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# OPERATOR VALUED PROBABILITY THEORY

Abstract. We outline an extention of probability theory based on positive operator valued measures. We generalize the main notions from probability theory such as random variables, conditional expectations, densities and mappings. We introduce a product of extended probability spaces and mappings, and show that the resulting structure is a monoidal category, just as in the classical theory.

Contents

## 1. Introduction

In this paper we present an extension of standard probability theory. An extended probability space is defined to be a normalized positive operator valued measure defined on a measurable space of events. This notion of extended probability space includes probability spaces and spectral measures as important special cases. The use of the word probability in this context is justified by showing that extended probability spaces enjoy properties analog to all the basic properties of classical probability spaces. Random vectors are defined as a generalization of the usual Hilbert space of square integrable functions. This generalization is well known in the literature and was first described by Naimark. Expectation and conditional expectation is defined for extended probability spaces by orthogonal projections in complete analogy with probability spaces.

The introduction of probability densities presents special problems in the context of extended probability spaces. For the case of probability spaces a probability density is any normalized positive integrable function, whereas for the case of extended probability spaces it turns out that the right notion is not a density but a half density. These half densities are elements in a Hilbert module of length one. Special cases of such half densities are well known in quantum mechanics where they are called wave functions. We define a random operator to be a linear operator on the space of half densities. The expectation of random operators are operators acting on the Hilbert space underlying the extended probability space. For the case of probability spaces the notion of random vectors and random operators coincide.

We introduce mappings or morphisms of extended probability spaces through a generalization of the notion of absolute continuity in probability theory. Half densities plays a pivotal role in this generalization. We show that the morphisms can be composed and that extended probability spaces and morphisms forms a category just as for probability spaces. The Naimark construction extends to morphisms and in fact defines a functor on the category of extended probability spaces.

Extended probability spaces can be multiplied and we furthermore show that this multiplication can be extended to morphisms in such a way that it defines a monoidal structure on the category of extended probability spaces. This is in complete analogy with the case of probability spaces and testify strongly to the naturalness of our constructions.

We do not in this paper attempt to give any interpretation of extended probabilities beyond the one implied by the strong structural analogies that we have shown to exists between the categories of probability spaces and extended probability spaces. It is well known that the interpretation of the classical Kolmogorov formalism for standard probability theory is not without controversy as the old debate between frequentists and Bayesians, among others, clearly demonstrate. Our theory of extended probability spaces is evidently a generalization of the Kolmogorov framework and it might be hoped that this enlarged framework will put some of the controversy in a different light. As a case in point note that extended probabilities are in general only partially ordered. The notion of partially ordered probabilities has been discussed and argued over for a very long time. In our theory of extended probability spaces, ordered and partially ordered probabilities lives side by side and enjoy the same formal categorical properties.

## 2. Extended probability spaces

In this section we will make some technical assumptions that will assumed to hold throughout this paper. These assumptions are not necessarily the most general ones possible.

A measurable space [5] is a pair $X = \langle \Omega_X, \mathcal{B}_X \rangle$ where $\Omega_X$ is a set and $\mathcal{B}_X$ is a $\sigma$-algebra on $\Omega_X$. A measurable map $f : X \to Y$ is a map of sets $\Omega_X \to \Omega_Y$ such that $f^{-1}(A) \in \mathcal{B}_X$ for all $A \in \mathcal{B}_Y$. Let $\Omega$ be a set and let $\tau$ be a topology on $\Omega$. In this paper the term topology is taken to mean a second countable,locally compact Hausdorff topology [3]. Note that any such space is metrizable,Polish and $\sigma$-compact. The Borel structure corresponding to a topology $\tau$ is the smallest $\sigma$-algebra containing the topology $\tau$ and is denoted by $\mathcal{B}(\tau)$. A Borel space is a measurable space where the $\sigma$-algebra is a Borel structure. Any continuous map $f : \langle \Omega_X, \tau_X \rangle \to \langle \Omega_Y, \tau_Y \rangle$ is measurable with respect to the Borel structures $\mathcal{B}(\tau_X)$ and $\mathcal{B}(\tau_Y)$. Borel sets are the observable events to which we must assign probabilities.

Let now $\langle \Omega_X, \mathcal{B}(\tau_X) \rangle$ be a Borel space and let $\mathcal{O}(H_X)$ be the real $C^*$ algebra [4] of bounded operators on the real Hilbert space $H_X$. A positive operator valued measure (POV) [1] defined on $\langle \Omega_X, \mathcal{B}(\tau_X) \rangle$ is a map $F_X$ from $\mathcal{B}(\tau_X)$ to $\mathcal{O}(H_X)$ such that $F_X(\emptyset) = 0$,$F_X(\Omega_X) = 1$. The map $F_X$ is assumed to be finitely additive on disjoint union of sets and for any increasing sequence of sets $\{V_i\}$ satisfy the following continuity condition

$$F_X(\lim_{i \longrightarrow \infty} V_i) = \sup\{F_X(V_i) \mid i = 1, 2, 3, ....\},$$

where the supremum is taken with respect to the usual partial ordering of self adjoint operators. The supremum always exists since the sequence

$\{F_X(V_i)\}$ is increasing and bounded above by $F_X(\lim_{i\longrightarrow\infty} V_i)$. The continuity condition implies that $F_X$ is additive on countable disjoint unions.

$$F_X(U_{i=1}^{\infty} V_i) = \sum_{i=1}^{\infty} F_X(V_i),$$

where the sum converges in the strong operator topology, that is, pointwise convergence in norm.

A positive operator valued measure is a spectral measure if $F_X(V)$ is a projector for all $V \in \mathcal{B}$. A necessary and sufficient condition for a POV, $F_X$,to be a spectral measure is that it is multiplicative

$$F_X(V_1 \cap V_2) = F_X(V_1)F_X(V_2).$$

We are now ready to define our first main object

**Definition 1.** *A extended probability space $X$ is a triple $X = \langle \Omega_X, \mathcal{B}(\tau_X), F_X\rangle$ where $F_X : \mathcal{B}(\tau_X) \longrightarrow \mathcal{O}(H_X)$ is a positive operator valued measure.*

Note that a probability space $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X\rangle$ can be identified with a extended probability space in many different ways. In fact for any given Hilbert space $H_X$ we can identify the probability space with a extended probability space $X = \langle \Omega_X, \mathcal{B}(\tau_X), F_X\rangle$ where $F_X(V) = \mu_X(V)I_{H_X}$.

## 3. Random vectors

In standard probability theory quadratic integrable random variables and their expectation plays an important role. We will now review the classical Naimark construction of the analog of such random variables for the case of extended probability spaces. We will call such random variables random vectors. The space of random vectors forms a Hilbert spaces and we use this structure to define expectation and conditional expectation by orthogonal projections in complete analogy with the standard case.

3.1. **The space of random vectors.** Let $\langle \Omega, \mathcal{B}, F\rangle$ be a extended probability space and let $S$ be the linear space of simple measurable functions $v : \Omega \to H$. The linear structure is defined through pointwise operations as usual. Elements in $S$ can be written as finite sums of characteristic functions.

$$v = \sum_i \xi_i \theta_{V_i} \ ,$$

where $\{V_i\}$ is a $\mathcal{B}$ -measurable partition of the set $\Omega$. We define a pseudo inner product on $S$ by

$$\langle v, w\rangle = \sum_{i,j} \langle F(V_i \cap W_j)\xi_i, \eta_j\rangle_H,$$

where $v = \sum_i \xi_i \theta_{V_i}$, $w = \sum_j \eta_j \theta_{W_j}$ and $\langle\ \rangle_H$ is the inner product in the Hilbert space $H$. The product is not definite. In fact we have

$$\langle v, v\rangle = 0$$

$$\Updownarrow$$

$$\sum_i \langle F(V_i)\xi_i, \xi_i\rangle_H = 0$$

$$\Updownarrow$$

$$\langle F(V_i)\xi_i, \xi_i\rangle = 0$$

for all $i$.

The last identity follows from the fact the $F(V_i)$ is a positive operator. So for any simple function $v = \sum \xi_i \theta_{V_i}$ we have $\langle v, v\rangle = 0$ if and only if $F(V_i)\xi_i \perp \xi_i$ for all $i$. This is of course true if $V_i$ is of $F$ measure zero but it can also be true if $F(V_i) \neq 0$ but $\xi_i$ is in the kernel of $F(V_i)$.

Since $\langle\ \rangle$ is a pseudo inner product the set of elements of length zero, $\langle v, v\rangle = 0$, form a linear subspace and we can divide $S$ by this subspace. and thereby get a, in general, incomplete inner product space. The completion of this space with respect to the associated norm is by definition the space of random vectors and is a Hilbert space. We will use the notation $L_2(\mathcal{B}, F)$ or just $L_2(F)$ for this space in analogy with the classical notation $L_2(\mu)$. The set of equivalence classes of simple functions $[v]$ evidently form a dense set in $L_2(F)$. Denote this dense subspace by $T(F)$. We have a well defined isometric embedding $\pi$ of $H$ into $L_2(F)$ defined by

$$\pi(\xi) = [\xi\theta_\Omega].$$

We also have a spectral measure $P : \mathcal{B} \to \mathcal{O}(L_2(F))$. On the dense set $T(F)$ the spectral measure is given by

$$P(\alpha)[v] = [\sum_i \xi_i \theta_{V_i \cap \alpha}],$$

where $v = \sum \xi_i \theta_{V_i}$.

In fact the existence of this spectral measure is the whole point of the Naimark construction. It show that by extending the Hilbert space one can turn any POV into a spectral measure. This idea has been generalized by Sz.-Nagy and J. Arveson into a theory for generating representations of $\acute{*}$-semigroups but we will not need any of these generalization in our work.

As our first example let $\mu$ be a measure on the measurable space $\langle\Omega, \mathcal{B}\rangle$ and let $H$ be a Hilbert space. Define a positive operator valued measure on $\langle\Omega, \mathcal{B}\rangle$ acting on $H$ by

$$F(U) = \mu(U)1_H.$$

For this case we have

$$\langle[v],[w]\rangle = \sum_{i,j}\langle\mu(V_i \cap W_j)\xi_i, \eta_j\rangle_H = \sum_{i,j}\langle\xi_i, \eta_j\rangle_H \mu(V_i \cap W_j) = \int\langle v, w\rangle_H d\mu,$$

where for any $H$ valued functions $f, g$ we define $\langle f, g\rangle_H(x) = \langle f(x), f(x)\rangle_H$. Thus for this case our space $L_2(F)$ will be the space of $H$ valued function elements such that $\displaystyle\int\langle f, f\rangle_H d\mu < \infty$. When $H = \mathbb{C}$ the space $L_2(F)$ turns into the space of square integrable complex valued functions $L_2(\mu)$.

As our second example let $H$ be two dimensional and let a basis $\{\xi_1, \xi_2\}$ be given. With respect to this basis we have

$$F(U) = \begin{bmatrix} \mu(U) & \omega(U) \\ \omega(U) & \nu(U) \end{bmatrix},$$

where $\mu$ and $\nu$ and $\omega$ are signed measures. In order for $F(U)$ to be positive for all $U$ it is easy to see that $\mu$ and $\nu$ must be positive measures and that the following inequality must hold

$$\omega(U)^2 \leq \mu(U)\nu(U).$$

Any function $f : \Omega \to H$ determines a pair of real valued functions $\{f_1, f_2\}$ through $f(x) = f_1(x)\xi_1 + f_2(x)\xi_2$. The inner product in $L_2(F)$ is given in terms of the measures $\mu$,$\nu$ and $\omega$ as

$$\begin{aligned}&\langle(f_1, f_2),(g_1, g_2)\rangle \\ &= \int f_1 g_1 d\mu + \int f_2 g_2 d\nu + \int(f_1 g_2 + f_2 g_1)d\omega.\end{aligned}$$

Similar expressions for the inner product in $L_2(F)$ exists for any finite dimensional Hilbert space $H$.

3.2. **The expectation of random vectors.** Recall that we have a isometric embedding $\pi : H \to L_2(F)$ defined by

$$\pi(\xi) = [\xi\theta_\Omega].$$

Note that the image $\pi(H) \subset L_2(F)$ is a closed subspace and therefore the orthogonal projection onto $\pi(H)$ exists. Let $Q_H$ be this orthogonal projection.

**Definition 2.** *The expectation of a random vector $f \in L_2(F)$ is the unique element $E(f) \in H$ such that*

$$\pi(E(f)) = Q_H(f).$$

The following result is a immediate consequence of the definition

**Proposition 3.** *The expectation is a surjective continuous linear map $: L_2(F) \to H$ and is the adjoint of the embedding $\pi$*

$$\langle f, \pi(\xi)\rangle = \langle E(f), \xi\rangle \quad \forall \xi \in H.$$

Note that adjointness condition uniquely determines the expectation. In fact we could define the expectation to be the adjoint of the embedding $\pi$.

Using this proposition it is easy to verify that the expectation of a simple function element $[v]$ where $v = \sum \xi_i \theta_{V_i}$ is given by

$$E([v]) = \sum_i F(V_i)(\xi_i).$$

This example makes it natural to introduce a integral inspired notation for the expectation

$$E(f) \stackrel{\text{def}}{=} \int dF f.$$

Note that it is natural to put the differential $dF$ in front of $f$ to emphasize the fact that $F$ is a operator valued measure that acts on the function valued of $f$.

Let $\{\xi_i\}$ be an orthonormal basis for $H$. For general elements $f$ the following formula holds

$$E(f) = \sum_i \langle f, \pi(\xi_i)\rangle \xi_i.$$

3.3. **Conditional expectation.** Let $\mathcal{A} \subset \mathcal{B}$ be a $\sigma$-subalgebra. We can restrict the POV $F$ to $\mathcal{A}$ and will in this way get the Hilbert space $L_2(\mathcal{A}, F)$ of $\mathcal{A}$ measurable random vectors. We obviously have a isometric embedding of $L_2(\mathcal{A}, F)$ into $L_2(\mathcal{B}, F)$. Thus $L_2(\mathcal{A}, F)$ can be identified with a closed subspace of $L_2(\mathcal{B}, F)$ and therefore the orthogonal projection $Q_{\mathcal{A}} : L_2(\mathcal{B}, F) \to L_2(\mathcal{A}, F)$ is defined. In complete analogy with the classical case we now define

**Definition 4.** *The conditional expectation of a element $f \in L_2(\mathcal{B}, F)$ is given by*

$$E_{\mathcal{A}}(f) = Q_{\mathcal{A}}(f) \in L_2(\mathcal{A}, F).$$

It is evident that $L_2(\mathcal{A}, F)$ is isomorphic to $H$ when $\mathcal{A} = \{\Omega, \emptyset\}$ and that for this case we have $E_{\mathcal{A}}(f) = \pi(E(f))$. Let us consider the next simplest case when $\mathcal{A}$ is generated by a partition $\{A_1 ... A_n\}$ where $\Omega = \cup A_i$ and $A_i \cap A_j = \emptyset$ when $i \neq j$. We need the following result

**Proposition 5.** *Let $F(A_i)$ for $i = 1..n$ have closed range. Then $L_2(\mathcal{A}, F) = T(\mathcal{A}, F)$.*

*Proof.* Let $[v_n]$ be a Cauchy sequence in the inner product space $T(\mathcal{A}, F)$. This means that $||[v_n] - [v_m]||^2 \to 0$ when $m$ and $n$ goes to infinity. But $v_n = \sum_i \xi_i^n \theta_{A_i}$ and since $F(A_i)$ are positive operators we get

$$\sum_i \langle F(A_i)(\xi_i^n - \xi_i^m), \xi_i^n - \xi_i^m \rangle \to 0$$
$$\Downarrow$$
$$\langle F(A_i)(\xi_i^n - \xi_i^m), \xi_i^n - \xi_i^m \rangle \to 0$$

for all $i$.

Let $L_i = F(A_i)(H)$ be the range of $F(A_i)$ and let $L^{\perp}$ be the orthogonal complement of $L_i$. We have $L_i^{\perp} = Ker(F(A_i))$ and since $L_i$ by assumption is a closed subspace we have the decomposition $H = L_i \oplus L_i^{\perp}$ . Write $\xi_i^n = r_i^n + t_i^n$ with $r_i^n \in L_i^{\perp}$ and $t_i^n \in L_i$. We then have by orthogonality

$$\langle F(A_i)(t_i^n - t_i^m), t_i^n - t_i^m \rangle \to 0.$$

Clearly $F(A_i)|_{L_i} : L_i \to L_i$ is a positive, bounded, injective and surjective map.

Let $T_i : L_i \to L_i$ be the square root of this operator. It is also a positive bounded injective and surjective map and therefore has a bounded inverse. From the previous limit we can conclude that

$$\langle T_i(t_i^n - t_i^m), T_i(t_i^n - t_i^m) \rangle \to 0.$$

Thus $\{T_i(t_i^n)\}$ is a Cauchy sequence in $L_i$ and since $L_i$ is closed there exists a element $y_i \in L_i$ such that $T_i(t_i^n) \to y_i$. From the previous remarks the element $\xi_i = T_i^{-1}(y_i) \in L_i$ exists and $\lim_{n\to\infty} t_i^n = \lim_{n\to\infty} T_i^{-1}(T_i(t_i^n)) =$

$T_i^{-1}(\lim_{n\to\infty} T_i(t_i^n)) = T_i^{-1}(y_i) = \xi_i$. If we let $v = \sum \xi_i\theta_{A_i}$we have

$$\begin{aligned}
&||[v_n] - [v]||^2 \\
&= \sum_i \langle F(A_i)(\xi_i^n - \xi_i), \xi_i^n - \xi_i\rangle \\
&= \sum_i \langle T_i(t_i^n - \xi_i), T_i(t_i^n - \xi_i)\rangle \\
&= \sum_i \langle T_i(t_i^n) - y_i, T_i(t_i^n) - y_i\rangle \\
&= \sum_i ||T_i(t_i^n) - y_i|| \to 0.
\end{aligned}$$

Therefore $T(\mathcal{A}, F)$ is complete. □

The assumption in the proposition holds for example if $H$ is finite dimensional or if $H$ is infinite dimensional but all the $F(A_i)$ are orthogonal projectors or isomorphisms. For the classical measure case $H \approx \mathbb{R}$ and the proposition is true.

Let $v = \sum \xi_j\theta_{V_j}$ be a simple function in $L_2(\mathcal{B}, F)$. Then by the previous proposition the conditional expectation must be of the form $Q_{\mathcal{A}}(v) = \sum \eta_i\theta_{A_i}$. It is uniquely determined by the conditions $\langle v - Q_{\mathcal{A}}(v), \xi\theta_{A_j}\rangle_H = 0$ for all $\xi \in H$ and $j = 1..n$. These conditions give us the following systems of equations for the unknown vectors $\eta_i$:

$$F(A_i)\eta_i = \sum_k F(V_k \cap A_i)\xi_k$$

for any $i$.

This systems does not have a unique solution in $H$ but all solutions represents the same element in $L_2(\mathcal{A}, F) = T(\mathcal{A}, F)$. For the special case $v = \xi_0\theta_C$ we get the simplified system

$$F(A_i)\eta_i = F(C \cap A_i)\xi_0.$$

When $\dim H = 1$ and $F(A_i) = \mu(A_i)$ we get the usual classical expression for the conditional expectation of $C$ given $\mathcal{A}$.

## 4. Densities and random operators

Densities are important for most applications of probability theory. For us they will make their appearance when we seek to generalize the relation of absolute continuity between measures to the context of positive operator valued measures. This generalization will play a pivotal role when we define maps between extended probability spaces. The generalization of the notion of density to the case of operator measures turns out to be surprisingly subtle.

4.1. **The Hilbert module of half densities.** Let $\nu$ be a measure. A density is a positive measurable function $\rho$ such that $\int \rho d\nu = 1$. Using this density we can define a new measure

$$\mu(V) = \int_V \rho d\nu.$$

If we try to generalize this formula directly to the case of POV measures we run into problems.

Let $F$ be a POV defined on a measurable space $\langle \Omega, \mathcal{B}(\tau) \rangle$ and let $\rho$ be a function as above. Then we can certainly define a new POV measure by the following formula

$$E(V) = \int_V \rho dF.$$

There is nothing inconsistent in this definition, the only problem is that it is very limited. In fact if $\Omega$ is a finite set then any POV measure on $\Omega$ is given by a finite set $\{F_i\}$ of positive operators between zero and the identity with the single condition $\sum F_i = 1$. If $E$ is the new POV determined by the above formula then we have $E_i = \rho_i F_i$ for some set of numbers $\{\rho_i\}$. Thus each $E_i$ is proportional to $F_i$.

Now if the numbers $\rho_i$ were changed into positive operators we could produce a much more general $E$ starting from a given $F$. We would thus be considering a formula like

$$E(V) = \int_V \rho dF,$$

where $\rho$ is a positive operator valued function. However even if we could make sense of the proposed integral we would have problems. This is because the product of positive operators is positive if and only if they commute. This would put a highly nontrivial constraint on the allowed densities, constraints it would be difficult to verify and keep track of.

There is however a natural way out of these problems. It is very simple to verify that if $F$ is a POV measure acting on $H$ and $Q$ a operator, then $QFQ^*$ is a new POV measure. This suggest that we consider a density to be a operator valued function $\varphi$ such that

$$\int_\Omega \varphi dF \varphi^* = 1. \tag{1}$$

We could then use this density to define a new POV measure by

$$E(V) = \int_V \varphi dF \varphi^*. \tag{2}$$

On a formal level this now looks fine, the only remaining problem is to make sense of the proposed integrals. We will now proceed to do this.

Let

$$V = \{s = \sum_i s_i \theta_{V_i} \ \mid \ s_i \in \mathcal{O}(H) \ \ V_i \in \mathcal{B}(\tau)\},$$

where $\{V_i\}$ form a measurable partition of $\Omega$. These are simple measurable operator valued functions. The set $V$ is a real linear space through pointwise operations as usual. We can define a left action of $\mathcal{O}(H)$ on $V$ in the following way

$$as = \sum_i (as_i)\theta_{V_i}.$$

This action clearly makes $V$ into a left module over the real $C^*$- algebra $\mathcal{O}(H)$. Define an $\mathcal{O}(H)$ valued product on $V$ through

$$\langle s, t\rangle = \sum_{i,j} s_i F(V_i \cap W_j) t_j^*,$$

where $s = \sum s_i \theta_{V_i}$ and $t = \sum t_j \theta_{W_j}$. This product is clearly bilinear over the real numbers.

**Proposition 6.** *The following properties*

$$\begin{aligned} \langle s, s\rangle &\geq 0, \\ \langle as, t\rangle &= a\langle s, t\rangle, \\ \langle s, t\rangle &= \langle t, s\rangle^*, \\ \langle s, at\rangle &= \langle s, t\rangle a^* \end{aligned}$$

*hold.*

Thus the product is like a Hermitian product where the role of complex numbers are played by the elements of the real $C^*$-algebra $\mathcal{O}(H)$. Such structures have been known and studied for a long time. They leads, as we will see, in a natural way to the idea that probability densities for operator measures are elements in a Hilbert module. Our main sources for the theory of Hilbert modules are the paper [10] and the book [2]. Chapters on Hilbert modules can also be found in the books [7] and [13].

Note that the product we have constructed is not positive definite. In fact, since the sum of positive operators in a real $C^*$-algebras is zero only if each operator is zero, the identity $\langle s, s\rangle = 0$ holds if and only if

$$s_i F(V_i) s_i^* = 0 \quad \text{for all } i.$$

These identities can easily be satisfied for nonzero operators $s_i$. In fact if $F(V_i)$ are projectors and $s_i$ are projectors orthogonal to $F(V_i)$ then the equations are clearly satisfied. In order to make the product definite we will need to divide out by the set of simple functions whose square is zero $\langle s, s\rangle = 0$. In order to do this we will need the analog of the Cauchy-Swartz inequality.

For any element $s \in V$ we know that $\langle s, s\rangle \geq 0$ and therefore there exists a positive operator $h$ such that $h^2 = \langle s, s\rangle$. Denote this operator by $|s|$. Thus we have $|s|^2 = \langle s, s\rangle$. Also for any element $s \in V$ define a real number $||s||$ by

$$||s||^2 = ||\langle s, s\rangle||$$

where $||\langle s, s\rangle||$ is the operator norm of the positive operator $\langle s, s\rangle$. With these definitions at hand we can now state the following Cauchy Swartz inequalities for $V$. The proof of this proposition is an adaption of the proof in [13] to the case of real $C^*$ algebras.

**Proposition 7.** *The following forms of the Cauchy-Swartz inequality*

$$\langle s, t\rangle\langle t, s\rangle \leq |s|^2 ||t||^2,$$
$$||\langle s, t\rangle|| \leq ||s||\ ||t||$$

*hold.*

*Proof.* A positive linear functional, $\omega$ ,on $\mathcal{O}(H)$ is a real valued linear functional such that $\omega(a) \geq 0$ whenever $a \geq 0$. A state on $\mathcal{O}(H)$ is a positive linear functional such that $\omega(1) = 1$ and $\omega(a) = \omega(a^*)$. The main property that makes states useful in $C^*$ algebra theory is that if $a \neq 0$ there exists a state such that $\omega(a) = ||a||$. From this it follows immediately that if $\omega(a) = 0$ for all states $\omega$ then $a = 0$ and this implies that if $\omega(a) \leq \omega(b)$ for all states then $a \leq b$. In this way verification of inequalities in a $C^*$ algebra is reduced to the verification of numerical inequalities. Also recall that in any real $C^*$ -algebra the following important inequality holds [4]

$$\omega(a^*b^*ba) \leq ||b^*b||\omega(a^*a)$$

For any given state $\omega$ define $(s, t)_\omega = \omega(\langle s, t\rangle)$. It is evident that $(\,,\,)_\omega$ is a pseudo inner product on $V$. It therefore satisfy the Cauchy-Swartz inequality $(s, t)_\omega^2 \leq (s, s)_\omega (t, t)_\omega$. Define $a = \langle s, t\rangle$. We clearly have

$$\omega(aa^*) = \omega(a\langle t, s\rangle) = \omega(\langle at, s\rangle) = (at, s)_\omega.$$

Therefore

$$\begin{aligned}\omega(aa^*) &\leq [(at,at)_\omega(s,s)_\omega]^{\frac{1}{2}}\\ &= [\omega(a\langle t,t\rangle a^*)(s,s)_\omega]^{\frac{1}{2}}\\ &= [\omega(a|t|^2a^*)(s,s)_\omega]^{\frac{1}{2}}\\ &\leq ||\langle t,t\rangle||^{\frac{1}{2}}\omega(aa^*)^{\frac{1}{2}}\omega(\langle s,s\rangle)^{\frac{1}{2}}.\end{aligned}$$

Dividing by $\omega(aa^*)^{\frac{1}{2}}$ we find

$$\omega(aa^*)^{\frac{1}{2}} \leq ||t||\omega(\langle s,s\rangle)^{\frac{1}{2}} = \omega(||t||\langle s,s\rangle).$$

The first inequality now follows since this numerical inequality holds for all states $\omega$. As for the second inequality recall that in any real $C^*$-algebra we have $||aa^*|| = ||a||^2$ and for any pair of operators $0 \leq a \leq b$ we have $||a|| \leq ||b||$. Using this we have

$$||\langle s,t\rangle||^2 = ||\langle s,t\rangle\langle s,t\rangle^*|| = ||\langle s,t\rangle\langle t,s\rangle|| \leq ||\ |s|^2||t||^2|| = ||s||^2||t|^2$$

and this proves the second inequality. □

From the second inequality we can in the usual way conclude that the triangle inequality holds for $||\ ||$.

**Corollary 8.** $||\ ||$ *is a pseudo norm on* $V$*.*

Let $N$ be the subset of elements in $V$ of pseudonorm zero.

$$N = \{s \mid ||s|| = 0\}.$$

For any operator $a \in \mathcal{O}(H)$ and a pair of elements $s$ and $t$ in $N$ we now have

$$\begin{aligned}&||as||^2 = ||\langle as,as\rangle|| = ||a\langle s,s\rangle a^*|| \leq ||a||\ ||s||^2\ ||a^*|| = 0\\ &||s+t|| \leq ||s|| + ||t|| = 0.\end{aligned}$$

Thus $N$ is a submodule and we can therefore define a quotient module

$$\widetilde{\mathcal{H}} = V/N.$$

Elements in $\widetilde{\mathcal{H}}$ are equivalent classes of simple operator valued functions denoted by $[s]$. Note that for any elements $[s],[t] \in \widetilde{\mathcal{H}}$ with $[s] = 0$ we have

$$||\langle s,t\rangle|| \leq ||s||\ ||t|| = 0,$$

and as a consequence of this $\langle s,t\rangle = 0$. We therefore have a well defined operator valued product on $\widetilde{\mathcal{H}}$ defined through

$$\langle [s],[t]\rangle = \langle s,t\rangle$$

This product enjoy the same properties as the product on $V$ and is in addition positive definite. Thus $\widetilde{\mathcal{H}}$ with this product is a pre-Hilbert module with a norm $|| \, ||$ defined on the underlying real vector space. In general this vector space is not complete with respect to the norm. We can however complete the vector space with respect to the norm. The resulting structure is a Hilbert module over the real $C^*$-algebra $\mathcal{O}(H)$. We will call it the Hilbert module corresponding to the extended probability space $\langle \Omega, \mathcal{B}(\tau), F \rangle$. With the analogy with Hilbert spaces in mind we will consider $\langle \varphi, \varphi \rangle$ to the the square length of $\varphi$. Note that for a general Hilbert module the length is a positive operator, not a positive number. Also note that in order to simplify the notation we use the same symbol $|| \, ||$ for the norm on $\mathcal{H}$ and for the operator norm on $\mathcal{O}(H)$. This is the sense of the formula $||\varphi||^2 = ||\langle \varphi, \varphi \rangle||$.

We have now made sense of equation (1). It just state that $\varphi$ should be a element in the Hilbert module $\mathcal{H}$ of length 1.

We will next proceed to make sense of equation (2). Note that what we do is in fact to prove the analog of the easy part of the classical Radon-Nikodym theorem.

For any $U \in \mathcal{B}(\tau)$ define a map $P_U : V \to V$ by

$$P_U(s) = \sum_i s_i \theta_{V_i \cap U}.$$

This map is clearly a $\mathcal{O}(H)$ module morphism.

**Proposition 9.** *The following properties*

$$\begin{aligned} P_U \circ P_U &= P_U, \\ P_U(as) &= aP_U(s), \quad \forall a \in \mathcal{O}(H), \\ P_{U \cap V} &= P_U \circ P_V, \\ \langle P_U(s), t \rangle &= \langle s, P_U(t) \rangle, \\ \langle s, P_U(s) \rangle &\geq 0, \\ P_V + P_W &= P_{V \cup W}, \quad \textit{if} \quad V \cap W = \emptyset, \\ \langle P_U(s), P_U(s) \rangle &\leq \langle s, s \rangle, \\ ||P_U(s)|| &\leq ||s|| \end{aligned}$$

*hold.*

The last property shows that if $||s|| = 0$ then $||P_U(s)|| = 0$. Therefore $P_U$ induce a well defined map, also denoted by $P_U$, on $\widetilde{\mathcal{H}}$ through

$$P_U([s]) = [P_U(s)].$$

The last property shows also that the map $P_U$ is bounded on $\widetilde{\mathcal{H}}$. It therefore extends to a unique bounded linear map on $\mathcal{H}$. This map clearly also enjoy the properties listed in the previous proposition.

Let now $\varphi$ be a element in the Hilbert module $\mathcal{H}$ of unit length $\langle\varphi,\varphi\rangle = 1$. For each set $U \in \mathcal{B}(\tau)$ define a operator $E_\varphi(U)$ on the Hilbert space $H$ by

$$E_\varphi(U) = \langle\varphi, P_U(\varphi)\rangle.$$

Clearly $E_\varphi(\Omega) = 1$ and $E_\varphi(U) \geq 0$ for all $U$. It is also evident from the previous proposition that $E_\varphi$ is finitely additive on disjoint sets. It is in fact also countably additive as we now show.

**Theorem 10.** *$E_\varphi : \mathcal{B}(\tau) \to \mathcal{O}(H)$ is a positive operator valued measure.*

*Proof.* Let first $s = \sum_i s_i\theta_{V_i}$ be a element in $V$ with $\langle s, s\rangle = 1$ and let $\{T_j\}$ be a increasing sequence of sets with limit $T = \cup_j T_j$. The set of operators $\{E_s(T_j)\}$ is a increasing sequence of positive operators. The supremum of this sequence exists [1]. Denote the supremum by $Sup\{E_s(T_j)\}$. In order to show that $E_s$ is a positive operator valued measure we only need to show that

$$E_s(\cup_j T_j) = Sup\{E_s(T_j)\}.$$

It is a fact [1] that the sequence $E_s(T_j)$ converges strongly to the limit $Sup\{E_s(T_j)\}$. Since the strong limit is unique when it exists we must only show that $E_s(T_j)(x) \to E_s(\cup_j T_j)(x)$ for all elements $x \in H$. We know that $F$ is a positive operator valued measure so $F(T_j \cap V_i) \to F(T \cap V_i)$ strongly. But then since all $s_i$ are bounded operators we have

$$\begin{gathered} s_i F(T_j \cap V_i) s_i^*(x) \to s_i F(T \cap V_i) s_i^*(x) \\ \Downarrow \\ \sum_i s_i F(T_j \cap V_i) s_i^*(x) \to \sum_i s_i F(T \cap V_i) s_i^*(x) \\ \Downarrow \\ E_s(T_j)(x) \to E_s(T)(x), \end{gathered}$$

for all $x \in H$. This proves that $E_s$ is a POV. Next for any element $[s]$ in $\widetilde{\mathcal{H}}$ we define $E_{[s]}(U) = \langle [s], P_U([s])\rangle$. It is trivial to verify that $E_{[s]} = E_s$ so that the previous proof show that $E_{[s]}$ is a POV. Finally let $\varphi$ be a arbitrary element in $\mathcal{H}$. Then there exists a sequence of elements $[s_n]$ in $\mathcal{H}$ such that $[s_n] \to \varphi$. Since $E_{[s_n]}$ is a POV we know that for all $x \in H$ $\mu_x^n(U) = \langle E_{[s_n]}(U)x, x\rangle_H$ is a measure.

Let $\mu_x$ be the positive set function defined by

$$\mu_x(U) = \langle E_\varphi(U)x, x\rangle_H.$$

By continuity we know that $E_{[s_n]}(U) \to E_\varphi(U)$ in the uniform norm and thus strongly. But then by continuity of the inner product on $H$ we can conclude that

$$\lim_{n\to\infty} \mu_x^n(U) = \mu_X(U),$$

for all sets $U \in \mathcal{B}(\tau)$. This implies through the Vitali-Hahn-Saks theorem [5] that $\mu_x$ is a measure and then it follows [1] that $E_\varphi$ is a POV. $\square$

We have now made sense of equation (2) and are now ready to define the symbolic expressions occurring in equation (1) and (2).

We define the integrals $\int \varphi dF\psi^*$and $\int_V \varphi dF\varphi^*$as follows:

$$\int \varphi dF\psi^* \stackrel{\text{def}}{=} \langle\varphi, \psi\rangle,$$

$$\int_V \varphi dF\varphi^* \stackrel{\text{def}}{=} \langle\varphi, P_V(\varphi)\rangle.$$

We have thus found that probability densities for operator valued measures are not functions but elements in a Hilbert module. They should in fact not be thought of as densities but as half densities, their square is a density in the above sense. This is a startling conclusion. Half densities are however not unfamiliar to anyone that has been exposed to quantum mechanics. Wave functions are half densities. In fact wave functions appear naturally in this scheme. If $F$ is a positive operator valued measure acting on a real two dimensional Hilbert space we are lead to define densities as functions whose values are operators on the plane. The complex numbers are isomorphic to a special subalgebra of operators on the plane (the conformal operators). Thus a large class of densities can be identified with complex valued functions of length one. Since self-adjoint operators are now naturally identified with real numbers the length can be considered to be a number. What we are describing are of course wave functions. Thus densities for positive operator valued measures acting on a two-dimensional plane are wave functions.

4.2. **Random operators.** Recall [2] that a map $A : \mathcal{H} \to \mathcal{H}$ is said to be adjointable if there exists a map denoted by $A^* : \mathcal{H} \to \mathcal{H}$ such that

$$\langle A^*\varphi, \psi\rangle = \langle\varphi, A\psi\rangle,$$

for all elements $\varphi$ and $\psi$ in $\mathcal{H}$. A map is self-adjoint if $A^* = A$. It follows directly from the algebraic properties of the inner product and the completeness of the underlying real vector space that any adjointable map is a bounded $\mathcal{O}(H)$ module morphism. In fact the set of all adjointable maps form a abstract real $C^*$-algebra that we denote by $\mathcal{A}$. We will call the elements in $\mathcal{A}$ random operators.

The *expectation of a random operator* $A$ with respect to a density $\varphi$ is by definition given by

$$\langle A\rangle = \langle \varphi, A\varphi\rangle.$$

The expectation of a random operator with respect to a density $\varphi$ is thus a operator on $H$. We can also use the density to define a POV acting on $H$ as we have seen. Note that the expectation of self-adjoint random operators is a self-adjoint operator in $\mathcal{O}(H)$.

Returning to the two dimensional example discussed above we see that in that case for complex valued densities the expectation of self-adjoint random operators can be identified with real numbers and thus the expectation of random operators can be thought of as numbers. In higher dimensions and for more general densities no such identification with real numbers is possible. Furthermore no such reduction should be expected. After all, the self-adjoint elements in a real $C^*$-algebra are the right analog of real numbers.

Let us assume that the real Hilbert space underlying the extended probability space $X$ is one dimensional. If we choose a basis we can identify the Hilbert space with $\mathbb{R}$ and the Hilbert module $\mathcal{H}_X$ with the real Hilbert space of square integrable functions on $\mathbb{R}$. A positive operator valued measure is through the basis identified with a probability measure and therefore for a half density $\varphi \in \mathcal{H}_X$ the formula $E(V) = \langle \varphi, P_V\varphi\rangle$ turns into

$$\mu(V) = \int \varphi^2 d\nu.$$

The half density $\varphi$ is of course not uniquely determined by the probability measures $\mu$ and $\nu$ unless we by convention always take the positive square root. If all our observables are random vectors then it does not matter which half density we choose, they will all produce the same expectation. Thus by restricting to random vectors as our observables the difference between the various half densities $\varphi$ are not observable. However there is really no rational reason to restrict to this class of observables. If we include random operators in our observables the difference between the half densities are readily observable.

## 5. The category of extended probability spaces

In classical probability theory the notion of morphisms of probability spaces plays a role at least as important as the notion of a probability space. In fact from the Categorical point of view morphisms are the most important element in any theory construction. All other entities should be defined in terms of the morphisms. In this section we review the notion of a morphism in the context of probability spaces and then define the corresponding notion for extended probability spaces. The naturalness of our definition is verified by proving that extended probability spaces and morphisms forms a category. We also show that just as for the case of probability spaces we get a functor mapping the category of extended probability spaces into the category of Hilbert spaces. The existence of this functor is a verification of the naturalness of our constructions.

Let $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X \rangle$ and $Y = \langle \Omega_Y, \mathcal{B}(\tau_Y), \mu_Y \rangle$ be probability spaces. A morphism $f : X \to Y$ is a measurable map $f : \Omega_X \to \Omega_Y$ such that $\mu_Y$ is absolutely continuous with respect to the push forward of the measure $\mu_X$ by $f$, $\mu_Y \leq f_* \mu_X$. By the Radon-Nikodym theorem this means that there exists a probability density $\rho : \Omega_Y \to \mathbb{R}$ such that

$$\mu_Y(V) = \int_{f^{-1}(V)} \rho d\mu_X.$$

There are several other possibilities for morphisms of probability spaces [11]. We could have required $f_*\mu_X \leq \mu_Y$ or $f_*\mu_X \approx \mu_Y$. They can all be composed and lead to a category structure. However the only possibility that generalize well to extended probability spaces is the first one $\mu_Y \leq f_*\mu_X$.

5.1. **Morphisms of extended probability spaces.** In this section we will introduce the notion of mapping between extended probability spaces and will then use mappings to define morphisms. This distinction between mappings and morphisms does not exist for probability spaces.

In order to define what a mapping is in the context of extended probability spaces, we must first generalize the notions of absolute continuity and push forward to positive operator valued measures. We will do this by combining them into a single entity.

**Definition 11.** *Let $X = \langle \Omega_X, \mathcal{B}(\tau_X), F_X \rangle$ be a extended probability space, $Y = \langle \Omega_Y, \mathcal{B}(\tau_Y) \rangle$ a measurable space and $h$ the 3 tuple $h = \langle f_h, g_h, \varphi_h \rangle$ where $f_h : \Omega_X \to \Omega_Y$ is a measurable map,$g_h : H_Y \to H_X$ is a isometry and $\varphi_h \in \mathcal{H}_X$ is a element in the Hilbert module corresponding to $X$. Then the push forward of $F_X$ by $h$ is the positive operator valued* *measure,* $h_*F_X$*, defined on the measurable space* $Y$ *by*

$$h_*F_X(V) = g_h^* \circ \langle \varphi_h, P_{f_h^{-1}(V)}\varphi_h \rangle \circ g_h,$$

*where* $g_h^*$ *is the adjoint of* $g_h$*.*

Note that we have $g_h^* = g_h^{-1} \circ Q_h$ where $Q_h$ is the orthogonal projection onto the closed subspace $g_h(H_Y) \subset H_X$ and therefore $g_h^* \circ g_h = 1$ and $g_h \circ g_h^* = Q_h$.We can now define mappings between extended probability spaces using push forward in a very simple way.

**Definition 12.** *Let* $X = \langle \Omega_X, \mathcal{B}(\tau_X), F_X \rangle$ *and* $Y = \langle \Omega_Y, \mathcal{B}(\tau_Y), F_Y \rangle$ *be extended probability spaces. A mapping* $h : X \to Y$ *is a 3 tuple,h, as in the previous definition such that*

$$h_*F_X = F_Y.$$

Let us assume that the real Hilbert spaces underlying the extended probability spaces $X$ and $Y$ are one dimensional. If we choose basis for these two spaces we can identify the Hilbert spaces with $\mathbb{R}$, the positive operator valued measures with probability measures $\mu$ and $\nu$ and the half density $\varphi$ with a real valued function on $\Omega_X$. We must have $g_h = 1$ and the condition for $h = \langle f_h, 1, \varphi_h \rangle$ to be a mapping is

$$\nu(V) = \int_{f_h^{-1}(V)} \varphi_h^2 d\mu.$$

This is of course the condition for $f_h$ to be a mapping between the probability spaces $\langle \Omega_X, \mathcal{B}(\tau_X), \mu \rangle$ and $\langle \Omega_X, \mathcal{B}(\tau_X), \mu \rangle$ if we identify the classical density with $\varphi_h^2$.

Our first goal is to show that the proposed mappings can be composed. In order to do this we must first define a certain pullback of half densities induced by a mapping. Let therefore mappings $h : X \to Y$ and $k : Y \to Z$ of extended probability spaces be given. Let us first define a measurable map $f_{k\circ h}$, a isometry $g_{k\circ h}$ and a linear map $h^*$ by

$$\begin{aligned} f_{k\circ h} &= f_k \circ f_h : \Omega_X \to \Omega_Z, \\ g_{k\circ h} &= g_h \circ g_k : H_Z \to H_X, \\ h^*(a) &= g_h \circ a \circ g_h^* : \mathcal{O}(H_Y) \to \mathcal{O}(H_X). \end{aligned}$$

The map $h^*$ has the following easily verifiable properties

**Proposition 13.** *The map* $h^*$ *is bounded and*

$$\begin{aligned} h^*(a+b) &= h^*(a) + h^*(b), \\ h^*(ab) &= h^*(a)h^*(b). \end{aligned}$$

Define a linear map $h^* : V_Y \to \mathcal{H}_X$ by

$$h^*(s) = \sum_j h^*(s_j) P_{f_h^{-1}(V_j)}(\varphi_h),$$

where $s = \sum s_j \theta_{V_j}$. The map $h^*$ has the following important properties

**Proposition 14.** *The map $h^*$ is bounded and*

$$\begin{aligned}
h^*(s+t) &= h^*(s) + h^*(t), \\
h^*(as) &= h^*(a) h^*(s), \\
\langle h^*(s), h^*(t) \rangle &= h^*(\langle s, t \rangle), \\
[s] = 0 &\Longrightarrow [h^*(s)] = 0, \\
h^*(P_V(s)) &= P_{f_h^{-1}(V)}(h^*(s)).
\end{aligned}$$

*Proof.* Let $s = \sum s_i \theta_{V_i}$ and $t = \sum t_j \theta_{W_j}$. Then it is easy to verify that $\{V_i \cap W_j\}$ form a partition of $\Omega_Y$ and that $s + t = \sum (s_i + t_j) \theta_{V_i \cap W_j}$. But then we have

$$\begin{aligned}
h^*(s+t) &= \sum_{i,j} \acute{h}^*(s_i + t_j) P_{f_h^{-1}(V_i \cap W_j)}(\varphi_h) \\
&= \sum_{i,j} h^*(s_i) P_{f_h^{-1}(V_i) \cap f_h^{-1}(W_j)}(\varphi_h) + \sum_{i,j} h^*(t_j) P_{f_h^{-1}(V_i) \cap f_h^{-1}(W_j)}(\varphi_h) \\
&= \sum_i h^*(s_i) P_{f_h^{-1}(V_i)}(\varphi_h) + \sum_j h^*(t_j) P_{f_h^{-1}(W_j)}(\varphi_h) = h^*(s) + h^*(t).
\end{aligned}$$

This proves the second statement. For the third statement we have

$$\begin{aligned}
h^*(as) = h^*(\sum_i a s_i \theta_{V_i}) &= \sum_i h^*(a s_i) P_{f_h^{-1}(V_i)}(\varphi_h) \\
&= \sum h^*(a) h^*(s_i) P_{f_h^{-1}(V_i)}(\varphi_h) = h^*(a) h^*(s),
\end{aligned}$$

and

$$\begin{aligned}
\langle h^*(s), h^*(t)\rangle &= \sum_{i,j} \langle h^*(s_i) P_{f_h^{-1}(V_i)}(\varphi_h), h^*(t_j) P_{f_h^{-1}(W_j)}(\varphi_h)\rangle \\
&= \sum_{i,j} h^*(s_i) \circ \langle \varphi_h, P_{f_h^{-1}(V_i \cap W_j)}(\varphi_h)\rangle \circ h^*(t_j)^* \\
&= g_h \circ \left( \sum_{i,j} s_i \circ g_h^* \circ \langle \varphi_h, P_{f_h^{-1}(V_i \cap W_j)}(\varphi_h)\rangle \circ g_h \circ t_j^* \right) \circ g_h^* \\
&= g_h \circ \left( \sum_{i,j} s_i \circ h_* F_X(V_i \cap W_j) \circ t_j^* \right) \circ g_h^* \\
&= g_h \circ \left( \sum_{i,j} s_i \circ F_Y(V_i \cap W_j) \circ t_j^* \right) \circ g_h^* \\
&= g_h \circ \langle s, t\rangle \circ g_h^* = h^*(\langle s, t\rangle)
\end{aligned}$$

proves the fourth statement. The first and last statement in the proposition follows from the fourth. Finally

$$\begin{aligned}
h^*(P_V(s)) = h^*(\sum_i s_i \theta_{V \cap V_i}) &= \sum_i h^*(s_i) P_{f_h^{-1}(V \cap V_i)}(\varphi) \\
&= \sum_i h^*(s_i) P_{f_h^{-1}(V)}(P_{f_h^{-1}(V_i)}(\varphi))) = P_{f_h^{-1}(V)}(h^*(s)).
\end{aligned}$$

□

Using this proposition we can extend the map $h^*$ to a continuous linear map from $\mathcal{H}_Y$ to $\mathcal{H}_X$ . This map is given on the dense set $\widetilde{\mathcal{H}_Y}$ by

$$h^*([s]) = h^*(s).$$

All the properties in the proposition holds for the extension. We are now ready to prove that our mappings can be composed

**Theorem 15.** *Let $h : X \to Y$ and $k : Y \to Z$ be mappings of extended probability spaces. Define $\varphi_{k \circ h} \in \mathcal{H}_X$ by $\varphi_{k \circ h} = h^*(\varphi_k)$. Then*

$$k \circ h = \langle f_{k \circ h}, g_{k \circ h}, \varphi_{k \circ h}\rangle$$

*is a mapping of extended probability spaces $k \circ h : X \to Z$ and we have*

$$(k \circ h)^* = h^* \circ k^*.$$

*Proof.* In order to show that $k \circ h$ is a mapping we must prove that $(k \circ h)_* F_X = F_Z$. But doing this is now a straight forward calculation if we use the previous proposition.

$$\begin{aligned}
&(k \circ h)_* F_X(V) \\
&= g^*_{k\circ h} \circ \langle \varphi_{k\circ h}, P_{f^{-1}_{k\circ h}(V)}(\varphi_{k\circ h})\rangle \circ g_{k\circ h} \\
&= g^*_k \circ g^*_h \circ \langle h^*(\varphi_k), P_{f_h^{-1}(f_k^{-1}(V))}(h^*(\varphi_k))\rangle \circ g_h \circ g_k \\
&= g^*_k \circ g^*_h \circ \langle h^*(\varphi_k), h^*(P_{f_k^{-1}(V)}(\varphi_k))\rangle \circ g_h \circ g_k \\
&= g^*_k \circ g^*_h \circ g_h \circ \langle \varphi_k P_{f_k^{-1}(V)}(\varphi_k)\rangle \circ g^*_h \circ g_h \circ g_k \\
&= g^*_k \circ \langle \varphi_k P_{f_k^{-1}(V)}(\varphi_k)\rangle \circ g_k = F_Z(V).
\end{aligned}$$

The last statement in the theorem is also proved by direct calculation. Let $s = \sum s_j \theta_{V_j} \in V_Z$. Then we have

$$\begin{aligned}
&(k \circ h)^*([s]) \\
&= \sum_j (k\circ h)^*(s_j) P_{f^{-1}_{k\circ h}(V_j)}(\varphi_{k\circ h}) \\
&= \sum_j h^*(k^*(s_j)) P_{f_h^{-1}(f_k^{-1}(V_j))}(h^*(\varphi_k)) \\
&= \sum_j h^*(k^*(s_j)) h^*(P_{f_k^{-1}(V_j)}(\varphi_k)) \\
&= h^*(\sum_j k^*(s_j) P_{f_k^{-1}(V_j)}(\varphi_k)) = h^*(k^*(s)).
\end{aligned}$$

Since the identity holds on a dense subset is also holds for all elements in $\mathcal{H}_Z$ and this proves the theorem. □

We now can use this Theorem to define composition of mappings

**Definition 16.** *Let $h : X \to Y$ and $k : Y \to Z$ be mappings of extended probability spaces. Then $k \circ h$ is the composition of $k$ and $h$.*

It is now straight forward to prove that composition of mappings is associative.

**Theorem 17.** *Let $h : X \to Y$ , $k : Y \to Z$ and $r : Z \to T$ be mappings of extended probability spaces. Then we have*

$$r \circ (k \circ h) = (r \circ k) \circ h.$$

*Proof.* Clearly we have $f_{r\circ(k\circ h)} = f_{(r\circ k)\circ h}$ and $g_{r\circ(k\circ h)} = g_{(r\circ k)\circ h}$. And from the previous theorem we have

$$\begin{aligned}
\varphi_{r\circ(k\circ h)} &= (k\circ h)^*(\varphi_r) = h^*(k^*(\varphi_r)) \\
\varphi_{(r\circ k)\circ h} &= h^*(\varphi_{r\circ k}) = h^*(k^*(\varphi_r))
\end{aligned}$$

□

Extended probability spaces and mappings of extended probability spaces does unfortunately not form a category, we will in general not have unit morphisms.

For a given extended probability space $X = \langle \Omega_X, \mathcal{B}(\tau_X), F_X \rangle$ the only reasonable candidate for a unit morphism is

$$1_X = \langle 1_{\Omega_X}, 1_{H_X}, 1_{H_X}\theta_{\Omega_X} \rangle.$$

For this mapping it is easy to show that

**Proposition 18.**

$$\begin{aligned} k \circ 1_X &= k, \\ 1_Y \circ h &= \langle f_h, g_h, Q_h \varphi_h \rangle. \end{aligned}$$

Thus the mapping is not a unit morphism in the categorical sense unless $g_h$ is a isomorphism. It is for this reason that we distinguish between mappings and the yet to be defined morphisms. Morphisms will be defined in terms of a equivalence relation on mappings.

Recall that for any mapping $h : X \to Y$ , $Q_h : H_X \to g_h(H_Y)$ is the orthogonal projection on the closed subspace $g_h(H_Y)$.

**Definition 19.** *Two mappings $h, k : X \to Y$ of extended probability spaces are equivalent if*

$$\begin{aligned} f_h &= f_k, \\ g_h &= g_k, \\ Q_h \varphi_h &= Q_k \varphi_k. \end{aligned}$$

*If $h$ and $k$ are equivalent we will write $h \approx k$.*

The defined relation is a equivalence relation. In order to define morphisms we must show that composition of mappings extends to equivalence classes of mappings. For this we need the following two lemmas.

**Lemma 20.** *Let $h : X \to Y$ and $k : Y \to Z$ be mappings of extended probability spaces. Then*

$$Q_{k \circ h} = h^*(Q_k).$$

*Proof.* For any $\xi \in H_X$ , $Q_{k \circ h}(\xi)$ is the unique vector in $g_h(g_k(H_Z))$ such that $\xi - Q_{k \circ h}(\xi)$ is orthogonal to $g_h(g_k(H_Z))$. But for any $\eta = g_h(g_k(\alpha))$

in $g_h(g_k(H_Z))$ we have

$$\begin{aligned}
&\langle \xi - h^*(Q_k)(\xi), \eta\rangle \\
&= \langle \xi - (g_h \circ Q_k \circ g_h^*)(\xi), g_h(g_k(\alpha)))\rangle \\
&= \langle g_k^*(g_h^*(\xi)) - (g_k^* \circ g_h^* \circ g_h \circ Q_k \circ g_h^*)(\xi), \alpha\rangle \\
&= \langle g_k^*(g_h^*(\xi)) - g_k^*(g_h^*(\xi)), \alpha\rangle = 0.
\end{aligned}$$

Therefore by uniqueness $Q_{k\circ h}(\xi) = h^*(Q_k)(\xi)$. □

**Lemma 21.** *Let $h, h' : X \to Y$ be equivalent. Then*

$$h^* = h'^*.$$

*Proof.* We only need to verify the identity on the dense subset $\widetilde{\mathcal{H}_X} \subset \mathcal{H}_X$. But for any $[s] \in \widetilde{\mathcal{H}_X}$ with $s = \sum s_i \theta_{V_i}$ we have

$$\begin{aligned}
h'^*([s]) &= \sum_i h'^*(s_i) P_{f_{h'}(V_i)}(\varphi_{h'}) \\
&= \sum_i (g_{h'} \circ s_i \circ g_{h'}^{-1} \circ Q_{h'}) P_{f_{h'}(V_i)}(\varphi_{h'}) \\
&= \sum_i (g_h \circ s_i \circ g_h^{-1}) Q_{h'} P_{f_h(V_i)}(\varphi_{h'}) \\
&= \sum_i (g_h \circ s_i \circ g_h^{-1}) P_{f_h(V_i)}(Q_{h'} \varphi_{h'}) \\
&= \sum_i (g_h \circ s_i \circ g_h^{-1}) P_{f_h(V_i)}(Q_h \varphi_h) = h^*([s]).
\end{aligned}$$

□

We can now prove that composition is well defined on classes.

**Proposition 22.** *Let $h, h' : X \to Y$ be equivalent and $k, k' : Y \to Z$ be equivalent. Then*

$$k \circ h \approx k' \circ h'.$$

*Proof.* We only need to prove that $\varphi_{k\circ h} = \varphi_{k'\circ h'}$. But using the previous two lemmas we have

$$Q_{k\circ h}\varphi_{k\circ h} = h^*(Q_k)h^*(\varphi_k) = h^*(Q_k\varphi_k) = h'^*(Q_{k'}\varphi_{k'}) = Q_{k'\circ h'}(\varphi_{k'\circ h'}).$$

□

**Definition 23.** *A morphism between extended probability spaces $X$ and $Y$ is a equivalence class, $[h]$,of mappings $h : X \to Y$.*

In order to keep the notation simple we will always denote a morphism $[h]$ by a representative mapping $h$. Thus when we speak of a morphism $h$ we mean the class $[h]$. The meaning will always be clear, we just have to make sure that any operations involving morphisms does not depend on choice of representative.

We can now formulate the main result of this subsection.

**Theorem 24.** *Extended probability spaces and morphisms form a category.*

*Proof.* We know that composition is well defined and associative. For any object $X$, let the unit mapping be $1_X = \langle 1_{\Omega_X}, 1_{H_x}, 1_{H_X}\theta_{\Omega_X}\rangle$. From proposition 18 we have for any morphisms $h : X \to Y$

$$\begin{aligned} &h \circ 1_X \approx h, \\ &1_Y \circ h = \langle f_h, g_h, Q_h\varphi_h\rangle \approx h \end{aligned}$$

because $Q_h$ is a projection. $\square$

We know that the category of probability spaces[11] has a terminal object, $T$ ,in the categorical sense, there is a unique morphism from any probability space $X$ to $T$. Here $T = \langle \Omega_T, \mathcal{B}_T, \mu_T\rangle$ with $\Omega_T = \{*\}$ , $\mathcal{B}_T = \{\emptyset, \{*\}\}$ and $\mu_T$ the only possible probability measure on $\mathcal{B}_T$. The existence of $T$ makes it possible to define points in probability spaces categorically. We will now see that the category of extended probability spaces does not have a terminal object and thus extended probability spaces will not have points in the categorical sense, but only generalized points. The only possible candidate for a terminal object in the category of extended probability spaces is the object $T = \langle \Omega_T, \mathcal{B}_T, F_T\rangle$ where $F_T : \mathcal{B}_T \to \mathcal{O}(\mathbb{R}) \approx \mathbb{R}$ is the only possible positive operator valued measure, $F_T(\Omega_T) = 1_{\mathbb{R}}$. We will now show that $T$ is in fact not a terminal object.

Let $h : X \to T$ be any morphism of extended probability spaces. We have $h = \langle f_h, g_h, \varphi_h\rangle$ and clearly $f_h : \Omega_X \to \Omega_T = \{*\}$ is unique. The map $g_h : \mathbb{R} \to H_X$ is a isometry and is therefore determined by a vector $\xi_h \in H_X$ where $\langle \xi_h, \xi_h\rangle = 1$ and $g_h(1) = \xi_h$. The vector $\xi_h$ and element $\varphi_h \in \mathcal{H}_X$ must satisfies the single condition

$$h_* F_X(\Omega_T) = F_T(\Omega_T) = 1_{\mathbb{R}}.$$

Using the definition of $h_*$ we find that the following identity must be satisfied

$$\langle \langle \varphi_h, \varphi_h\rangle(\xi_h), \xi_h\rangle = 1,$$

and clearly this identity will be satisfied by many choices of $\varphi_h$ and $\xi_h$. Thus the morphism $h$ is not uniquely determined and therefore $T$ is not a terminal object.

5.2. **The Naimark functor.** In probability theory there is a certain functor that plays a major role in the theory. We will now review the construction of this functor and show that a analog functor is defined on the category of extended probability spaces. The existence of this functor testify to the naturalness of our constructions. The functor will be called the Naimark functor since the Naimark dilatation construction plays a major role in its construction.

Let us start with a review of the functor for the case of probability spaces. For any probability space $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X \rangle$ define a Hilbert space,denoted by $L_2(X)$, by $L_2(X) = L_2(\mu_X)$. Let $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X \rangle$ and $Y = \langle \Omega_Y, \mathcal{B}(\tau_Y), \mu_Y \rangle$ be two probability spaces and let $f : \Omega_X \to \Omega_Y$ be a morphism of probability spaces in the sense that

$$\mu_Y(V) = \int_{f^{-1}(V)} \rho d\mu_X$$

Define a mapping $L_2(f) : L_2(Y) \to L_2(X)$ by

$$L_2(f)(\xi) = \sqrt{\rho}(\xi \circ f)$$

It is easy to verify, using the Radon Nikodym theorem, that $L_2(f)$ is in fact a isometry and moreover that $L_2$ is a functor from the category of probability spaces to the category of Hilbert spaces. We will now show that it is possible to define a functor, also denoted by $L_2$, from the category of extended probability spaces to the category of Hilbert spaces that for probability spaces reduce to the functor discussed above.

Let $X$ and $Y$ be extended probability spaces and let $L_2(X)$ and $L_2(Y)$ be the corresponding Hilbert spaces of random vectors. Informally to any morphism $h : X \to Y$ of extended probability spaces we will define a isometry $L_2(h) : L_2(Y) \to L_2(X)$ by the formula

$$L_2(h)(\xi)(x) = \varphi_h^*(x)(g_h((\xi \circ f_h)(x)))$$

It is easy to see that the mapping $L_2(f)$ is a special case of this general formula. Of course we can not use this formula to actually define $L_2(h)$ since elements in $L_2(Y)$ are not vector functions and elements in $\mathcal{H}_X$ are not operator valued functions. The action of elements in $\mathcal{H}_X$ on $L_2(X)$ implied by the formula must also be made sense of and since morphisms are classes of mappings we need to prove independence of representative.. We will now prove that the map $L_2(h)$ exists and that it defines a functor.

Recall that if $S_Y$ denote the space of simple $H_Y$ valued functions with inner product $\langle v, w\rangle = \sum_{i,j}\langle F_Y(V_i \cap T_j)\xi_i, \eta_j\rangle_{H_Y}$ then $L_2(Y)$ is the closure of $T_Y = \{[v] \mid v \in S_Y\}$ where $[v] = 0$ iff $\langle v, v\rangle = 0$. For any extended probability space, $V_X$ is the linear space of simple operator valued functions occurring in the construction of the Hilbert module $\mathcal{H}_X$. For a measurable map $f : \Omega_X \to \Omega_Y$,a isometry $g : H_Y \to H_X$ and a element $v = \sum_i \xi_i\theta_{V_i} \in S_Y$ define a linear map $t_v^{f,g} : V_X \to L_2(X)$ by

$$t_v^{f,g}(s) = [\sum_{i,j} s_j^*(g(\xi_i))\theta_{f^{-1}(V_i)\cap W_j}]$$

where $s = \sum_j s_j\theta_{W_j} \in V_X$.

**Lemma 25.** *For the linear map $t_v^{f,g}$ the following property*

$$\langle t_v^{f,g}(s), t_v^{f,g}(s)\rangle \leq c_{v,g}||s||^2$$

*holds.*

*Proof.* Let $v = \sum_i \xi_i\theta_{V_i}$ and $s = \sum_j s_j\theta_{W_j}$. Then we have

$$\begin{aligned}
&\langle t_v^{f,g}(s), t_v^{f,g}(s)\rangle \\
&= \sum_{i,j}\langle F_X(W_j \cap f^{-1}(V_i))s_j^*(g(\xi_i)), s_j^*(g(\xi_i))\rangle_{H_X} \\
&= \sum_{i,j}\langle (s_jF_X(W_j \cap f^{-1}(V_i))s_j^*)(g(\xi_i)), g(\xi_i)\rangle_{H_X} \\
&= \sum_{i}\langle \langle s, P_{f^{-1}(V_i)}(s)\rangle(g(\xi_i)), g(\xi_i)\rangle_{H_X} \\
&\leq \sum_{i}\langle \langle s, s\rangle(g(\xi_i)), g(\xi_i)\rangle_{H_X} \leq c_{v,g}||s||^2.
\end{aligned}$$

In the last line we used the Cauchy-Swartz inequality and the definition of the norm in the Hilbert module. □

This lemma implies that if $[s] = 0$ then $[t_v^{f,g}(s)] = 0$ and therefore we can extend $t_v^{f,g}$ to a bounded linear map $t_v^{f,g} : \mathcal{H}_X \to L_2(X)$. It is defined on the dense subset $\widetilde{\mathcal{H}_X}$ by $t_v^{f,g}([s]) = [t_v^{f,g}(s)]$.

The following proposition sets the stage for proving the existence of the Naimark functor.

**Proposition 26.** *Let $h : X \to Y$ be a mapping of extended probability spaces. Then there exists a isometry $L_2(h) : L_2(Y) \to L_2(X)$ that is defined on the dense subset $T_Y$ by*

$$L_2(h)([v]) = t_v^{f_h,g_h}(\varphi_h),$$

*and that satisfy*

$$L_2(k \circ h) = L_2(h) \circ L_2(k),$$
$$L_2(1_X) = 1_{L_2(X)}.$$

*Proof.* We will start by showing that $t_v^{f_h,g_h}$ only depends on the class of $v$. Let $\{s_n\}$ be a sequence of elements in $\mathcal{H}_X$ converging to $\varphi_h$. For each $n$ we can define a positive operator valued measure on $\langle \Omega_Y, \mathcal{B}(\tau_Y)\rangle$ acting on the Hilbert space $H_Y$ by

$$F_Y^n(V) = g^* \circ \langle s_n, P_{f^{-1}(V)}(s_n)\rangle \circ g.$$

By continuity $F_Y^n(V) \to F_Y(V)$ strongly and thus weakly. But then we have

$$\begin{aligned}
&\langle t_v^{f_h,g_h}(\varphi_h), t_v^{f_h,g_h}(\varphi_h)\rangle \\
&= \lim_{n\to\infty} \langle t_v^{f_h,g_h}(s_n), t_v^{f_h,g_h}(s_n)\rangle \\
&= \lim_{n\to\infty} \sum_i \langle \langle s_n, P_{f^{-1}(V_i)}(s_n)\rangle (g(\xi_i)), g(\xi_i)\rangle_{H_X} \\
&= \lim_{n\to\infty} \sum_i \langle (g^* \circ \langle s_n, P_{f^{-1}(V_i)}(s_n)\rangle \circ g)(\xi_i)), \xi_i\rangle_{H_Y} \\
&= \lim_{n\to\infty} \sum_i \langle F_Y^n(V_i)(\xi_i), \xi_i\rangle_{H_Y} \\
&= \sum_i \langle F_Y(V_i)\xi_i, \xi_i\rangle_{H_Y} = \langle v, v\rangle.
\end{aligned}$$

The assumption $[v] = 0$ means that $\langle v, v\rangle = 0$, so $t_v^{f_h,g_h}$ depends only on the class of $v$. Therefore $L_2(h)$ is well defined on the dense subset $T_Y$ and the argument just given show that it is a isometry. It therefore extends to a isometry from $L_2(Y)$ to $L_2(X)$.

For the last part of the Theorem let $[s_n]$ and $[t_m]$ be sequences in $\mathcal{H}_X$ and $\mathcal{H}_Y$ converging to $\varphi_h$ and $\varphi_k$. Here $s_n = \sum_l s_{nl}\theta_{W_{nl}}$ and $t_m = \sum_j t_{mj}\theta_{T_{mj}}$. For $[v] \in T_Z \subset L_2(Z)$ with $v = \sum_i \xi_i\theta_{V_i}$ we have by continuity of all maps involved that if we define $[u] \in T_Y \subset L_2(Y)$ by

$u_m = \sum_{i,j} t^*_{mj}(g_k(\xi_i))\theta_{f_k^{-1}(V_i)\cap T_{mj}}$ then we have

$$
\begin{aligned}
&L_2(h)\circ L_2(k)([v])\\
&= L_2(h)(t_v^{f_k,g_k}(\varphi_k)) = L_2(h)(t_v^{f_k,g_k}(\lim_{m\to\infty}[t_m]))\\
&= \lim_{m\to\infty} L_2(h)(t_v^{f_k,g_k}([t_m]))\\
&= \lim_{m\to\infty} L_2(h)(\sum_{i,j} t^*_{mj}(g_k(\xi_i))\theta_{f_k^{-1}(V_i)\cap T_{mj}}) = \lim_{m\to\infty} L_2(h)([u_m])\\
&= \lim_{m\to\infty} t_{u_m}^{f_h,g_h}(\varphi_h) = \lim_{m\to\infty}\lim_{n\to\infty} t_{u_m}^{f_h,g_h}([s_n])\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i,j,l}(s^*_{nl}\circ g_h\circ t^*_{mj}\circ g_k)(\xi_i)\theta_{f_h^{-1}(f_k^{-1}(V_i)\cap T_{mj})\cap W_{nl}}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&h^*([t_m])\\
&= \sum_j h^*(t_{mj})P_{f_h^{-1}(T_{mj})}(\varphi_h)\\
&= \sum_j h^*(t_{mj})P_{f_h^{-1}(T_{mj})}(\lim_{n\to\infty}[s_n])\\
&= \lim_{n\to\infty}\sum_{j,l} h^*(t_{mj})s_{nl}\theta_{f_h^{-1}(T_{mj})\cap W_{nl}}.
\end{aligned}
$$

We have

$$
\begin{aligned}
L_2(k\circ h)([v]) &= t_v^{f_{k\circ h},g_{k\circ h}}(\varphi_{k\circ h}) = t_v^{f_{k\circ h},g_{k\circ h}}(h^*(\varphi_k))\\
&= t_v^{f_{k\circ h},g_{k\circ h}}(h^*(\lim_{m\to\infty}[t_m])) = \lim_{m\to\infty} t_v^{f_{k\circ h},g_{k\circ h}}(h^*([t_m]))\\
&= \lim_{m\to\infty} t_v^{f_{k\circ h},g_{k\circ h}}(\lim_{n\to\infty}\sum_{j,l} h^*(t_{mj})s_{nl}\theta_{f_h^{-1}(T_{mj})\cap W_{nl}})\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i,j,l}(h^*(t_{mj})s_{nl})^*(g_{k\circ h}(\xi_i))\theta_{f_{k\circ h}^{-1}(V_i)\cap f_h^{-1}(T_{mj})\cap W_{nl}}\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i,j,l}(s^*_{nl}\circ g_h\circ t^*_{mj}\circ g^*_h\circ g_h\circ g_k)(\xi_i)\theta_{f_h^{-1}(f_k^{-1}(V_i)\cap T_{mj})\cap W_{nl}}\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\sum_{i,j,l}(s^*_{nl}\circ g_h\circ t^*_{mj}\circ g_k)(\xi_i)\theta_{f_h^{-1}(f_k^{-1}(V_i)\cap T_{mj})\cap W_{nl}}.
\end{aligned}
$$

The last statement of the theorem is verified by a trivial calculation. □

We are now finally ready to prove the existence of the Naimark functor.

**Theorem 27.** *$L_2(h)$ is a well defined functor from the category of extended probability spaces to the category of Hilbert spaces.*

*Proof.* We only need to prove that $L_2(h)$ is well defined for a given morphism $h$. The functorial properties follows from the previous proposition. Assume $h \approx h'$. Let us first assume that the densities of $h$ and $h'$ are $[s]$ and $[s']$. We can without loss of generality assume that $s$ and $s'$ are of the form

$$s = \sum_i s_i \theta_{W_i},$$
$$s' = \sum_i s'_i \theta_{W_i},$$

since we can bring it to this form by the same construction as in lemma 29. The equivalence then amounts to $Q_h s_i = Q_{h'} s'_i$ for all $i$. Then on the dense subset $T_Y \subset L_2(Y)$ we have for $v = \sum \xi_i \theta_{V_i}$ that

$$\begin{aligned} L_2(h)([v]) &= \sum_{i,j} s_j^*(g_h(\xi_i))\theta_{f_h^{-1}(V_i)\cap W_j} \\ &= \sum_{i,j} (s_j^* \circ Q_h \circ g_h)(\xi_i)\theta_{f_h^{-1}(V_i)\cap W_j} \\ &= \sum_{i,j} (s_j'^* \circ Q_{h'} \circ g_{h'})(\xi_i)\theta_{f_{h'}^{-1}(V_i)\cap W_j} = L_2(h')([v]). \end{aligned}$$

The case for general densities follows by continuity. □

The Naimark functor $L_2$ is not the only functor occurring in this theory. In fact if we recall the properties of the pullback operation $h \to h^*$ defined earlier in this section we can define a second functor.

**Theorem 28.** *For any extended probability space $X$, define a Hilbert module $\mathcal{H}(X) = \mathcal{H}_X$ and for any morphism $h : X \to Y$ of extended probability spaces define a morphism of Hilbert modules $\mathcal{H}(h) = h^*$. Then $\mathcal{H}$ is a functor from the category of extended probability spaces to the category of Hilbert modules.*

For the case of probability spaces the Hilbert module $\mathcal{H}(X)$ and the space of random vectors $L_2(X)$ are both isomorphic to the Hilbert space of square integrable real valued function. This is why random variables and densities appear to be taken from the same space in probability theory. But this is a very special situation. If the underlying Hilbert space is not one dimensional but two dimensional the densities and random vectors start to reveal their different nature. As we have discussed previously for this case a important subclass of densities are the one whose values are contained in the conformal group of the plane. These densities form a sub-Hilbert module that is actually a isomorphic to the complex Hilbert space of complex valued functions.

## 6. Monoidal structure on the category of extended probability spaces

In probability theory the notion of product measures and product densities play a major role. It is through these that dependence and independence for random variables are defined. From a categorical point of view the situation is summarized by saying that the category of probability spaces supports a monoidal structure. We will now show that the category of extended probability spaces also supports a monoidal structures and that as a consequence the notions of dependence and independence can be defined.

Let us start by reviewing the notion of a monoidal structure for a category. A monoidal structure in a category is basically a product in the category that is associative up to natural isomorphism and has a unit object up to natural isomorphism. What this means is that if $X$,$Y$ and $Z$ are objects in the category and if the product is denoted by $\otimes$ then we require that there exists a isomorphism $\alpha_{XYZ} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$. Similarly if $I$ is the unit object we require that there exists isomorphisms $\beta_X : I \otimes X \to X$ and $\gamma_X : X \otimes I \to X$. The isomorphisms can not be arbitrarily chosen for different objects, they must form the components of a natural transformation. In addition they must satisfies a set of equations known as the MacLane coherence conditions. These equations ensure that associativity and unit isomorphisms can be extended consistently to products of finitely many objects. The conditions that must be satisfied by $\alpha$,$\gamma$ and $\beta$ are the following.

For all objects $X$,$Y$,$Z$ and $T$ we must have

$$\begin{aligned}
\alpha_{X\otimes Y,Z,T} \circ \alpha_{X,Y,Z\otimes T} &= (\alpha_{X,Y,Z} \otimes 1_T) \circ \alpha_{X,Y\otimes Z,T} \circ (1_X \otimes \alpha_{Y,Z,T}),\\
(\gamma_X \otimes 1_Y) \circ \alpha_{X,I,Y} &= 1_X \otimes \beta_Y,\\
\gamma_I &= \beta_I.
\end{aligned}$$

These are the MacLane coherence conditions. The naturality conditions are expressed as follows. For any arrows $f : X \to X'$,$g : Y \to Y'$

and $h : Z \to Z'$ we must have

$$((f \otimes g) \otimes h) \circ \alpha_{X,Y,Z} = (f \otimes (g \otimes h)) \circ \alpha_{X',Y',Z'},$$
$$f \circ \beta_X = \beta_{X'} \circ (1_I \otimes f),$$
$$f \circ \gamma_X = \gamma_{X'} \circ (f \otimes 1_I).$$

In general such equations are difficult to solve, there is a very large number of variables and equations. However in some simple situations the naturality conditions can be used to reduce the system of equations to a much smaller set.

The reader not familiar with categories,natural transformations and Coherence conditions might want to consult the book [8] for a elementary introduction to the categorical view of mathematics, a more advanced introduction can be found in the book [9]

The notion of product measures in probability theory has of course been known for a long time. The corresponding monoidal structure in the category of probability spaces is described in detail in [11]. The main features are as follows. For two probability spaces $X = \langle \Omega_X, \mathcal{B}(\tau_X), \mu_X \rangle$ and $Y = \langle \Omega_Y, \mathcal{B}(\tau_Y), \mu_Y \rangle$ their product is the probability space $X \otimes Y = \langle, \Omega_X \times \Omega_Y, \mathcal{B}(\tau_X \otimes \tau_Y), \mu_X \otimes \mu_Y \rangle$, where $\mu_X \otimes \mu_Y$ is the product measure. The product of two morphisms $f : X \to Y$ and $g : X' \to Y'$ is a morphism $f \otimes g : X \otimes X' \to Y \otimes Y'$ where $f \otimes g = f \times g$ is just the Cartesian product of the maps $f$ and $g$. The associativity and unit isomorphisms are just the usual one from the category of sets. $\alpha_{XYZ}((x,(y,z))) = ((x,y),z), \beta_X((*,x)) = x$, and $\gamma_X((x,*)) = x$. For the category of probability spaces this choice of $\alpha$, $\beta$ and $\gamma$ are the only possible ones as we show in [11]. The unit object for the monoidal structure is the trivial, one-point probability space.

6.1. **Product of extended probability spaces and morphisms.** We will now define the product of extended probability spaces and morphisms and show that this product is a bifunctor on the category of extended probability spaces.

Let $X = \langle \Omega_X, \mathcal{B}(\tau_X), F_X \rangle$ and $Y = \langle \Omega_Y, \mathcal{B}(\tau_Y), F_Y \rangle$ be two extended probability spaces. The product of the two positive operator valued measures $F_X$ and $F_Y$ always exists and is uniquely determined [1] by its value on measurable boxes by

$$(F_X \otimes F_Y)(C \times D) = F_X(D) \otimes F_Y(D).$$

The product measure acts on the Hilbert space $H_X \otimes H_Y$. The tensor product is the Hilbert tensor product. We now need to extend the product to morphisms and show that it is a bifunctor. Before we do this we must specify the relationship between the Hilbert modules $\mathcal{H}_X \otimes \mathcal{H}_Y$ and $\mathcal{H}_{X\otimes Y}$. We will show that, as expected, we can map the first into the second using a continuous injective module morphism. We will start by constructing this morphism.

Recall that for any extended probability space $X$, $\mathcal{H}_X$ is the completion of the dense subspace $\widetilde{\mathcal{H}_X} = \{[s] \mid s \in V_X\}$ and

$$V_X = \{s = \sum_i s_i\theta_{V_i} | s_i \in \mathcal{O}(H_X), \{V_i\} \text{ is a } \mathcal{B}(\tau_X) \text{ measurable partition of } \Omega_X\}$$

is the real linear space of simple $\mathcal{O}(H_X)$ valued measurable functions on $\Omega_X$.

For a pair of extended probability spaces define a map $\gamma_{XY} : V_X \times V_Y \to V_{X\otimes Y}$ by

$$\gamma_{XY}(s,t) = \sum_{i,j}(s_i \otimes t_j)\theta_{V_i\times W_j},$$

where $s = \sum s_i\theta_{V_i}$ and $t = \sum t_j\theta_{W_j}$.

For this map we have the following

**Lemma 29.** *The map $\gamma$ is bilinear and if $[s] = 0$ or $[t] = 0$ then $[\gamma(s,t)] = 0$.*

*Proof.* We evidently have $\gamma_{XY}(as,t) = \gamma_{XY}(s,at)$ for all real numbers $a$. Let $s = \sum_{i=1}^n s_i\theta_{V_i}$ and $r = \sum_{k=1}^m r_k\theta_{C_k}$ be two elements in $V_X$. Define a new sequence of sets $\{A_l\}$ where $A_l = V_l$ for $l = 1..n$ and $A_l = C_{l-n}$ for $l = n+1, ....n+m$ and let $L = \{1, 2, ...n+m\}$. Let $S = \{\sigma : L \to \mathbb{Z}_2\}$ be the set of all $\mathbb{Z}_2 = \{-1, +1\}$ valued functions on the index set $L$. The set $S$ is a index set for a new partition, $\{T^\sigma\}_{\sigma\in S}$ of the set $\Omega_X$ defined by

$$T^\sigma = \cap_{l\in L} A_l^{\sigma(l)},$$

where for any set $U$ we define $U^{+1} = U$ and $U^{-1} = U^c$, the complement of $U$. We evidently have

$$V_i = \cup_{\{\sigma|\sigma(i)=1\}} T^\sigma,$$
$$C_k = \cup_{\{\sigma|\sigma(n+k)=1\}} T^\sigma.$$

Therefore

$$s + r = \sum_\sigma \left( \sum_{\{i|\sigma(i)=1\}} s_i + \sum_{\{k|\sigma(k+n)=1\}} r_k \right) \theta_{T^\sigma}.$$

But then we have for any $t = \sum t_j \theta_{W_j} \in V_Y$ that

$$
\begin{aligned}
&\gamma_{XY}(s+r,t)\\
&= \sum_{\sigma,j} \left( \left( \sum_{\{i|\sigma(i)=1\}} s_i + \sum_{\{k|\sigma(k+n)=1\}} r_k \right) \otimes t_j \right) \theta_{T^\sigma \times W_j}\\
&= \sum_{\sigma,j} \sum_{\{i|\sigma(i)=1\}} (s_i \otimes t_j)\theta_{T^\sigma \times W_j} + \sum_{\sigma,j} \sum_{\{k|\sigma(k+n)=1\}} (r_k \otimes t_j)\theta_{T^\sigma \times W_j}\\
&= \sum_{i,j} (s_i \otimes t_j) \sum_{\{\sigma|\sigma(i)=1\}} \theta_{T^\sigma \times W_j} + \sum_{k,j} (r_k \otimes t_j) \sum_{\{\sigma|\sigma(n+k)=1\}} \theta_{T^\sigma \times W_j}\\
&= \sum_{i,j} (s_i \otimes t_j)\theta_{V_i \times W_j} + \sum_{k,j} (r_k \otimes t_j)\theta_{C_k \times W_j} = \gamma_{XY}(s,t) + \gamma_{XY}(r,t).
\end{aligned}
$$

This show that $\gamma$ is bilinear. For the second part of the statement in the lemma we have

$$
\begin{aligned}
&\langle \gamma_{XY}(s,t), \gamma_{XY}(s,t) \rangle\\
&= \sum_{i,j,k,l} (s_i \otimes t_j) F_{X\otimes Y}((V_i \times W_j) \cap (V_k \times W_l))(s_k \otimes t_l)^*\\
&= \sum_{i,j,k,l} (s_i \otimes t_j)(F_X(V_i \cap V_k) \otimes F_Y(W_j \cap W_l))(s_k^* \otimes t_l^*)\\
&= \sum_{i,j} (s_i \otimes t_j)(F_X(V_i) \otimes F_Y(W_j))(s_i^* \otimes t_j^*)\\
&= \left( \sum_i s_i F_X(V_i) s_i^* \right) \otimes \left( \sum_j t_j F_Y(W_j) t_j^* \right) = \langle s,s \rangle \otimes \langle t,t \rangle.
\end{aligned}
$$

But $[s] = 0$ implies that $\langle s,s \rangle = 0$ and the identity just derived then implies that $\langle \gamma_{XY}(s,t), \gamma_{XY}(s,t) \rangle = 0$ and therefore by definition $[\gamma_{XY}(s,t)] = 0$. □

Using the lemma we have a well linear map, also denoted by $\gamma_{XY}$, from $\widetilde{\mathcal{H}_X} \otimes \widetilde{\mathcal{H}_Y}$ to $\widetilde{\mathcal{H}_{X\otimes Y}}$

$$\gamma_{XY}([s] \otimes [t]) = [\gamma_{XY}(s,t)].$$

The map $\gamma_{XY}$ satisfy the following important identity

**Lemma 30.**

$$\langle \gamma_{XY}(v), \gamma_{XY}(v) \rangle = \langle v,v \rangle.$$

*Proof.* Any $v \in \widetilde{\mathcal{H}_X} \otimes \widetilde{\mathcal{H}_Y}$ is of the form $v = \sum_i s_i \otimes t_i$ where $s_i = \sum_j s_{ij}\theta_{V_{ij}}$ and $t_i = \sum_k t_{ik}\theta_{W_{ik}}$. But then we have

$$\begin{aligned}
&\langle \gamma_{XY}(v), \gamma_{XY}(v)\rangle \\
&= \sum_{i,j,k,l,m,n} (s_{ij} \otimes t_{ik}) F_{X\otimes Y}((V_{ij} \times W_{ik}) \cap (V_{lm} \times W_{ln}))(s_{lm} \otimes t_{ln})^* \\
&= \sum_{i,j,k,l,m,n} (s_{ij} \otimes t_{ik})(F_X(V_{ij} \cap V_{lm}) \otimes F_Y(W_{ik} \cap W_{ln}))(s^*_{lm} \otimes t^*_{ln}) \\
&= \sum_{i,l} \left( \sum_{j,m} s_{ij} F_X(V_{ij} \cap V_{lm}) s^*_{lm} \right) \otimes \left( \sum_{k,n} t_{ik} F_Y(W_{ik} \cap W_{ln}) t^*_{ln} \right) \\
&= \sum_{i,l} \langle s_i, s_l\rangle \otimes \langle t_i, t_l\rangle = \sum_{i,l} \langle s_i \otimes t_i, s_l \otimes t_l\rangle = \langle v, v\rangle.
\end{aligned}$$

□

We can now state and prove the main property of $\gamma_{XY}$. First we will recall some facts about (external) tensor products of Hilbert modules. Let $\mathcal{H}_X \otimes_{\mathcal{H}} \mathcal{H}_Y$ denote the tensor product of $\mathcal{H}_X$ and $\mathcal{H}_Y$,as real vector spaces, with topology determined by the norm induced from the operator valued inner product $\langle \varphi \otimes \psi, \varphi' \otimes \psi'\rangle = \langle \varphi, \varphi'\rangle \otimes \langle \psi, \psi'\rangle$. The completion of $\mathcal{H}_X \otimes_{\mathcal{H}} \mathcal{H}_Y$ is the external tensor product [2] of the Hilbert modules $\mathcal{H}_X$ and $\mathcal{H}_Y$ and will be denoted by $\mathcal{H}_X \otimes \mathcal{H}_Y$. It is a module over the spatial tensor product $\mathcal{O}(H_X) \otimes \mathcal{O}(H_Y)$ [12] of the represented $C^*-$ algebras $\mathcal{O}(H_X)$ and $\mathcal{O}(H_X)$.

**Proposition 31.** *There exists an injective morphism of Hilbert modules* $\gamma_{XY} : \mathcal{H}_X \otimes \mathcal{H}_Y \to \mathcal{H}_{X\otimes Y}$ *such that*

$$\langle \gamma_{XY}(v), \gamma_{XY}(v)\rangle = \langle v, v\rangle.$$

$\widetilde{\mathcal{H}_X} \otimes_{\mathcal{H}} \widetilde{\mathcal{H}_Y}$ *is a dense subspace of* $\mathcal{H}_X \otimes \mathcal{H}_Y$ *and on this dense subspace* $\gamma_{XY}$ *is given by*

$$\gamma_{XY}([s] \otimes [t]) = [\gamma_{XY}(s, t)].$$

*Proof.* Let $\widetilde{\mathcal{H}_X} \otimes_\pi \widetilde{\mathcal{H}_Y}$ and $\mathcal{H}_X \otimes_\pi \mathcal{H}_Y$ be the projective tensor products [6] of the underlying real vector spaces. Note that the tensor product spaces have not been completed with respect to the projective norm. The embedding $\widetilde{\mathcal{H}_X} \otimes_\pi \widetilde{\mathcal{H}_Y} \hookrightarrow \mathcal{H}_X \otimes_\pi \mathcal{H}_Y$ is know to exist and be dense [6]. The norm on $\widetilde{\mathcal{H}_X} \otimes_{\mathcal{H}} \widetilde{\mathcal{H}_Y}$ and $\mathcal{H}_X \otimes_{\mathcal{H}} \mathcal{H}_Y$ induced by the operator valued inner product is evidently a cross norm and it is know that the projective norm is the largest possible cross norm. Therefore we can conclude that

$\widetilde{\mathcal{H}_X} \otimes_{\mathcal{H}} \widetilde{\mathcal{H}_Y}$ is a dense subspace of $\mathcal{H}_X \otimes_{\mathcal{H}} \mathcal{H}_Y$ and thus by completion in $\mathcal{H}_X \otimes \mathcal{H}_Y$. By the previous lemma $\gamma_{XY}$ is bounded and therefore extends uniquely to a bounded map $\gamma_{XY} : \mathcal{H}_X \otimes \mathcal{H}_Y \to \mathcal{H}_{X \otimes Y}$. The first identity in the statement of the proposition follows from the previous lemma and the continuity of the operator valued inner product. □

In order to introduce tensor product of morphisms between extended probability spaces we need the previous proposition and the following lemma

**Lemma 32.** *For any measurable sets $C \in \mathcal{B}(\tau_X)$ and $D \in \mathcal{B}(\tau_Y)$ we have the identity*

$$\gamma_{XY} \circ (P_C \otimes P_D) = P_{C \times D} \circ \gamma_{XY}$$

*Proof.* For $C \in \mathcal{B}(\tau_X)$ and $D \in \mathcal{B}(\tau_Y)$ we have

$$\begin{aligned}
&(\gamma_{XY} \circ (P_C \otimes P_D))([s] \otimes [t]) \\
&= \gamma_{XY}([P_C(s)] \otimes [P_D(t)]) \\
&= \sum_{i,j} (s_i \otimes t_j)\theta_{(V_i \cap C) \times (W_j \cap D)} \\
&= \sum_{i,j} (s_i \otimes t_j)\theta_{(V_i \times W_j) \cap (C \times D)} = P_{C \times D}(\gamma_{XY}([s] \otimes [t]).
\end{aligned}$$

By continuity and density we can conclude that the identity $\gamma_{XY} \circ (P_C \otimes P_D) = P_{C \times D} \circ \gamma_{XY}$ holds on $\mathcal{H}_X \otimes \mathcal{H}_Y$. □

Let now $h : X \to Y$ and $k : X' \to Y'$ be morphisms of extended probability spaces. We thus have $h = \langle f_h, g_h, \varphi_h \rangle$ and $k = \langle f_k, g_k, \varphi_k \rangle$ where $\varphi_h \in \mathcal{H}_X$ and $\varphi_k \in \mathcal{H}_{X'}$. Define a 3-tuple $h \otimes k$ by

$$h \otimes k = \langle f_{h \otimes h}, g_{h \otimes k}, \varphi_{h \otimes k} \rangle,$$

where $f_{h \otimes k} = f_h \times f_k$ , $g_{h \otimes k} = g_h \otimes g_k$ and $\varphi_{h \otimes k} = \gamma_{XX'}(\varphi_h \otimes \varphi_k)$. Then we have

**Proposition 33.** *$h \otimes k : X \otimes X' \to Y \otimes Y'$ is a morphism of extended probability spaces.*

*Proof.* We need to prove that $(h\otimes k)_*F_{X\otimes X'} = F_{Y\otimes Y'}$. But this is true because

$$
\begin{aligned}
&(h\otimes k)_*F_{X\otimes X'}(C\times D)\\
&= g^*_{h\otimes k}\circ\langle\varphi_{h\otimes k}, P_{f^{-1}_{h\otimes k}(C\times D)}(\varphi_{h\otimes k})\rangle\circ g_{h\otimes k}\\
&= (g_h\otimes g_k)^*\circ\langle\gamma_{XX'}(\varphi_h\otimes\varphi_k), (P_{f^{-1}_{h\otimes k}(C\times D)}\circ\gamma_{XX'})(\varphi_h\otimes\varphi_k))\rangle\circ(g_h\otimes g_k)\\
&= (g^*_h\otimes g^*_k)\circ\langle\gamma_{XX'}(\varphi_h\otimes\varphi_k), (\gamma_{XX'}\circ(P_{f_h^{-1}(C)}\otimes P_{f_k^{-1}(D)}))(\varphi_h\otimes\varphi_k))\rangle\circ(g_h\otimes g_k)\\
&= (g^*_h\otimes g^*_k)\circ\langle\varphi_h\otimes\varphi_k, P_{f_h^{-1}(C)}(\varphi_h)\otimes P_{f_k^{-1}(D)}(\varphi_k))\rangle\circ(g_h\otimes g_k)\\
&= (g^*_h\circ\langle\varphi_h, P_{f_h^{-1}(C)}(\varphi_h)\rangle\circ g_h)\otimes(g^*_k\circ\langle\varphi_k, P_{f_k^{-1}(D)}(\varphi_k)\rangle\circ g_k)\\
&= (h_*F_X)(C)\otimes(k_*F_{X'})(D) = F_Y(C)\otimes F_{Y'}(D) = F_{Y\otimes Y'}(C\times D),
\end{aligned}
$$

where we have used the previous lemma. This proves that $h\otimes k$ is a mapping of extended probability spaces. In order to show that it is also a morphism we must show that it is independent of choice of representatives. Thus assume that $h\approx h'$ and $k\approx k'$. We need to show that $h\otimes k\approx h'\otimes k'$ and this amounts to proving that $Q_{h\otimes k}\varphi_{h\otimes k} = Q_{h'\otimes k'}\varphi_{h'\otimes k'}$. But from the identity $(g_h\otimes g_k)(H_X\otimes H_{X'}) = g_h(H_X)\otimes g_k(H_{X'})$ we have $Q_{h\otimes k} = Q_h\otimes Q_k$ and the rest of the proof is a simple calculation. □

Having proved that $h\otimes k$ is a morphism our next goal is to prove that it behaves as a functor under composition. For this we need the following lemma.

**Lemma 34.**

$$\gamma_{XX'}\circ(h^*\otimes k^*) = (h\otimes k)^*\circ\gamma_{YY'}$$

*Proof.* By continuity we only need to prove the identity on the dense subset $\widetilde{\mathcal{H}_Y}\otimes_{\mathcal{H}}\widetilde{\mathcal{H}_{Y'}}\subset\mathcal{H}_Y\otimes\mathcal{H}_{Y'}$. But on this subset we have

$$
\begin{aligned}
&((h\otimes k)^*\circ\gamma_{YY'})([s]\otimes[t])\\
&= (h\otimes k)^*(\gamma_{YY'}(s,t))\\
&= \sum_{i,j}(h\otimes k)^*(s_i\otimes t_j)P_{(f_h\times f_k)^{-1}(V_i\times W_j)}(\varphi_{h\otimes k})\\
&= \sum_{i,j}(h^*(s_i)\otimes k^*(t_j))(P_{(f_h\times f_k)^{-1}(V_i\times W_j)}\circ\gamma_{XX'})(\varphi_h\otimes\varphi_k)\\
&= \sum_{i,j}(h^*(s_i)\otimes k^*(t_j))(\gamma_{XX'}\circ(P_{f_h^{-1}(V_i)}\otimes P_{f_k^{-1}(W_j)}))(\varphi_h\otimes\varphi_k)\\
&= \gamma_{XX'}(\sum_{i,j}(h^*(s_i)P_{f_h^{-1}(V_i)}(\varphi_h))\otimes(k^*(t_j)P_{f_k^{-1}(W_j)}(\varphi_k)))\\
&= (\gamma_{XX'}\circ(h^*\otimes k^*))([s]\otimes[t]).
\end{aligned}
$$

□

We can now prove our first main result in this section

**Theorem 35.** *The operation* $\otimes$ *is a bifunctor on the category of extended probability spaces.*

$$(h' \otimes k') \circ (h \otimes k) = (h' \circ h) \otimes (k' \circ k),$$
$$1_X \otimes 1_Y = 1_{X \otimes Y}.$$

*Proof.* The unit property is trivial to verify and for the first identity we only need to prove that $\gamma_{XX'}(\varphi_{k \circ h} \otimes \varphi_{k' \circ h'}) = (h \otimes h')^*(\varphi_{k \otimes k'})$. But using the previous lemma we have

$$\begin{aligned}
&\gamma_{XX'}(\varphi_{k \circ h} \otimes \varphi_{k' \circ h'}) \\
&= \gamma_{XX'}(h^*(\varphi_k) \otimes h'^*(\varphi_{k'})) \\
&= (\gamma_{XX'} \circ (h^* \otimes h'^*))(\varphi_k \otimes \varphi_{k'}) \\
&= ((h^* \otimes h'^*) \circ \gamma_{YY'})(\varphi_k \otimes \varphi_{k'}) \\
&= (h^* \otimes h'^*)(\varphi_{k \otimes k'}).
\end{aligned}$$

□

6.2. **The monoidal structure.** Showing that $\otimes$ exists and is a bifunctor is the only hard part in proving that there is a monoidal structure on the category of extended probability spaces.

The only reasonable candidate for a unit object is clearly the extended probability space $T$ discussed previously. For any objects $X, Y$ and $Z$ define

$$\begin{aligned}
\eta_X &= \langle f_{\eta_X}, g_{\eta_X}, \varphi_{\eta_X} \rangle, \\
\gamma_X &= \langle f_{\gamma_X}, g_{\gamma_X}, \varphi_{\gamma_X} \rangle, \\
\alpha_{XYZ} &= \langle f_{\alpha_{XYZ}}, h_{\alpha_{XYZ}}, \varphi_{\alpha_{XYZ}} \rangle,
\end{aligned}$$

where

$$
\begin{aligned}
f_{\eta_X}(*, x) &= f_{\gamma_X}(x, *) = x, \\
f_{\alpha_{XYZ}}((x,(y,z)) &= ((x,y),z), \\
g_{\eta_X}(\xi) &= 1 \otimes \xi, \\
g_{\gamma_X}(\xi) &= \xi \otimes 1, \\
g_{\alpha_{XYZ}}(\xi \otimes (\xi' \otimes \xi'')) &= (\xi \otimes \xi') \otimes \xi'', \\
\varphi_{\eta_X} &= 1_{H_{T\otimes X}}, \\
\varphi_{\gamma_X} &= 1_{H_{X\otimes T}}, \\
\varphi_{\alpha_{XYZ}} &= 1_{H_{X\otimes(Y\otimes Z)}}.
\end{aligned}
$$

These are obviously the simplest choices we can make and it is a tedious but simple exercise prove the following theorem. This is the second main result of this section.

**Theorem 36.** *$\eta_X, \gamma_X$ and $\alpha_{XYZ}$ are morphisms of extended probability spaces*

$$
\begin{aligned}
\eta_X &: T \otimes X \to X, \\
\gamma_X &: X \otimes T \to X, \\
\alpha_{XYZ} &: X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z,
\end{aligned}
$$

*and are the components of natural isomorphisms. Furthermore $\langle \otimes, T, \eta, \gamma, \alpha \rangle$ is a monoidal structure on the category of extended probability spaces.*

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