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QUIVERS, VECTOR BUNDLES AND COVERINGS OF SMOOTH CURVES

(submitted by B. N. Shapukov)

Abstract. Fix a finite quiver Q and consider quiver-bundles on smooth and connected projective curves. Let $f : X \to Y$ be a degree m morphism between such curves and $\tilde{E}$ a quiver bundle on Y. We prove that $\tilde{E}$ is semistable (resp. polystable) if and only if $f^*(\tilde{E})$ is semistable. Then we construct many stable quiver-bundles on bielliptic curves.

1. Introduction

Here we consider a problem related to stable and semistable quiver-bundles on a smooth and connected projective curve X, i.e. representations of a finite quiver into the the category of all coherent sheaves on X ([1], [6], [11], [12], [13], [14]). We assume the existence of a finite covering $f : X \to Y$ between smooth and connected projective curves and we want to use informations on quiver-bundles on Y to obtain informations on quiver-bundles on X (for the same quiver). For the case of plain vector bundles this approach was used several times (see e.g. [3] for hyperelliptic curves and [10] for bielliptic curves). In the case of plain vector bundles the starting point is the following result ([8], Lemmas 3.2.2

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and 3.2.3), which was a key step in the usual proof of the (generalized) Grauert-Mülich restriction theorem ([8], [9]).

Proposition 1. *Let X, Y integral projective varieties, $\mathcal{O}_Y(1)$ an ample line bundle on Y, E a vector bundle on Y and $f : X \to Y$ a finite covering. Set $\mathcal{O}_X(1) := f^*(\mathcal{O}_Y(1))$.*

(a) *$f^*(E)$ is $\mathcal{O}_X(1)$-semistable if and only if E is $\mathcal{O}_Y(1)$-semistable.*
(b) *$f^*(E)$ is $\mathcal{O}_X(1)$-polystable if and only if E is $\mathcal{O}_Y(1)$-polystable.*

Of course, only the “ if ” parts are not trivial. For our quiver-bundles we do not study or use the existence of moduli spaces (see e.g. [1], [5], [6], [11], [12], [13], [14]) and hence we may allow fairly general oriented finite quivers (e.g. with multiple paths). Here we give our set-up. Let Z be a smooth and connected projective curve and $Q = (S, A, s, t)$ (or just $Q = (S, A)$ for short) a finite quiver, i.e. two finite sets V, A, $V \neq \emptyset$, equipped with two functions $s : A \to V$ (the source), $t : A \to V$ (the target). Contrary to the assumptions made in [1], [6], [12] and [13] we allow multiple arrows, i.e. we allow the existence of $a, b \in A$ such that $s(a) = s(b)$, $t(a) = t(b)$ and $a \neq b$: recall that our aims are more modest: we only consider curves and do not consider moduli spaces. A quiver-bundle $\tilde{E}$ of Z is given by a finite set $\{E_v\}_{v \in V}$ of vector bundles on Z and a finite set $f_a : E_{s(a)} \to E_{t(a)}$ of homomorphisms. For every $v \in V$ fix $m_v \in \mathbb{R}$, $m_v > 0$, and call m_v the weight of the node v. We will require that each E_v has non-zero rank, so that its slope $\mu_{E_v} := \deg(E_v)/\mathrm{rank}(E_v)$ is well-defined and call $\mu(\tilde{E}) := \sum_{v \in V} m_v \mu(E_v)$ the slope (or the total slope or the total weighted slope) of $\tilde{E}$. A subobject $\tilde{F} = \{F_v, {f'}_a\}$ of $\tilde{E}$ will be said to be strict if all bundles F_v, $v \in V$, have non-zero rank and hence the total slope $\mu(\tilde{F})$ is well-defined. By the very definition of slope stability (or stability in the sense of Mumford and Takemoto) to check if $\tilde{E}$ is semistable or stable or polystable it is sufficient to check all its strict subobjects. This is one of the two reasons why we do not use the Hilbert polynomials of the bundles E_v to define total stability. The second reason is that degrees and slopes work very well when making a pull-back by a degree m coverings (they are just multiplied by m), while in general the genus is not multiplied by m. Hence with our definition of total slope the “ only if ” part in Theorem 1 below is obvious.

In section 2 we will prove the following result.

Theorem 1. *Let $f : X \to Y$ a degree m covering between smooth and connected projective curves and $\tilde{E}$ a quiver-bundle on Y.*

(a) *$f^*(\tilde{E})$ is semistable if and only if $\tilde{E}$ is semistable.*

(b) *$f^*(\tilde{E})$ is polystable if and only if $\tilde{E}$ is polystable.*

On X we take $m \cdot m_v$, $v \in V$, as weights to define the total weighted slope, but m_v, $v \in V$, defines the same notion of stability and semistability, because for any $\lambda > 0$ the weights $(m_v, v \in V)$ and $(\lambda m_v, v \in V)$ give the same notion of stability. We stress that in the definition of polystability any vector bundle appearing in a direct factor must have positive rank, otherwise its slope is not defined.

If $f : X \to Y$ is a degree m covering and E a vector bundle on Y, then $\deg(f^*(E)) \equiv 0 \pmod{m}$. This is a very strong restrictions for the semistable quiver-bundles on X obtained using Theorem 1. In section 3 we will show how to overcome this restriction (when $m = 2$ and $p_a(Y) = 1$, i.e. for bielliptic curves) for certain quivers and how to obtain stable (not just semistable or polystable) quiver-bundles on any bielliptic curve X with large genus (again, only for certain very specific quivers). In Theorem 2 we will consider the case of a multiple arrow, i.e. we fix an integer $n \geq 2$ and take $V := \{0, 1\}$, $A := \{f_1, \dots, f_n\}$, $s(f_i) = 0$ and $t(f_i) = 1$ for all i. In Theorem 3 we will consider the case of a source, i.e. we fix an integer $n \geq 1$ and take $V := \{0, 1, \dots, n\}$, $A := \{f_1, \dots, f_n\}$ with $s(f_i) = 0$ and $t(f_i) = i$, for all i. Taking duals, from Theorem 3 one gets the case of a so-called sink. In Theorem 4 we will consider the case of an oriented chain, i.e. we take $V := \{1, \dots, n\}$, $A := \{f_1, \dots, f_{n-1}\}$ with $s(f_i) = i$ and $t(f_i) = i + 1$, $1 \leq i \leq n - 1$. In Theorem 5 we will consider the case of a fork, i.e. we take $V := \{-1, 0, 1, \dots, n\}$, $A := \{f_0, f_1, \dots, f_n\}$, $s(f_0) = -1$, $t(f_0) = 0$, $s(f_i) = 0$ and $t(f_i) = 1$ for all $1 \leq i \leq n$.

2. Proof of Theorem 1

Let $Q = (V, S)$ be a finite quiver in the sense of section 1. We fix $m_v \in \mathbb{R}$, $m_v > 0$, for all $v \in V$. All quiver-bundles will be with respect to the quiver Q and semistability (resp. stability, resp. polystability) will be the total slope semistability (resp. stability, resp. polystability) with respect to the weights $m_v, v \in V$ discussed in section 1. Let Z be any smooth and connected projective curve.

Remark 1. Let $\tilde{E} = \{E_v, f_a\}$ and $\tilde{F} = \{F_v, f'_a\}$ be stable quiver-bundles with the same total slope and $\tilde{u} : \tilde{E} \to \tilde{F}$ a non-zero morphism. Call $u_v : E_v \to F_v$, $v \in V$, the associated morphism. By the definition of stability $\tilde{u}$ is an isomorphism if $u_v \neq 0$ for all $v \in V$. We do not know if this is true in general. It is true in certain cases, e.g. the case of triples ([7], Cor. 2.1).

Lemma 1. *Let $\tilde{E}$ be a quiver-bundle on Z. Then $\tilde{E}$ has a Harder-Narasimhan filtration; we do not claim its uniqueness.*

Proof. If $\tilde{E}$ is semistable, then there is nothing to prove. Hence we may assume that $\tilde{E}$ is not semistable. Since Q is finite and each bundle E_v, $v \in V$, associated to $\tilde{E}$ has a well-defined and finite rank and degree, there is a quiver-subsheaf $\tilde{F} = \{F_v, f_a\}_{v\in V, a\in A}$ of $\tilde{E}$ with $\mu(\tilde{F}) > \mu(\tilde{E})$ and with $\mu(\tilde{F})$ maximal; we stress that we may get $\tilde{F}$ such that to no F_v has rank zero and hence each $\mu(F_v)$ is well-defined, because only strict subobjects are used to test the semistability of a quiver-bundle. Since Z is smooth, the kernel of any homomorphism $m : G \to G'$ of vector bundles on Z is saturated in G, i.e. either $m \equiv 0$ or $\mathrm{Coker}(m) \cong \mathrm{Im}(m)$ is locally free. Hence, by the very definition of subobject and the maximality of $\mu(\tilde{F})$ we get that each F_v is a saturated subbundle of E_v, $v \in V$. If $F_v = E_v$ for at least one v, then we stop: this chain cannot be refined and we stop. If $F_v \subsetneq E_v$ for all v, then the family $\{f_a\}_{a\in A}$ induces a quiver-structure on the set of vector bundles $\{E_v/F_v\}$, $v \in V$. Now we may use induction on the total rank $\sum_{v\in V} \mathrm{rank}(E_v)$. □

Similarly, we have the following result.

Lemma 2. *Let $\tilde{E}$ be a semistable quiver-bundle on Z. Then $\tilde{E}$ has a Jordan-Hölder filtration.*

Although we do not claim uniqueness in Lemma 1, we are able to prove the following key lemma.

Lemma 3. *Let $\tilde{E} = \{E_v, f_a\}$ be a quiver-bundle on Z and $\tilde{A} := \{A_v, f'_a\}$, $\tilde{B} := \{B_v, f''_v\}$ strict subobjects of $\tilde{E}$ with maximal total slope and maximal total rank. Then $\tilde{A} = \tilde{B}$*

Proof. Since $\tilde{A}$ and $\tilde{B}$ have maximal slope, they are semistable. It is easy to define the subobject $\tilde{A} + \tilde{B}$ of $\tilde{E}$, because $f_a(A_{s(a)}) + f_a(B_{s(a)}) \subseteq A_{t(a)} + B_{t(a)}$ for every $a \in A$. Furthermore, $\tilde{A} + \tilde{B}$ is a quotient of $\tilde{A} \oplus \tilde{B}$, which is semistable. Since $\tilde{A} + \tilde{B}$ is a strict subobject of $\tilde{E}$ (with our definition of " strict " $\tilde{E}$ is a strict subobject of itself!), we easily conclude, by the maximality of the total rank of both $\tilde{A}$ and $\tilde{B}$ and the fact (proved in the proof of Lemma 1) that A_v and B_v are saturated in E_v for all $v \in V$. □

Proof of Theorem 1. Since $\mu(\tilde{F}) = m \cdot \mu(\tilde{F})$ for every quiver bundle $\tilde{F}$ on Y, the " only if " parts of both (a) and (b) are obvious. Furthermore, to check the " if " parts, it is sufficient to prove the semistabily or the polystability of a pull-back of $f^*(\tilde{E})$ by another finite covering. Hence we may reduce to the case in which f is a Galois covering ; here we use the assumption $\mathrm{char}(\mathbb{K}) = 0$. Call G the Galois group of the covering f. Now assume $\tilde{E}$ semistable and that $f^*(\tilde{E})$ is not semistable. Let $\tilde{F}$ be a strict subobject of $f^*(\tilde{E})$ with maximal slope and (among the strict subobjects with maximal slope) with maximal total rank. Hence $h^*(\tilde{F})$, $h \in G$, has the same properties. Hence $\tilde{F} = h^*(\tilde{F})$ (Lemma 3). Hence G acts on each F_v. Fix any $v \in V$. Notice that G acts trivially on the fiber of $f^*(E_v)$ over any ramification point, P, of f. Since F_v is saturated, in $f^*(E_v)$, G acts trivially also the fiber $F_v|\{P\}$ of F_v at P. By descent theory we get the existence of a subbundle M_v of E_v such that $F_v = f^*(M_v)$. Also the maps f'_a descend and hence we get a strict subobject $\tilde{M}$ of $\tilde{E}$ such that $\mu(\tilde{M}) = \mu(\tilde{F})/m$, contradicting the semistability of $\tilde{E}$. Now assume $\tilde{E}$ polystable. We just proved that $f^*(\tilde{E})$ is semistable. Let $\tilde{A}$ be a strict subobject of $f^*(\tilde{E})$ with maximal slope and minimal total rank. Hence $\tilde{A}$ is stable. Hence $\mu(\tilde{A}) = \mu(f^*(\tilde{E}))$. We just proved that $\sum_{h\in G} h^*(\tilde{A})$ descends to a quiver-bundle on Y. By the polystability of $\tilde{E}$ we get that $\tilde{C} := \sum_{h\in G} h^*(\tilde{A})$ is stable and that, calling C_v, $v \in V$, the vector bundles associated to $\tilde{C}$, either C_v is a proper saturated subbundle of $f^*(E_v)$ for all v, or $C_v = f^*(E_v)$ for all v, i.e. that either $f^*(\tilde{A}) = \tilde{C}$ is polystable or we may define a quotient quiver-bundle $f^*(\tilde{E})/\tilde{C}$ with all associated bundles with non-zero rank. Hence we may define the slopes of all bundles associated to $f^*(\tilde{E})/\tilde{C}$ and get part (b) by induction on the total rank of $\tilde{E}$.

3. Examples

We recall the following well-known lemma.

Lemma 4. *Let $f : X \to Y$ be a double covering between smooth and projective curves and E a stable vector bundle on Y. Assume that f is not étale, i.e. assume $p_a(X) \geq 2p_a(Y)$. Then $f^*(E)$ is stable.*

Proof. Since $\mathrm{char}(\mathbb{K}) \neq 2$, we have $f_*(\mathcal{O}_X) \cong \mathcal{O}_Y \oplus R$ for some $R \in \mathrm{Pic}(Y)$ (use the trace map). By Riemann-Hurwitz and the assumption $p_a(X) \geq 2p_a(Y)$ we get $\deg(R) < 0$. By Proposition 1 $f^*(E)$ is polystable and hence $f^*(E)$ is stable if and only if it is simple. By the projection formula we have $h^0(X, End(f^*(E))) = h^0(Y, f_*(End(f^*(E)))) = h^0(Y, End(E)) + h^0(Y, End(E) \otimes R) = 1 + h^0(Y, End(E) \otimes R)$. Since $End(E)$ is semistable ([8], §3.2) and $\deg(R) < 0$, we get $h^0(Y, End(E) \otimes R) = 0$, concluding the proof. □

From now on, in this paper we fix the following notation. Let Y be an elliptic curve, X a smooth and connected projective curve and

$f : X \to Y$ a double covering. We assume that f is not étale, i.e. we assume $g := p_a(X) \geq 2$ (Riemann-Hurwitz). Let $\sigma : X \to X$ denote the order two automorphism associated to f.

Theorem 2. *Fix an integer $n \geq 2$ and take $V := \{0,1\}$, $A := \{f_1, \dots, f_n\}$, $s(f_i) = 0$ and $t(f_i) = 1$ for all i; this quiver is called a multiple arrow. Fix vector bundles E_i on Y, $i \in \{0,1\}$, and set $r_i := rank(E_i)$ and $d_i := \deg(E_i)$. Assume $g := p_a(X) \geq 6$, $(d_0+1)/r_0 < (d_1-1)/r_1$, that each E_i, $i = 0,1$, is polystable and that no two of the indecomposable factors of any E_i are isomorphic. Fix integers a_i, $i = 0,1$, such that $a_0 \in \{2d_0-1, 2d_0-2\}$ and $a_1 \in \{2d_1+1, 2d_1+2\}$. Then there exists a stable quiver-bundle $\tilde{F} = \{F_0, F_1, \phi_1, \phi_n\}$ such that $rank(F_i) = r_i$ and $\deg(F_i) = a_i$ for all i.*

Proof. By Lemma 4 all vector bundles $f^*(E_i)$ are polystable and E_i and $f^*(E_i)$ have the same number of indecomposable factors. Fix a general $P \in Y$. Hence $\sharp(f^{-1}(P)) = 2$. Set $f^{-1}(P) = \{P', P''\}$.

(a) Here we assume $a_0 = 2d_0 - 1$ and $a_1 = 2d_1 + 1$. Let F_0 (resp. F_1) be the general bundle obtained from $f^*(F_0)$ (resp. $f^*(F_1)$) making a general negative (resp. positive) elementary transformation supported by P'. Call A_0 (resp. A_1) the general bundle obtained from E_0 making a negative (resp. positive) elementary transformation supported by P. Since the set of all bundles on X (resp. Y) obtained from a fixed bundle making a positive elementary transformation supported by P' (resp. P) is irreducible, to prove Theorem 2 using the bundles F_0, F_1 just defined we may assume $f^*(A_0) \subsetneqq F_0 \subsetneqq f^*(E_0)$ and $f^*(E_1) \subsetneqq F_1 \subsetneqq f^*(A_1)$; essentially, $f^*(A_i)$, $i = 0,1$, is obtained from $f^*(E_i)$ making two " exchanged by the involution σ " negative elementary transformations at P' and $P'' := \sigma(P')$, and we impose that F_i is obtained from one of them, say the one supported by P'. By [4], Cor. 2.4 and its dual, A_0 and A_1 are polystable and no two of the indecomposable factors of of one of them are isomorphic. Hence we may apply Lemma 4 to their indecomposable factors. Since A_0 is a subsheaf of E_0 and E_1 is a subsheaf of A_1, the maps $f_i : E_0 \to E_i$, $1 \leq i \leq n$, induce maps $u_i : A_0 \to A_1$. Set $v_i := f^*(u_i)$. Since F_0 is a subsheaf of $f^*(E_0)$ and $f^*(E_1)$ of F_1, each map $f^*(f_i)$ induces a map $\phi_i : F_0 \to F_i$. Set $\tilde{F} := \{F_0, F_1, \phi_1, \dots, \phi_n\}$ in order to obtain a contradiction we assume that $\tilde{F}$ is not stable with respect to the parameters $2m_0$ and $2m_1$. Take a strict subobject $\tilde{G} = \{G_0, G_1, \tau_1, \dots, \tau_n\}$ of $\tilde{F}$ with maximal slope.

First Claim: We may find F_0, F_1 as above with both F_0 and F_1 stable

Proof of the First Claim: We will only prove the stability of F_0, because the case of F_1 is very similar. Assume that F_0 is not stable and take a proper subsheaf A of F_0 with maximal slope and (among these subsheaves) with minimal rank. Since A has maximal slope, it is semistable and saturated in F_0, i.e. F_0/A is locally free. Since A has minimal rank among the subsheaves of F_0 with maximal slope, it is stable. We see F_0 as a subsheaf of the σ-invariant vector bundle $f^*(E_0)$. Hence we may see A as a subsheaf of $f^*(E_0)$. With this identification the subsheaf $\sigma^*(A)$ of $f^*(E_0)$ is defined. Since F_0 is obtained from $f^*(E_0)$ making a negative elementary transformation supported by a non-ramification point of f, for any ramification point O of f the fiber $F_0|\{O\} \cong \mathbb{K}^{\oplus r_{E_0}}$ of F_0 at O maps isomorphically onto $f^*(E_0)|\{O\}$. Since A is saturated in F_0, the fiber $A|\{O\}$ maps injectively into $f^*(E_0)|\{O\}$. Similarly, we see that the fibers of $A \cap \sigma^*(A)$ and $A + \sigma^*(A)$ over O maps injectively into the vector space $f^*(E_0)|\{O\}$. Since $f^*(E_0)$ comes from Y, σ acts trivially on the vector space $f^*(E_0)|\{O\}$ and hence on each of linear subspaces. Let B be any σ-invariant subsheaf B of $f^*(E_0)$ such that the natural map $B|\{O\} \to f^*(E_0)|\{O\}$ is injective for all ramification point O of f. Descent theory implies the existence of a subsheaf B' of E_0 such that $B = f^*(B')$. In particular B has even degree. The semistability of E_0 implies $\mu(B') \le \mu(E_0)$ with equality if and only if B' is a direct factor of E_0 and hence $\mu(B) \le \mu(f^*(E_0))$ with equality if and only if B is a direct factor of $f^*(E_0)$ (for the " equality " part of the latter statement we use Lemma 2). At this point we have all the ingredients to copy [4], pp. 543–544, using our assumption on $p_a(X)$, i.e. that there are " sufficiently many " ramification points on f. However, we may take a shortcut. Set $r := \mathrm{rank}(E_0)$, $\delta := \deg(A)$ and $\rho := \mathrm{rank}(A)$. By [4], Lemmas 3.2 and 3.1, the vector bundles F_i, $i = 0, 1$, are semistable. Hence $\mu(A) = \mu(F_0) = (2d_0 - 1)/r$. First assume that A is not saturated in $f^*(E_0)$ and call $\bar{A}$ its saturation. Hence $\mu(\bar{A}) = \mu(A) + 1/\rho > \mu(f^*(E_0))$, contradicting the semistability of $f^*(E_0)$. Hence A is saturated in $f^*(E_0)$. Thus $\sigma^*(A)$ is saturated in $f^*(E_0)$. Notice that $f^*(G_0) = F_0 \cap \sigma^*(F_0)$ (as subsheaves of $f^*(E_0)$). Since $A \oplus \sigma^*(A)$ is polystable and $D := A + \sigma^*(A)$ is a quotient of it, $\mu(D) \ge \mu(A)$, with equality if and only if D is isomorphic to a direct factor of $A \oplus \sigma^*(A)$; since $A \oplus \sigma^*(A)$ has only two direct factors, we get $\mu(A) = \mu(D)$ if and only if either $A = \sigma^*(A)$ or $A \cap \sigma^*(A) = 0$. Let K be the saturation of the sheaf D inside $f^*(E_0)$. Since D is σ-invariant, K is σ-invariant. Since both $A + \sigma^*(A)$ and K are σ-invariant, $\deg(K) - \deg(A + \sigma^*(A))$ is an even integer. Since $r > \rho$, $\mu(A) = (2d_0 - 1)/r = \mu(f^*(E_0)) - 1/r$, and $f^*(E_0)$ is polystable, we get

$K = D$, i.e. D is saturated in $f^*(E_0)$. Since it is also σ-invariant, there is a subbundle D' of E_0 such that $D = f^*(D')$. First assume $D \ne f^*(E_0)$. Hence E_0/D' is a non-zero vector bundle.

Second Claim: $h^0(X, f^*(E_0/D')) = h^0(Y, E_0/D')$.

Proof of the Second Claim: Since $\mathrm{char}(\mathbb{K}) \ne 2$, there is $R \in \mathrm{Pic}(Y)$ such that $\deg(R) = 1 - g$ and $f_*(\mathcal{O}_X) = \mathcal{O}_Y \oplus R$ (Riemann-Hurwitz). Hence $h^0(X, f^*(E_0/D')) = h^0(Y, E_0/D') + h^0(Y, (E_0/D') \otimes R$ (projection formula). Hence by Atiyah's classification of indecomposable vector bundles on any elliptic curve ([2], Part II), it is sufficient to show that every indecomposable factor of E_0/D' has slope $< 1 - g$. Let t denote the maximal slope of an indecomposable factor of E_0/D'. Since $g \ge 4$, it is sufficient to check the inequality $t < 3$. In the proof of the semistability of F_0 given in [4] the statement corresponding to the Claim is [4], Prop. 3.5; in that set-up it was proved the inequality $t < 2$, but with the stronger assumption (with our notation) $\mu(A) > \mu(F_0)$; here instead we only have $\mu(A) = \mu(F_0)$. Following the proof of [4], Prop. 3.5, in our set-up we get $t \le 2$, concluding the proof of the Second Claim.

Consider the exact sequence

$$0 \to A \cap \sigma^*(A) \to A \oplus \sigma^*(A) \to A + \sigma^*(A) \to 0 \qquad (1)$$

By construction the vector bundle $f^*(E_0/D')$ splits into two direct factors, $A/(A\cap\sigma^*(A))$ and $\sigma^*(A)/(A\cap\sigma^*(A))$. The projection of $f^*(E_0/D')$ onto its factor $A/(A\cap\sigma^*(A))$ does not come from an element of $H^0(Y, E_0/D')$, contradicting the Claim.

By the Second Claim we have $\mu(G_i) \le \mu(F_i)$ for $i = 0, 1$, with strict inequality for at least one index i. Since $m_0 > 0$ and $m_1 > 0$, we have $\mu(\tilde{G}) > \mu(\tilde{F})$, contradiction.

Now assume $D = f^*(E_0)$, i.e $D' = D$. Hence $2\rho \ge r$. Since $\deg(A) = \deg(\sigma^*(A)) = \rho(2d_0 - 1)/r$, while $\deg(A + \sigma(A)) = 2d_0/r$, we obtain $2\rho \ne r$. Hence $\rho > r/2$, i.e. $A \cap \sigma^*(A) \ne 0$. Notice that $A \cap \sigma^*(A) \subset f^*(A_0)$. Since $f^*(A_0)$ is polystable, we get $\deg(A \cap \sigma^*(A)) = \mathrm{rank}(A \cap \sigma^*(A)) \cdot \mu(A \cap \sigma^*(A)) \le (2\rho - r)\mu(f^*(A_0)) = (2\rho - r)(2d_0 - 2)/r$. By Lemma 4 every direct factor of $f^*(E_0)$ is σ-invariant. Apply the proof of the Claim directly to the vector bundles $A + \sigma^*(A)$ and $A \cap \sigma^*(A)$ and use again the exact sequence (1).

(b) Here we assume $a_0 = 2d_0 - 2$ and $a_1 = 2d_1 + 2$. Fix two general $P_1, P_2 \in Y$. Hence $\sharp(f^{-1}(P_1)) = \sharp(f^{-1}(P_2)) = 2$. Set $\{P_i', P_i''\} := f^{-1}(P_i)$, $i = 1, 2$. Copy the proof of part (a) with the following modifications. Here A_0 (resp. A_1) is obtained from E_0 (resp. E_1) making two general negative (resp. positive) elementary transformations supported by P_1 and P_2. F_0 (resp. F_1) is obtained from $f^*(E_0)$ (resp. $f^*(E_1)$) making two general negative (resp. positive) elementary transformations supported by P'_1 and P'_2. Here we use that g is large to obtain the Claim proved in part (a) under our numerical assumptions.

(c) The cases " $a_0 = 2d_0 - 1$ and $a_1 = 2d_1 + 2$ " and " $a_0 = 2d_0 - 2$ and $a_1 = 2d_1 + 1$ " are similar and may be done as in part (b). □

Theorem 3. *Fix an integer $n \geq 1$ and take $V := \{0, 1, \dots, n\}$, $A := \{f_1, \dots, f_n\}$ with $s(f_i) = 0$ and $t(f_i) = i$, for all i; this quiver is called a source. Fix vector bundles E_i on Y, $i \in \{0, 1, \dots, n\}$, and set $r_i :=$ rank(E_i), and $d_i := \deg(E_i)$. Assume $g \geq 6$, $(d_0 + 1)/r_0 < (d_i - 1)/r_i$ for all $i \in \{1, \dots\}$, that each E_i is polystable and that no two of the indecomposable factors of any E_i are isomorphic. Fix integers a_i, $0 \leq i \leq n$, such that $a_0 \in \{2d_0 - 1, \}$ and $a_j \in \{2d_j, 2d_j + 1\}$ for all $1 \leq j \leq n$. Then there exists a stable quiver-bundle $\tilde{F} = \{F_v, \phi_a\}$ such that rank$(F_i) = r_i$ and $\deg(F_i) = a_i$ for all $i \in \{0, 1, \dots, n\}$. Furthermore, we may find such a quiver-bundle with all F_v, $v \in V$, stable.*

Proof. By Lemma 4 all vector bundles $f^*(E_i)$ are polystable and E_i and $f^*(E_i)$ have the same number of indecomposable factors. It is sufficient to copy the proof of Theorem 2 with the following modification. If $a_j = 2d_j$ for some j, then from E_j and $f^*(E_j)$ we make first a sufficiently general negative elementary transformation and then we apply to this bundle a new sufficiently general positive elementary transformation (based on a different point of the curve). With our assumptions on g the Second Claim made in the proof of Theorems 2 is OK in our different set-up. □

In the same way we get the following two examples.

Theorem 4. *Take $V := \{1, \dots, n\}$, $A := \{f_1, \dots, f_{n-1}\}$ with $s(f_i) = i$ and $t(f_i) = i + 1$, $1 \leq i \leq n - 1$; this quiver is called an oriented chain. Fix vector bundles E_i on Y, $i \in \{1, \dots, n\}$, and set $r_i :=$ rank(E_i), and $d_i := \deg(E_i)$. Assume $p_a(Y) = 1$, $g := p_a(X) \geq 6$, $(d_i + 1)/r_i < (d_{i+1} - 1)/r_{i+1}$ for all $i \in \{1, \dots, n - 1\}$, that each E_i is polystable, and that no two of the indecomposable factors of any E_i are isomorphic. Fix $a_j \in \{2d_j - 1, 2d_j\}$ for all $j \in \{1, \dots, n - 1\}$. Then there exists a stable quiver-bundle $\tilde{F} = \{F_v, \phi_a\}$ such that rank$(F_i) = r_i$ and $\deg(F_i) = a_i$ for all $i \in \{1, \dots, n\}$. Furthermore, we may find such a quiver-bundle with all F_v, $v \in V$, stable.*

Theorem 5. *Take $V := \{-1, 0, 1, \dots, n\}$, $A := \{f_0, f_1, \dots, f_n\}$, $s(f_0) = -1$, $t(f_0) = 0$, $s(f_i) = 0$ and $t(f_i) = 1$ for all $1 \leq i \leq n$; this quiver is often called a fork. Fix vector bundles E_i on Y, $i \in \{-1, 01, \dots, n\}$,* *and set* $r_i := rank E_i$ *and* $d_i := \deg(E_i)$*. Assume* $g \ge 6$*,* $(d_{-1}+1)/r_{-1} < (d_0-1)/r_0$ *and* $(d_0+1)/r_0 < (d_i-1)/r_i$ *for all* $i \in \{1, \dots, n\}$*. Fix integers* $a_i \in \{2d_i-1, 2d_i\}$*,* $i \in \{-1, 0, 1, \dots, n\}$*. Then there exists a stable quiver-bundle* $\tilde{F} = \{F_v, \phi_a\}$ *such that rank*$(F_i) = r_i$ *and* $\deg(F_i) = a_i$ *for all* $i \in \{1, \dots, n\}$*. Furthermore, we may find such a quiver-bundle with all* F_v*,* $v \in V$*, stable.*

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