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Complex Reflections and Polynomial Generators of Homotopy Groups

Lucas M. Chaves and A. Rigas *

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Abstract. Starting from suitable maps of the form $\psi : \mathbb{S}^n \to \mathrm{SU}(m)$, for large m, we obtain functions $\theta : \mathbb{S}^n \longrightarrow \mathrm{SU}(k)$ that generate $\pi_n \, \mathrm{SU}(k)$, for certain n and k. We have used this method in elementary algebraic topology courses for purposes of illustration, since it requires no machinery beyond the homotopy exact sequence of fibrations and provides an insight about the Bott periodicity theorems.

We begin with the natural identification

$$\mathbb{S}^3 \overset{\alpha}{\to} \mathrm{SU}(2), \quad \mathbb{S}^3 \subseteq \mathbb{C}^2$$

with $\alpha\binom{z}{w} = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$ representing the generator of $\pi_3 \, \mathrm{SU}(2)$. Suspending it we obtain

$$\begin{pmatrix} z \\ w \\ x \end{pmatrix} \overset{\psi}{\mapsto} \begin{pmatrix} \alpha\binom{z}{w} & -\overline{x} I \\ xI & \left(\alpha\binom{z}{w}\right)^* \end{pmatrix} = \begin{pmatrix} z & -\overline{w} & -\overline{x} & 0 \\ w & \overline{z} & 0 & -\overline{x} \\ x & 0 & \overline{z} & \overline{w} \\ 0 & x & -w & z \end{pmatrix}$$

from $\mathbb{S}^5$ to $\mathrm{SU}(4)$.

FOMENKO used this construction in the context of the Bott periodicity theorem ([3], vol.III, p. 271) and obtained the following result.

Theorem 1. *If $f_{i-1} : \mathbb{S}^{i-1} \to \mathrm{U}(m)$ is a generator of $\pi_{i-1} U(m)$, then the map $f_{i+1} : \mathbb{S}^{i+1} \to \mathrm{SU}(2m)$ obtained by suspension from f_{i-1} is a generator of $\pi_{i+1} \, \mathrm{SU}(2m)$.* ■

As a corollary, ψ is the generator of $\pi_5 \, \mathrm{SU}(4)$.

We prove this result through elementary methods.

Observe that the first column of the matrices in the image of ψ never assumes the value $(0, 0, 0, 1)$. The idea [5] is then to reflect the column

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vectors of $\psi(\mathbb{S}^5)$ with respect to the plane normal to $(0,\ 0,\ 0,\ 1)$ and obtain a map $\theta\colon \mathbb{S}^5 \to \mathrm{SU}(3)$. We review first a few general facts about hermitian reflections: Given a hermitian space $\{V, (\ ,\ )\}$ and a unit vector v in V, we define the complex reflection of ξ at the hyperplane $v^\perp$, denoted $R_v\colon V \to V$, by $R_v(\xi) = \xi - 2\overline{(v,\xi)}v$. We then have

i) $R_v(\lambda\xi) = \lambda R_v(\xi)$,
ii) $R_v(v) = -v$,
iii) if $(v,\xi) = 0$, then $R_v(\xi) = \xi$,
iv) $R_v \circ R_v = \mathrm{id}$, and
v) $(R_v(\xi), R_v(\eta)) = (\xi, \eta)$.

Returning to $\psi\colon \mathbb{S}^5 \to \mathrm{SU}(4)$, let $x_1 = \begin{pmatrix} z \\ w \\ x \\ 0 \end{pmatrix}$ be the first column vector of $\psi\begin{pmatrix} z \\ w \\ x \end{pmatrix}$, $1 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, and $v = \dfrac{\sqrt{2}}{2}(1 - x_1)$. Applying R_v we obtain

$$R_v(x_1) = x_1 - 2\overline{(v, x_1)}v = \begin{pmatrix} z \\ w \\ x \\ 0 \end{pmatrix} - 2\frac{\sqrt{2}}{2}(-1)\frac{\sqrt{2}}{2}\begin{pmatrix} -z \\ -w \\ -x \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and for x_2, x_3, x_4, the respective column vectors of $\psi\begin{pmatrix} z \\ w \\ x \end{pmatrix}$ we have

$$R_v(x_2) = \begin{pmatrix} -\overline{w} + xz \\ \overline{z} + xw \\ x^2 \\ 0 \end{pmatrix},\quad R_v(x_3) = \begin{pmatrix} -\overline{x} - wz \\ -w^2 \\ \overline{z} - wx \\ 0 \end{pmatrix},\quad \text{and } R_v(x_4) = \begin{pmatrix} z^2 \\ -\overline{x} + zw \\ \overline{w} + zx \\ 0 \end{pmatrix}.$$

Therefore we get the matrix

$$\begin{pmatrix} 0 & & & \\ 0 & & B & \\ 0 & & & \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where B is the obvious 3×3 matrix with columns $R_v(x_2), R_v(x_3)$, and $R_v(x_4)$ without the last entry which is equal to zero.

From $\det(R_v) = -1$ we get $\det(R_v\psi) = -1$. Therefore $\det B = 1$. So B is in $\mathrm{SU}(3)$. Now we define the map $\theta\colon \mathbb{S}^5 \to \mathrm{SU}(3)$ by

$$\theta\begin{pmatrix} z \\ w \\ x \end{pmatrix} := \begin{pmatrix} -\overline{w} + xz & -\overline{x} - wz & z^2 \\ \overline{z} + xw & -w^2 & -\overline{x} + zw \\ x^2 & \overline{z} - wx & \overline{w} + zx \end{pmatrix}.$$

To emphasize the dependence of v on the point $y = \begin{pmatrix} z \\ w \\ x \end{pmatrix}$ we write $v(y)$ and therefore have $\theta(y) = R_{v(y)}\psi(y)$.

Proposition 2. *θ is a generator of $\pi_5\,\mathrm{SU}(3)$.*

Proof. The exact homotopy sequence of the fibration $\mathrm{SU}(2)\cdots\mathrm{SU}(3) \xrightarrow{\pi} \mathbb{S}^5$ yields exact sequences

$$
\begin{array}{cccccccccccc}
\to & \pi_7\mathbb{S}^5 & \to & \pi_6(\mathrm{SU}(2)) & \to & \pi_6\,\mathrm{SU}(3) & \to & \pi_6\mathbb{S}^5 & \to & \pi_5\,\mathrm{SU}(2) & \to \\
 & \mathbb{Z}_2 & \to & \mathbb{Z}_{12} & \to & \mathbb{Z}_6 & \to & \mathbb{Z}_2 & \xrightarrow{\cong} & \mathbb{Z}_2 & \to \\
 & & & & & & & & & & \\
\to & \pi_5\,\mathrm{SU}(3) & \xrightarrow{\pi_*} & \pi_5\mathbb{S}^5 & \to & \pi_4\,\mathrm{SU}(2) & \to & \pi_4\,\mathrm{SU}(3) & \to & \pi_4\mathbb{S}^5 & \\
\to & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \to & \mathbb{Z}_2 & \to & 0 & \to & 0. &
\end{array}
$$

Thus a generator of $\pi_5\,\mathrm{SU}(3)$ is mapped to twice a generator of $\pi_5\mathbb{S}^5$. A generator of $\pi_5\mathbb{S}^5$ is just a map $\mathbb{S}^5 \to \mathbb{S}^5$ of degree one, and π is the projection on the first column. We must only note that $\pi\circ\theta\colon \mathbb{S}^5 \to \mathbb{S}^5$ (the first column of θ) is a map of degree 2. ∎

As the SU(3) principal bundles over $\mathbb{S}^6$ are classified by $\pi_5\,\mathrm{SU}(3)$, one can ask which of these bundles is defined by θ.

Corollary 3. *θ defines the bundle $\mathrm{SU}(3)\cdots G_2 \to \mathbb{S}^6$, where G_2 is the exceptional compact Lie group of the automorphisms of the algebra of Cayley numbers.*

Proof. We conclude this immediately from the exact homotopy sequence of the fibration, observing that $\pi_6 G_2 \cong \mathbb{Z}_3$, $\pi_5\,\mathrm{SU}(3) \cong \mathbb{Z}$, and $\pi_5 G_2 = 0$ [4]. ∎

An elementary, geometrically direct way of obtaining $\pi_6(G_2) \cong \mathbb{Z}_3$ is described in [2]. As a consequence of Corollary 3 we can construct G_2 from two copies of $D^6 \times \mathrm{SU}(3)$ by glueing along the common boundary using θ (see [6], p. 62).

Using $\mathbb{Z} \cong \pi_5\,\mathrm{SU}(3) \cong \pi_5\,\mathrm{SU}(4) \cong \pi_5\mathrm{U}(4)$ one can show as an easy exercise that the maps $\theta, \psi\colon \mathbb{S}^5 \to \mathrm{SU}(4)$ are homotopic. We thus obtain a simple explicitly described generator of $\pi_5\,\mathrm{SU}(4)$ as follows from Theorem 1. The same construction can be used to find an explicit generator of $\pi_7\,\mathrm{SU}(4) \cong \mathbb{Z}$ (see [1]). For this purpose one performs another suspension of ψ getting a map $\gamma\colon \mathbb{S}^7 \to \mathrm{SU}(8)$, and by four consecutive reflections $\mathrm{SU}(8) \to \mathrm{SU}(7) \to \mathrm{SU}(6) \to \mathrm{SU}(5) \to \mathrm{SU}(4)$ we obtain a generator η of $\pi_7\,\mathrm{SU}(4)$. In this case we have

$$
\begin{array}{ccccc}
\mathrm{SU}(3) & \overset{i}{\cdots} & \mathrm{SU}(4) & \xrightarrow{\ \pi\ } & \mathbb{S}^7 \\
 & & {\scriptstyle\eta}\big\downarrow & & \big\| \\
 & & \mathbb{S}^7 & \xrightarrow[\eta_1]{\quad} & \mathbb{S}^7,
\end{array}
$$

where η_1 is a map of degree 6 that is no longer polynomial. This calculation follows the same elementary steps defined above, but η_1 is not particularly simple.

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Universidade Federal de Lavras
Cx. Postal 37
37.200-000 Lavras, MG, Brasil

Instituto de Matemática
Universidade Estadual de Campinas
Cx. Postal 6065
13.081-970 Campinas, SP, Brasil

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