels developed by G. Sviridyuk [15]. Such a mixture of diverse methods has given the possibility of investigating a wide class of singular ordinary differential operator equations and classes of partial differential operator equations with the Noether operator in the main part. Some recent general results for singular linear partial differential operator equations have been included to this paper.

Let $x = (t, x')$ be a point in the space R^{m+1}, $x' = (x_1, \dots, x_m)$, $D = (D_t, D_{x_1}, \dots, D_{x_m})$, $\alpha = (\alpha_0, \dots, \alpha_m)$, $\mid \alpha \mid = \alpha_0 + \alpha_1 + \cdots \alpha_m$, α_i are integer non-negative indices, $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial t^{\alpha_0} \dots \partial x_m^{\alpha_m}}$.

We also suppose that $B_\alpha : D_\alpha \subset E_1 \to E_2$ are closed linear operators with dense domains in E_1, $x \in \Omega$, where $\Omega \subset R^{m+1}$, $\mid t \mid \leq T$, $x' \in R^m$, E_1, E_2 are Banach spaces.

It is assumed that $\forall u \in E_1$ the function $B_\alpha(x)u$ is analytical with respect to x' and sufficiently smooth with respect to t.

Consider the following differential operator $L(D) = \sum_{|\alpha|\leq l} B_\alpha(x)D^\alpha$. We call the operator $\sum_{|\alpha|=l} B_\alpha D^\alpha$ the main part of $L(D)$.

We consider the equation

$$L(D)u = f(x), \tag{1}$$

where $f : \Omega \to E_2$ is an analytical function of x' sufficiently smooth with respect to t. The initial value problem for (1), when $E_1 = E_2 = R^n$ and the matrix $B = B_{l0\ldots0}$ is not singular, has been thoroughly investigated in fundamental papers by I.G. Petrovsky (see [8]). In the case when the operator B is not invertible the theory of initial and boundary value problems for (1) has not been developed even for the case of finite dimensions. The case with the Fredholm operator B with $dimN(B) \geq 1$ is of special interest. This case, when $x \in R^1$, has been considered from different viewpoints in [11], [7], [15] etc. The case, when $x \in R^{m+1}$, $dimN(B) \geq 1$ has attracted our attention only lately [13]. In general, the standard initial value problem with conditions $D_t^i u|_{t=0} = g_i(x')$, $i = 0, \ldots, l-1$ for (1) has no classical solutions for an arbitrary right-hand side $f(x)$.

This does not mean that in the present case we do not have a "correctly" stated problem for (1), which has a unique solution for any right-hand side $f(x)$. For example, the positive result can be obtained by decomposing the space E_1 into a direct sum of subspaces in accordance with the properties of operator coefficients B_α and assigning initial conditions on these subspaces separately. This technique applied in a different situation [16] has been also used in the present work. It is assumed that B is a constant Fredholm operator, and among the coefficients B_α there is a constant operator $A \overset{\text{def}}{\equiv} B_{l_1 0\ldots0}$, $l_1 < l$, with respect to which B has a complete $A-$Jordan set.

In Section 2 the sufficient conditions of existence of the unique solution for (1) with the initial conditions

$$D_t^i u|_{t=0} = g_i(x'), \quad i = 0, 1, \ldots, l_1 - 1, \tag{2}$$

$$(I - P)D_t^i u|_{t=0} = g_i(x'), \quad i = l_1, \ldots, l-1, \tag{3}$$

are obtained, where $g_i(x')$ are analytical functions with values in E_1, $Pg_i(x') = 0, \quad i = l_1, \ldots, l-1$, and the left and right regularizators of singular operators in Banach spaces are constructed. Here P is the projector of E_1 onto the corresponding A-root subspace (see [17] chapter 7). In Section 3 a method of fundamental operators for constructing the generalized solution in the class of Schwarz distributions [9] is considered. These investigations can be useful for the new applications [14], [15], [6] of singular differential systems in mechanics and physics and for the development of the new numerical methods in these areas.

2. Continuous Solutions

The first part of this section gives some auxiliary information from [13], the second part suggests the reduction of (1) to the form of Cauchy–Kovalevskaya, whereas in the third part the theorems of existence and uniqueness of solutions of the problem (1), (2), (3) are proved. In conclusion of the first section, left and right regularizators of singular operators in Banach spaces are constructed.

2.1. **Decomposition of Banach spaces,** $(P, Q)-$**commutativity of linear operators.** Let M_i and N_i be mutually complementary subspaces of Banach spaces E_1 and E_2, i.e. $E_1 = M_1 + N_1$, $E_2 = M_2 + N_2$, P is a projector onto M_1 parallel to N_1, Q is a projector onto M_2 parallel to N_2.

Let A be a linear and, generally speaking, unbounded operator from E_1 in E_2 with the domain of definition dense in E_1.

Definition 1. *Let* $A : D \subseteq E_1 \to E_2$. *If*

$$PD \subseteq D,\ AM_1 \subseteq M_2,\ A(N_1 \cap D) \subseteq N_2$$

then the operator A *is said to be* $(P, Q)-$*reducible.*

Definition 2. *If, for any* $u \in D(A)$, *the vector* $Pu \in D(A)$ *and* $APu = QAu$, *then the operator* A *is said to be* (P, Q)*-commutating.*

The operator A is (P, Q)-commutating if and only if A is $(P, Q)-$reducible.

Property 1. *Let an operator* A *be* (P, Q)*-commutating, an operator* Γ *be* (Q, P)*-commutating,* $R(\Gamma) \subseteq D(A)$, $R(A) \subseteq D(\Gamma)$. *Then,*

1. *the operator* $A\Gamma$ *is* Q*-commutating,* $M_2 \cap D(\Gamma)$ *and* $N_2 \cap D(\Gamma)$ *are its invariant subspaces;*
2. *the operator* ΓA *is* P*-commutating,* $M_1 \cap D(A)$ *and* $N_1 \cap D(A)$ *are its invariant subspaces.*

Let us further assume that M_1 and M_2 are some finite-dimensional subspaces, $M_1 \subseteq D(A)$, $\tilde{P} = \sum_1^n \langle \cdot, \gamma_i \rangle \varphi_i$, $\tilde{Q} = \sum_1^n \langle \cdot, \psi_i \rangle z_i$, furthermore, $\langle \varphi_i, \gamma_k \rangle = \delta_{ik}$, $\langle z_i, \psi_k \rangle = \delta_{ik}$, $\{\varphi_i\} \in M_1$, $\{z_i\} \in M_2$. Then the condition of $(\tilde{P}, \tilde{Q})-$commutativity of the operator A implies that $AM_1 \subseteq M_2$. Hence, there exists a matrix $\aleph_A : R^n \to R^n$, such that $A\Phi = \aleph_A Z$, $\Phi =$

$(\varphi_1, \dots, \varphi_n)'$, $Z = (z_1, \dots, z_n)'$. This matrix will be called the matrix of $(\tilde{P}, \tilde{Q})$-commutation of the operator A.

Property 2. *If* $A\Phi = \aleph_A Z$, $A^*\Psi = \aleph_B \Upsilon$, $\aleph_A, \aleph_B : R^n \to R^n$, *where* $\Psi = (\psi_1, \dots, \psi_n)'$, $\Upsilon = (\gamma_1, \dots, \gamma_n)'$, *then* A $(\tilde{P}, \tilde{Q})$*-commutates if and only if* $\aleph_B = \aleph_A'$.

Consider now a special case when the basis in M_1 consists of the elements $\{\varphi_i^{(j)}\}$, $i = \overline{1,n}$, $j = \overline{1,p_i}$, which form a complete A-Jordan set of the operator B, where B is a Fredholm operator.

Hence $B\varphi_i^{(1)} = 0$, $B\varphi_i^{(j)} = A\varphi_i^{(j-1)}$, $i = \overline{1,n}$, $j = \overline{2,p_i}$, and there exist $\{\psi_i^{(j)}\}$ such that $B^*\psi_i^{(1)} = 0$, $B^*\psi_i^{(j)} = A^*\psi_i^{(j-1)}$. The system $\{z_i^{(j)}\}$ biorthogonal to $\{\psi_i^{(j)}\}$ will be taken as the basis in $M_2 \subset E_2$.

Let us introduce the projectors

$$P = \sum_{i=1}^{n} \sum_{j=1}^{p_i} \langle \cdot, \gamma_i^{(j)} \rangle \varphi_i^{(j)}, \quad Q = \sum_{i=1}^{n} \sum_{j=1}^{p_i} \langle \cdot, \psi_i^{(j)} \rangle z_i^{(j)}. \tag{4}$$

Property 3. *Let the projectors* P *and* Q *be defined by the formulas* (4)*. Hence operators* B *and* A *be* (P, Q)*-commutating, furthermore, the corresponding matrices of* (P, Q)*-commutation are symmetric cell-diagonal ones:* $\aleph_B = diag(B_1, \dots, B_n)$, $\aleph_A = diag(A_1, \dots, A_n)$, *where*

$$B_i = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & \dots & 1 \\ \dots & \dots & \dots \\ 1 & \dots & 0 \end{bmatrix}, \quad i = \overline{1,n},$$

if $p_i \geq 2$, *and* $B_i = 0$, $A_i = 1$ *if* $p_i = 1$.

2.2. **Reduction of equation (1) to the form of Cauchy-Kovalevskaya.** Let $B \overset{\text{def}}{=} B_{l0\dots0}$, $A \overset{\text{def}}{=} B_{l_1 0\dots0}$, where $B_{l0\dots0}$, $B_{l_1 0\dots0}$ are constant operators, $l_1 < l$, $D(B) \subseteq D(A)$.

Condition 1. $D(B) \subseteq D(B_\alpha)$ $\forall \alpha$, *the Fredholm operator* B *has a complete* A*-Jordan set* $\varphi_i^{(j)}$, B^* *has a complete* A^**-Jordan set* $\psi_i^{(j)}$, $i = \overline{1,n}$, $j = \overline{1,p_i}$, *and the systems* $\gamma_i^{(j)} \equiv A^*\psi_i^{(p_i+1-j)}, z_i^{(j)} \equiv A\varphi_i^{(p_i+1-j)}$, $i = \overline{1,n}, j = \overline{1,p_i}$, *corresponding to them, are biorthogonal,* $k = p_1 + \dots p_n$ *is a root number.*

Hence, the formulas (4) define the projectors P and Q respectively onto the root subspaces $E_{1k} = span\{\varphi_i^{(j)}\}$, $E_{2k} = span\{z_i^{(j)}\}$.

Since $E_1 = E_{1k} \oplus E_{1\infty-k}$, any solution of (1) can be represented in the form

$$u(x) = \Gamma v(x) + (C(x), \Phi), \tag{5}$$

where $\Gamma = (B+\sum_{i=1}^{n}\langle\cdot,\gamma_i^{(1)}\rangle z_i^{(1)})^{-1}$ is a bounded operator from E_2 in E_1,

$$C(x) = (C_1(x),\dots C_n(x))',\ C_i(x) = (C_{i1}(x),\dots,C_{ip_i}(x)),$$

$$\Phi = (\Phi_1,\dots,\Phi_n)',\ \Phi_i = (\varphi_i^{(1)},\dots,\varphi_i^{(p_i)}),$$

$$v : \Omega\subset R^{m+1} \to E_{2\infty-k},\ C : \Omega\subset R^{m+1}\to R^k.$$

Since

$$\Gamma z_i^{(j)} = \varphi_i^{(p_i+2-j)},\ \Gamma^*\gamma_i^{(j)} = \psi_i^{(p_i+2-j)},\ j=\overline{1,p_i},$$

$$\varphi_i^{(p_i+1)} \stackrel{\text{def}}{=} \varphi_i^{(1)},\ \psi_i^{(p_i+1)} \stackrel{\text{def}}{=} \psi_i^{(1)},$$

the operator Γ is (P,Q)-commutating.

Substituting the function (5) into (1), we obtain the equality

$$D_t^l v + \sum_{|\alpha|\le l,\ \alpha\ne(l,0,\dots,0)} B_\alpha(x)\Gamma D^\alpha v + \sum_{|\alpha|\le l} B_\alpha(x)(D^\alpha C,\Phi) = f(x). \tag{6}$$

Condition 2. *Each of the coefficients B_α satisfy just one of the following three conditions:*

(1) *B_α is (P,Q)-commutating; briefly, $B_\alpha\in\alpha^0$;*
(2) *$QB_\alpha = 0$; briefly, $B_\alpha\in\alpha^1$;*
(3) *$(I-Q)B_\alpha = 0$; briefly, $B_\alpha\in\alpha^2$.*

Now, by projecting (6) onto $E_{2\infty-k}$, we obtain the equation

$$D_t^l v + \sum_{|\alpha|\le l,\ \alpha\notin\alpha^2,\ \alpha\ne(l,0,\dots,0)} B_\alpha(x)\Gamma D^\alpha v = \tag{7}$$

$$= (I-Q)f(x) - \sum_{|\alpha|\le l,\ \alpha\in\alpha^1} B_\alpha(x)(D^\alpha C,\Phi).$$

By projecting the equation (6) onto E_{2k}, we obtain the system

$$\aleph_{l0\dots0}D_t^l C + \sum_{|\alpha|\le l,\ \alpha\notin\alpha^1,\ \alpha\ne(l,0,\dots,0)} \aleph'_\alpha D^\alpha C = b(x,v). \tag{8}$$

Here the vector function $b:\Omega\to R^k$ is defined by the formula

$$\langle f(x) - \sum_{|\alpha|\le l,\ \alpha\notin\alpha^2} B_\alpha(x)\Gamma D^\alpha v,\Psi\rangle.$$

Therefore, equation (6) is reduced to equation (7) and system (8). This equation (7), as a differential equation with respect to v, has the form of Cauchy-Kovalevskaya.

2.3. **Selection of initial conditions. Theorems of existence and uniqueness.** Let us find the solution of (1) which would satisfy the initial conditions (2), (3). Since $\Gamma E_{2\infty-k} \subset E_{1\infty-k}$, the solution (5) satisfies the initial conditions (2), (3) if and only if

$$D_t^i v|_{t=0} = \begin{cases} B(I-P)g_i(x'),\ i=0,\dots,l_1-1, \\ Bg_i(x'),\ i=l_1,\dots,l-1, \end{cases} \tag{9}$$

$$D_t^i C|_{t=0} = \beta_i(x'),\ i=0,\dots,l_1-1. \tag{10}$$

Here $\beta_i(x')$ are coefficients of projections $Pg_i(x'),\ i=0,\dots,l_1-1$. Hence, the desired $v(x)$ satisfies the initial-value problem (7), (9) in the Cauchy-Kovalevskaya form, and the desired vector function $C(x)$ satisfies, respectively, the initial-value problem (8), (10).

Consider the following two cases when the initial-value problem (8), (10) also has the Cauchy-Kovalevskaya form.

Case 1. $k=n$.

Hence, in system (8), $\aleph_{l0\dots0}=0,\ \aleph_{l_10\dots0}=E$ is a unique matrix. If $\aleph_\alpha = 0$ for $l_1 <\mid \alpha \mid\leq l$, condition 2 is satisfied for $P=\sum_1^n\langle\cdot,\gamma_i^{(1)}\rangle\varphi_i^{(1)},\ Q=\sum_1^n\langle\cdot,\psi_i^{(1)}\rangle z_i^{(1)}$,

$$\{\alpha^1=\emptyset\}\vee\{\alpha^2=\emptyset\}\vee\{\max_{\alpha\in(\alpha^1\cup\alpha^2)}\mid\alpha\mid< l_1\}, \tag{11}$$

then system (8) has the order of l_1 and the Cauchy-Kovalevskaya form.

In this connection, the corresponding initial-value problems (7), (9); (8), (10) have unique solutions.

If $\aleph_\alpha$ are triangular $n\times n-$matrices with zeros on the main diagonal and to the right of it, and condition (11) holds, then system (8) turns out to be a recurrent sequence of equations of the order of l_1 in the Cauchy-Kovalevskaya form.

The above reasoning implies the following

Theorem 1. *Let B be a Fredholm operator, $\langle A\varphi_i^{(1)},\psi_k^{(1)}\rangle=\delta_{ik},\ i,k=\overline{1,n}$, and let condition 2 for $P=\sum_1^n\langle\cdot,\gamma_i^{(1)}\rangle\varphi_i^{(1)},\ Q=\sum_1^n\langle\cdot,\psi_i^{(1)}\rangle z_i^{(1)}$ and condition (11) be satisfied. If, for $l_1<|\alpha|\leq l$, the matrices $\aleph_\alpha$ are either equal to zero or all the matrices have zeros to the right of the main diagonal, and for $l_1<|\alpha|$ these have zeros also on the main diagonal, then the problem (1), (2), (3) has unique solution.*

Case 2. $k>n$.

Now, in the system (8) $\aleph_{l0\dots0}=\aleph_B,\ \aleph_{l_10\dots0}=\aleph_A$, where the matrices $\aleph_B,\ \aleph_A$ are as defined above (see section 1.2).

Theorem 2. *Let*

(1) *conditions* 1, 2 *be satisfied, furthermore, in condition* 2 $\alpha^1 = \emptyset$ *or* $\alpha^2 = \emptyset$*;*
(2) *matrices* $\aleph'_\alpha = [\aleph^\alpha_{ik}]^n_{i,k=1}$ *are lower block-triangular, i.e.* $\aleph^\alpha_{ik} = 0$ *for* $i < k$;
(3) *there are zeros in each diagonal block* $\aleph^\alpha_{ii}$ *to the left of the nonmain diagonal, and for* $|\alpha| > l_1$ *there are zeros also on the nonmain diagonal.*

Then the initial-value problem (1), (2), (3) has a unique solution.

For the purpose of proving it is sufficient to note that under the conditions of Theorem 2 system (8) turns out to be a recurrent sequence of linear differential equations of the order of l_1 in the Cauchy–Kovalevskaya form, and (7) is a differential equation of the order of l_1 in the Cauchy–Kovalevskaya form with the bounded operator coefficients. Note that due to the structure of the matrices $\aleph'_\alpha$ components of the vector function $C : \Omega \to R^k$ are defined in the following sequence $c_{1p_i}, \ldots, c_{11}, c_{2p_2}, \ldots, c_{21}, c_{np_n}, \ldots, c_{n1}$. For a more special situation, details of proof can be found in [12].

2.4. **The left and right regularizators of singular operators in Banach spaces.** Let A and B be constant linear operators from E_1 to E_2, where E_1 and E_2 are Banach spaces, $x(t)$ is an abstract function, $t \in R_n$ with the values in $E_1(E_2)$. The set of such functions is denoted by $X_t(Y_t)$. Now introduce the operator L_t, defined on X_t and Y_t and which is commutable with operators B, A. The examples of such an operator L_t are differential and integral operators, difference operators and their combinations. Note that if operators are solved with respect to higher order derivatives, then they usually generate correct initial and boundary value problems. In other cases, when operators are unsolved according to higher order derivatives, we encounter singular problems (see subsec. 1.1).

Consider the operator $L_tB - A$, which acts from X_t to Y_t, where B, A are closed linear operators from E_1 to E_2 with dense domains, and $D(B) \subseteq D(A)$. If B is invertable, then the operator $L_tB - A$ can be reduced to regular operator by multiplication on B^{-1}. If B is uninvertible, then $L_tB - A$ is called the singular operator. Let operator B in $L_tB - A$ be Fredholm and $\dim N(B) = n \geq 1$. If $\lambda = 0$ is an isolated singular point of the operator-function $B - \lambda A$, then the operators $L_tB - A, BL_t - A$ admit some regularization. For the purpose of explicit representation of the regularizer we use Schmidt's pseudo resolvent $\Gamma = \hat{B}^{-1}$, where $\hat{B} = B + \sum_{i=1}^n < ., A^*\psi_i^{(p_i)} > A\phi_i^{(p_i)}$. On account of condition 1 (sect.

1) and using the equalities $\phi_i^{(j)} = \Gamma A \phi_i^{(j-1)}, \quad \psi_i^{(j)} = \Gamma^* A^* \psi_i^{(j-1)}, \quad j = 2, \dots, p_i, \quad i = 1, \dots, n$ it is easy to verify the following equalities

$$(\Gamma - \sum_{i=1}^{n} \sum_{j=1}^{p_i} L_t^j < ., \psi_i^{(p_i+1-j)} > \phi_i)(L_t B - A) = L_t - \Gamma A,$$

$$(L_t B - A)(\Gamma - \sum_{i=1}^{n} \sum_{j=1}^{p_i} L_t^{p_i+1-j} < ., \psi_i > \phi_i^{(j)}) = L_t - A\Gamma.$$

As a result, we have the following

Theorem 3. *Suppose condition* 1 *in section* 1.2 *be satisfied. Then*

$$\Gamma - \sum_{i=1}^{n} \sum_{j=1}^{p_i} L_t^j < ., \psi_i^{(p_i+1-j)} > \phi_i \text{ and } \Gamma - \sum_{i=1}^{n} \sum_{j=1}^{p_i} L_t^{p_i+1-j} < ., \psi_i > \phi_i^{(j)}$$

are the left and right regularizators of $L_t B - A$, *respectively.*

3. Generalized solutions

Since problems (1), (2), (3) with the Fredholm operator $B_{l0\dots0}$ in the general case are not solvable in the class of continuous functions, it is of doubtless interest to obtain solutions of such problems in the class of distributions. In this connection it is reasonable to suggest some initial information on generalized functions in Banach spaces before proceeding to the main results of the paragraph (subsection 2.1). The most interesting is the construction of the fundamental operator functions for the singular differential operators in Banach spaces which help to obtain the generalized solutions in closed forms (subsection 2.2).

3.1. **Generalized functions in Banach spaces.** Let E be a Banach space, E^* be a conjugate Banach space. Let all the finite class C^∞ functions with the values in E^* be attributed to the set of *main functions* $K(E^*)$; these functions be denoted by $s(x)$; the closure in R^n of the set of points x for which $s(x) \neq 0$ be called the *carrier supps*(x) of the main function $s(x)$. The main set $K(E^*)$ is a vector space. In order to make this space topological let us define the convergence on it as follows.

Definition 3. *The sequence of functions* $s_n(x)$ *from* $K(E^*)$ *converges to the function* $s(x) \in K(E^*)$ *if*

a) there exists $R > 0$ *such that* $supp\, s_n(x) \subset U_R(0) \quad \forall n \in N$, *where* $U_R(0)$ *is a closed ball in* R^n, *having its center in the origin and its radius equal to* R;

b) $\forall \alpha \quad \| D^\alpha s_n(x) - D^\alpha s(x) \| \Longrightarrow 0$ *uniformly with respect to* $x \in U_R$ *for* $n \to \infty$.

The set $K(E^*)$ with the convergence introduced for it will be called *the main space.* Any linear continuous functional on $K(E^*)$ will be called *the generalized function.* Let us define the convergence for the set of generalized functions as weak. The carrier, the equality for two generalized functions, the operations of addition and multiplication by a number of generalized functions will be defined as usual. The Bochner locally integrable function $u(x)$ with its values in E generates a *regular* generalized function in accordance with the following rule

$$(u(x),\ s(x)) \stackrel{\text{def}}{=} \int_{R^n} \langle u(x),\ s(x)\rangle dx, \ \ \forall s(x) \in K(E^*).$$

Let all the rest of the generalized functions be called *singular.*

Let us separate a special class $K'_+(E) \equiv K'_+(R^n; E)$ of generalized functions – from the whole set of generalized functions $K'(E)$ – which turn zero if at least one of the variables $x_i < 0,\ i = 1, \dots, n$. The set of such functions will include, for example, the functions of the form $u(x)g(x)$, where $u(x) \in C^\infty(E),\ g(x) \in D'(R^n_+)$ or $u(x) \in C^\infty_+(E),\ g(x) \in D'$, acting in accordance with the rule

$$(u(x)g(x), s(x)) \stackrel{\text{def}}{=} (g(x),\ \langle u(x), s(x)\rangle), \ \ \forall s(x) \in K(E^*).$$

Let E_1, E_2 be Banach spaces, $K(x) \in L(E_1; E_2)$ be a strongly continuous operator function of class C^∞, furthermore, $K^*(x) \in L(E_2^*; E_1^*)$ exist for almost all $x,\ f(x) \in D'_+$, then the formal symbol $K(x)f(x)$ will be called *the generalized operator function.*

Definition 4. *The generalized function $K(x)v(x) \in K'(E_2)$, defined by*

$$(K(x)v(x), s(x)) \stackrel{\text{def}}{=} (u(x), K^*(x)s(x)),$$

where $s(x) \in K(E_2^*)$ is a main function, is called the effect of the operator function $K(t) \in L(E_1, E_2)$ on the generalized function $v(t) \in K'(E_1)$.

Definition 5. *The generalized function $K(x)f(x) * v(x) \in K'_+(E_2)$, acting in accordance with the formula*

$$(K(x)f(x){*}v(x),\ s(x)) \stackrel{\text{def}}{=} (f(x),\ (v(y),\ K^*(x)s(x{+}y))), \ \ \forall s(x) \in K(E_2^*),$$

will be called the convolution of the generalized operator function $K(x)f(x)$ and the generalized function $v(x) \in K'_+(E_1)$.

Hence, we obtain the following equality (see [2], or p.340 in [14])

$$A\frac{\partial^k \delta(x)}{\partial x_i^k} * K(x)f(x) * v(x) = \frac{\partial^k (AK(x)f(x))}{\partial x_i^k} * v(x), \ \ R(K(\cdot)) \subset D(A),$$

which will be considered as the definition for the case of closed linear operator A.

Consider the differential operator $L(D) = \sum_{|\alpha|\le l} B_\alpha D^\alpha$ where B_α are closed linear operators from E to E, $\overline{\bigcap_{|\alpha|\le l} D(B_\alpha)} = E$, and the generalized operator function $L(\delta(x)) = \sum_{|\alpha|\le l} B_\alpha D^\alpha \delta(x)$ corresponding to it.

Definition 6. *The generalized operator function $E(x)$, such that $\forall u(x) \in K'_+(E)$ on the main space $K(E^*)$ the equality*

$$L(\delta(x)) * E(x) * u(x) = u(x)$$

holds, is called the fundamental operator function of the differential operator $L(D)$ in the class $K'_+(E)$.

Example 1. For the operator

$$(I\frac{\partial^{2N}\delta(x,y)}{\partial^N x \partial^N y} - A\delta(x,y)) = I\delta^{(N)}(x)\cdot\delta^{(N)}(y) - A\delta(x)\cdot\delta(y),$$

containing a bounded operator A, the generalized operator function $E_N(x,y) = U_N(A)(x,y)\theta(x,y)$, where

$$U_N(A)(x,y) \equiv \sum_{i=1}^{\infty} A^{i-1}\cdot\frac{x^{i\cdot N-1}}{(i\cdot N-1)!}\cdot\frac{y^{i\cdot N-1}}{(i\cdot N-1)!}$$

is the fundamental operator function of class $K'_+(E)$.

Example 2. For the operator $I\delta'(t)\cdot\delta(x) - A\delta(t)\cdot(\delta(x-\mu) - \delta(x))$ the generalized operator function

$$E(t,x) \equiv \sum_{k=1}^{\infty} e^{-At}\frac{(At)^k}{k!}\theta(t)\cdot\delta(x-k\mu)$$

is the fundamental operator function of class $K'_+(E)$.

Proposition 1. *If $E(x)$ is a fundamental operator function of the differential operator $L(D)$ of class $K'_+(E)$, then $\forall g(x) \in K'_+(E)$ the generalized function $u(x) = E(x) * g(x) \in K'_+(E)$ in the main space $K(E^*)$ satisfies the convolution equation $L(\delta(t)) * u(t) = g(t)$.*

3.2. **Fundamental operator functions of singular differential operators.** In this section our consideration will be reduced to the differential operator of the form

$$L(D) = B\frac{\partial^{2N}}{\partial x^N \partial y^N} - A$$

and the differential-difference operator

$$L(D)u = B\frac{\partial u}{\partial t} - A(u(t,x-\mu) - u(t,x))$$

where B is Fredholmian. For such operators, their fundamental operator functions are constructed in explicit form.

Theorem 4. *Suppose condition* 1 *in section* 1.2 *be satisfied, then the mapping* $B\delta^{(N)}(x)\cdot\delta^{(N)}(y)\ -A\delta(x)\cdot\delta(y))$ *in the space* $K'_+(E_2)$ *has the fundamental operator function of the form*

$$E_N(x,y)=\Gamma U_N(A\Gamma)(x,y)[I-Q]\theta(x,y)$$

$$-\sum_{i=1}^{n}\sum_{k=0}^{p_i-1}\left\{\sum_{j=1}^{p_i-k}\langle\cdot,\psi_i^{(j)}\rangle\varphi_i^{(p_i-k+1-j)}\right\}\delta^{(k\cdot N)}(x)\cdot\delta^{(k\cdot N)}(y).$$

Proof. See proof of theorem 1 in [2], or of theorem 1.1 on p.343-344 in [14]. □

As one of the corollaries for Theorem 4 we obtain

Corollary 1. *Suppose that condition* 1 *holds true, the function* $f(x,y)\in C(R_+^2)$ *has its values in* E_2. *Then the boundary value problem*

$$B\frac{\partial^2 u}{\partial x\partial y}=Au+f(x,y),\quad u|_{x=0}=\alpha(y),\ u|_{y=0}=\beta(x),$$

$u(x,y)\in C^2(R_+^2)$, $\alpha(x),\beta(x)\in C^1(R_+^1)$, $\alpha(0)=\beta(0)$, *has a generalized solution of the form*

$$u=E_1(x,y)*(f(x,y)\theta(x,y)+B\alpha'(y)\delta(x)\cdot\theta(y)$$

$$+B\beta'(x)\theta(x)\cdot\delta(y)+B\alpha(0)\delta(x)\cdot\delta(y)).$$

If, in addition, the singular components of the generalized solutions are equal to zero then, first, generalized solutions coincide with the continuous (classical) solutions, and, second, we can determine a set of boundary values $\alpha(y)$ and $\beta(x)$ as well as the right sides $f(x,y)$, for which such problems are solvable in the class of functions $C^2(R_+^2)$.

Theorem 5. *Suppose that condition* 1 *in section* 1.2 *is satisfied. Then the mapping* $B\delta'(t)\cdot\delta(x)-A\delta(t)\cdot(\delta(x-\mu)-\delta(x))$ *in the space* $K'(E_2)$ *has the fundamental operator function of the form*

$$E(t,x)=\sum_{k=0}^{\infty}\Gamma e^{-A\Gamma t}\frac{(A\Gamma t)^k}{k!}\theta(t)\cdot\delta(x-k\mu)*$$

$$\left\{I\delta(t)\cdot\delta(x)+\sum_{i=1}^{n}\sum_{j=1}^{p_i}(-1)^{j-1}\langle\cdot,\psi_i^{(p_i+1-j)}\rangle z_i\delta^{(j)}(t)\cdot\sum_{l=0}^{\infty}C_{l+j-1}^{j-1}\delta(x-l\mu)\right\}.$$

Proof. The proof of the theorem will be conducted for the case when $p_1=p_2=\ldots=p_n=1$. The proof of the general case is rather bulky, and so it will not be given. In accordance with the definition it is necessary to verify the validity of the equality

$$(B\delta'(t)\cdot\delta(x) - A\delta(t)\cdot(\delta(x-\mu)-\delta(x))) * E(t,x) * u(t,x) = u(t,x)$$

in the basic space $K(E_2^*)$. Let us substitute the expression for $E(t,x)$ into the left-hand side of this equality

$$(B\delta'(t)\cdot\delta(x) - A\delta(t)\cdot(\delta(x-\mu)-\delta(x))) * E(t,x) * u(t,x)$$
$$= \Big[I\delta(t)\cdot\delta(x) + F(t,x)\Big] * u(t,x),$$

where

$$F(t,x) = -\sum_{k=0}^{\infty} Q\left(e^{-t}\frac{t^k}{k!}\theta(t)\right)'\cdot\delta(x-k\mu) + \Bigg[I\delta(t)\cdot\delta(x)$$
$$-\sum_{k=0}^{\infty} Q\left(e^{-t}\frac{t^k}{k!}\theta(t)\right)'\cdot\delta(x-k\mu))\Bigg] * Q\delta'(t)\cdot\sum_{j=0}^{\infty}\delta(x-j\mu) = 0.$$

□

Corollary 2. *Suppose that the assumption of Theorem* 5 *is satisfied, the function* $f(t,x)\in BUC(R^1)$ [3] $\forall t \geq 0$, *has its values in* E_2. *Then the initial-value problem for the differential-difference equation*

$$B\frac{\partial u}{\partial t} = A(u(t,x-\mu)-u(t,x)) + f(t,x), \quad u|_{t=0} = u_0(x),$$

where $u_0(x)\in BUC(R^1)$, *has a generalized solution of the form*

$$u = E(t,x) * (f(t,x)\theta(t) + Bu_0(x)\delta(t)).$$

Remark 3. *In Theorem* 5, x *can be vector. Moreover, Theorem* 5 *assumes its generalization onto the case of differential-difference operators of the following form* $B\delta^{(N)}(t)\cdot\delta(x) - A\delta(t)\cdot(\delta(x-\mu)-\delta(x))$. The corresponding theorems have been proved by E.Grazhdantseva [4].

Conclusion. The approach presented in the paper employs essentially the technique of generalized Jordan sets [16], stable pseudoconverses of Noether operators and (P,Q)-commutativity of the operators [13] (in accordance with the Jordan structure of the equation's operator coefficients). This is right the technique that makes it possible to state correct initial-boundary-value problems for the differential equations with partial derivatives and with the Noether (unbounded) operator in the main part, as well as to reduce these problems to regular ones. This approach has given the possibility to construct generalized solutions with the finite singular part and to obtain solutions of a number of classes of singular differential equations in closed form [14], [2]. For the first time such an approach was applied by Sidorov [10] in 1972 for the purpose of constructing the asymptotic of branching solutions of nonlinear singular differential and integro-differential equations. Later the method was developed in a number of works and applied to different problems (see the bibliography in [14]). For the case of matrix coefficients, the technique of pseudoconverses of matrices and differential regularizers was developed in detail in the works by Yu.Ye.Boyarintsev, M.V.Bulatov, V.F.Chistyakov and others on the basis of classical methods of linear algebra, This technique was applied by these authors for the purpose of numerical solving algebro-differential equations. Our method can be applied in a more general situation of unbounded operator coefficients, and so, it can be employed not only for constructing the asymptotic of accurate solutions but also for development of stable numerical methods for some classes of Sobolev-type [15] singular differential equations with partial derivatives for which a theory of numerical methods still does not exist.

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