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Cauchy–Szegö kernels for Hardy spaces on simple Lie groups

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Résumé. Pour tout groupe simple réel G possédant des représentations unitaires de plus haut poids on peut definir l'espace de Hardy $H^2(G)$. C'est un espace hilbertien formé des fonctions holomorphes dans un tube 'non–commutatif' Γ^0 satisfaisant une condition de type Hardy (Γ^0 est l'intérieur d'un semi–groupe complexe Γ contenant le groupe G). L'espace $H^2(G)$ s'identifie au sous–espace bi–invariant de $L^2(G)$ portant la série discrète holomorphe. Le noyau reproduisant K de l'espace de Hardy est appelé le noyau de Cauchy–Szegö. Le résultat principal de l'article est un calcul explicite de ce noyau pour 3 familles de groupes G: groupes symplectiques $\mathrm{Sp}(n,\mathbb{R})$, groupes métaplectiques $Mp(n,\mathbb{R})$, et groupes pseudo–unitaires $SU(p,q)$.

Abstract. For any simple real group G possessing unitary highest weight representations one can define the Hardy space $H^2(G)$. This is a Hilbert space formed by holomorphic functions in a 'non–commutative' tube domain Γ^0 satisfying a Hardy–type condition (Γ^0 is the interior of a non–commutative complex semigroup Γ containing the group G). The space $H^2(G)$ is identified with the bi–invariant subspace of $L^2(G)$ carrying the holomorphic discrete series. The reproducing kernel K of the Hardy space is called the Cauchy–Szegö kernel. The main result of the paper is an explicit calculation of this kernel for 3 families of groups G: the symplectic groups $\mathrm{Sp}(n,\mathbb{R})$, the metaplectic groups $Mp(n,\mathbb{R})$, and the pseudo–unitary groups $SU(p,q)$.

Introduction

Let $\mathcal{H}$ be a Hilbert space whose elements f are holomorphic functions in a complex domain $\mathcal{D}$ [1]. Then one can associate to $\mathcal{H}$ a *kernel function* K: if

[1] More exactly, one should assume that $\mathcal{H}$ is continuously embedded into $\mathcal{O}(\mathcal{D})$, the space of all holomorphic functions, equipped with the topology of uniform convergence on compact subsets.

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$f_1, f_2, \ldots$ is an arbitrary orthonormal basis in $\mathcal{H}$ then

$$K(z,w) = f_1(z)\overline{f_1(w)} + f_2(z)\overline{f_2(w)} + \ldots, \qquad z,\, w \in \mathcal{D},$$

and the definition does not depend on the choice of the basis. The kernel K possesses the following properties: K is holomorphic in z and anti–holomorphic in w; $K(z,w) = \overline{K(w,z)}$; K is a positive definite kernel on $\mathcal{D} \times \mathcal{D}$; for any $w \in \mathcal{D}$ the function $K(\cdot, w)$ lies in $\mathcal{H}$; for any function $f \in \mathcal{H}$ and any $w \in \mathcal{D}$

$$f(w) = (f, K(\cdot, w))_{\mathcal{H}}$$

(the reproducing property); in particular, for any $z,\, w \in \mathcal{D}$

$$(K(\cdot, w), K(\cdot, z))_{\mathcal{H}} = K(z,w);$$

finally, the initial Hilbert space $\mathcal{H}$ is uniquely determined by its kernel function K.

Classical examples are the Bergman and Hardy (H^2) spaces in the unit disc $|z| < 1$ and the (say, right) half–plane $\operatorname{Re} z > 0$.

For the Bergman space, the square of norm $\|f\|^2$ is obtained by integrating $|f(z)|^2$ over the domain with respect to the Lebesgue measure, and the corresponding kernel function has the form

$$(1 - z\overline{w})^{-2} \quad \text{(the disc)} \qquad \text{or} \qquad (z + \overline{w})^{-2} \quad \text{(the half–plane)}.$$

For the Hardy space, $\|f\|^2$ is defined as the integral of $|f(z)|^2$ over the boundary,[2] and the corresponding kernel function is

$$(1 - z\overline{w})^{-1} \quad \text{(the disc)} \qquad \text{or} \qquad (z + \overline{w})^{-1} \quad \text{(the half–plane)}.$$

It is well–known that these examples can be generalized to multidimensional bounded symmetric domains and to tube domains $\mathbb{R}^n + iC$ (where C is an open convex cone in $\mathbb{R}^n$); further generalizations involve interpolation between the Bergman and Hardy cases, vector–valued holomorphic functions etc. Note that various Hilbert spaces of holomorphic functions and their kernel functions naturally arise in connection with unitary highest weight representations. (See, e.g., Faraut and Korányi [3], Inoue [10], Stein and Weiss [22], Vergne and Rossi [23].)

The present paper deals with multidimensional complex domains of another kind, which may be viewed as 'noncommutative' tube domains.

Let D be an irreducible bounded symmetric domain and G be a connected group, locally isomorphic to the automorphism group of D. Then G is (locally isomorphic to) one of the groups $\mathrm{SU}(p,q)$, $\mathrm{Sp}(n,\mathbb{C})$, $\mathrm{SO}^*(2n)$, $\mathrm{SO}_0(2,n)$ or certain 2 exceptional groups of type E. Let $G_{\mathbb{C}}$ be the complexification of G.[3] Then in $G_{\mathbb{C}}$ there exist closed subsemigroups Γ with nonempty interior Γ^0

[2] The boundary value of $f(z)$ must be defined in a suitable way.

[3] To simplify the discussion we tacitly assume here that G is linear, i.e., admits a global complexification; however, this assumption is not essential.

such that $\Gamma \supset G$ and $\Gamma \neq G_{\mathbb{C}}$ (in particular, Γ and Γ^0 are invariant with respect to the two–sided action of the group G). The existence of such semigroups is closely related to the existence of proper closed invariant convex cones C in $\mathfrak{g}$, the Lie algebra of G: there is a bijective correspondence $\Gamma \leftrightarrow C$ between semigroups and cones, and each open semigroup Γ^0 has the form $G \exp iC^0$, where C^0 is the interior of the cone C. (See Olshanski [19], Hilgert and Neeb [6].) Thus, Γ^0 indeed looks as a 'noncommutative' tube domain with skeleton G.

It turns out that for any semigroup Γ one can define a Hilbert space $H^2(\Gamma) \subset \mathcal{O}(\Gamma^0)$, which is an analogue of the Hardy H^2 space. The space $H^2(\Gamma)$ admits a canonical isometric embedding into $L^2(G)$ as a two–sided G-invariant subspace. When Γ is the so–called minimal semigroup, $H^2(\Gamma)$ just carries the holomorphic discrete series of G (in general case $H^2(\Gamma)$ carries a part of the holomorphic discrete series). Let $K(z,w)$ be the kernel function of the space $H^2(\Gamma)$ (it is also called the *Cauchy–Szegö* kernel). Then the kernel $K(z,g)$, where $z \in \Gamma$, $g \in G$,[4] defines the orthoprojection $L^2(G) \to H^2(\Gamma)$. In particular, if the semigroup is minimal then this is the orthoprojection onto the holomorphic discrete series.

The idea of Hardy spaces carrying the holomorphic discrete series is due to Gelfand and Gindikin [4]. A construction of the spaces $H^2(\Gamma)$ was given in author's paper [20]. Further works in this direction are Hilgert and Ólafsson [7], Hilgert, Ólafsson, and Ørsted [8].

A natural problem is to compute the Cauchy–Szegö kernel explicitly, at least for the minimal semigroups. To this end, one can try to start from the following presentation of the Cauchy–Szegö kernel:

Assume the center of G is finite. Let λ range over the set of the highest weights of the holomorphic discrete series representations T_λ occuring in the decomposition of $H^2(\Gamma)$ (recall that if Γ is minimal, the whole holomorphic discrete series occurs). Let $\operatorname{fdim}\lambda$ stand for the formal dimension of T_λ, let χ_λ denote the character of T_λ, which admits a canonical holomorphic extension to Γ^0, and let $w \mapsto w^\#$ be the antilinear antiautomorphism of the semigroup that extends the antiautomorhism $g \mapsto g^{-1}$ of the group G. Then we have

$$K(z,w) = \sum_\lambda \operatorname{fdim}(\lambda) \cdot \chi_\lambda(w^\# z), \qquad z, w \in \Gamma^0. \tag{0.1}$$

Since for both $\operatorname{fdim}(\lambda)$ and χ_λ there exist nice formulas, the problem consists in evaluating the series

$$\sum_\lambda \operatorname{fdim}(\lambda) \cdot \chi_\lambda(\gamma), \qquad \gamma \in \Gamma^0. \tag{0.2}$$

The idea of this approach to computing the Cauchy–Szegö kernel was indicated in [4]; there it was illustrated on the simplest example of the group $\mathrm{SU}(1,1) = \mathrm{Sp}(1,\mathbb{R})$. In the present paper, the sum (0.2) (and hence the Cauchy–Szegö kernel) is computed for three families of groups G: the symplectic groups

[4] Note that the kernel is still well–defined if one of the argument lies in the interior Γ^0 of the semigroup and another argument – on the boundary $\Gamma \setminus \Gamma^0$.

$\mathrm{Sp}(n,\mathbb{R})$, their two–sheeted coverings (the metaplectic groups) $Mp(n,\mathbb{R})$, and the pseudo–unitary groups $\mathrm{SU}(p,q)$ (note that for the first and second families the semigroup Γ is unique, and for the third family only the minimal semigroup is considered). These are the main results of the paper.

Two methods to sum the series (0.2) are presented. The first method is based on a certain preliminary transformation of (0.2), which is effectued in general terms of root data and holds for any G. The second method consists in reducing the problem to a certain combinatorial identity involving finite–dimensional characters of classical groups (a Littlewood–type formula).

The case of $G=\mathrm{Sp}(n,\mathbb{R})$ seems to be the simplest one. It can be easily handled by both methods. Moreover, in this case the result also can be obtained by a direct elementary computation: in fact, this computation, performed by my student V. Ivanov, was the starting point of the present work.

The other two cases require more efforts. For $G=Mp(n,\mathbb{R})$ I apply the first method, and for $G=\mathrm{SU}(p,q)$ — the second one.

Note that an analogue of the Hardy space $H^2(\Gamma)$ also can be defined when the groups G are replaced by certain pseudo–Riemannian symmetric spaces G/H; then the role of $\Gamma^0\subset G_{\mathbb{C}}$ is played by a domain in $G_{\mathbb{C}}/H_{\mathbb{C}}$ (see [8]). In the particular case of the hyperboloids $G/H=\mathrm{SO}_0(1,n)/\mathrm{SO}_0(1,n-1)$ the Cauchy–Szegö kernel was found by Molchanov . The paper [13] by Koufany and Ørsted contains (among other things) a calculation of the Cauchy–Szegö kernel for the group $U(1,1)$, and in the next papers [14], [15] these authors computed the 'odd' part of the kernel for the groups $G=Mp(n,\mathbb{R})$ and $G=Spin^*(2n)$ and also the whole kernel for $G=U(p,q)$. Note that the approach of [13]–[15] is quite different from that of the present paper (it does not use summation of characters).

It would be interesting to continue the study of Cauchy–Szegö kernels for groups and symmetric spaces and to single out the cases when the kernel admits a closed expression.

A related open problem, raised by J. Faraut, is to study (for the same groups and symmetric spaces) Bergman–type spaces of holomorphic functions and the corresponding kernels (as was observed by J. Faraut, the Bergman space on a semigroup Γ can be defined by integrating $|f(z)|^2$ with respect to the Haar measure of $G_{\mathbb{C}}$, restricted to Γ^0, and a similar definition also holds when $G_{\mathbb{C}}$ is replaced by $G_{\mathbb{C}}/H_{\mathbb{C}}$).

The paper is organized as follows.

– Sections 1–2 contain basic notations and definitions.

– In Sections 3, I use Harish–Chandra's result [5] about the formal dimensions of the holomorphic discrete series representations to write down the Cauchy–Szegö kernel as a series involving finite–dimensional characters only (Corollary 3.5).

– Section 4 is devoted to a useful transformation of the series (0.2) (Theorem 4.2). As a corollary, on obtains that the series is essentially a rational function (Theorem 4.3): this is the only result that I could prove for general groups G.

– Sections 5–7 contain proofs of the main results (Theorems 5.1, 6.1, and 7.1).

– In Section 8, I prove two character identities (Lemma 5.2 and Lemma 7.4), which are used in the second proof of Theorem 5.1 and in Theorem 7.1, respectively.

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1. Basic notation

Let $\mathfrak{g}$ be a simple real Lie algebra, let $\mathfrak{k}\subset\mathfrak{g}$ be a maximal compact subalgebra, and let $\mathfrak{h}\subset\mathfrak{k}$ be a Cartan subalgebra. One supposes that $\mathfrak{k}$ is not semisimple: $\mathfrak{k}\neq[\mathfrak{k},\mathfrak{k}]$; then $\mathfrak{k}$ has a one–dimensional center and $\mathfrak{h}$ also is a Cartan subalgebra for $\mathfrak{g}$.

The Lie algebras $\mathfrak{g}$ with $\mathfrak{k}\neq[\mathfrak{k},\mathfrak{k}]$ are called *Hermitian Lie algebras* [5]; recall that there exist four infinite series of classical Hermitian algebras ($\mathfrak{su}(p,q)$, $\mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{o}^*(2n)$, $\mathfrak{o}(2,n)$) and two exceptional ones.

Let the symbol $(\cdot)_{\mathbb{C}}$ denote complexification. Fix a system Δ^+ of positive roots for $(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$; then Δ^+ is disjoint union of Δ^+_c and Δ^+_n, the compact and noncompact roots, respectively.[6] Let ρ (respectively, ρ_c, ρ_n) be the half–sum of the roots in Δ^+ (respectively, Δ^+_c, Δ^+_n); then $\rho=\rho_c+\rho_n$. Let W and W_c be the Weyl groups for $(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$ and $(\mathfrak{k}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$, respectively.

Put $\mathfrak{h}_{\mathrm{Re}}=i\mathfrak{h}\subset\mathfrak{h}_{\mathbb{C}}$, let $\mathfrak{h}^*_{\mathrm{Re}}$ be the dual space, and let $\langle\cdot,\cdot\rangle$ be the pairing between $\mathfrak{h}^*_{\mathrm{Re}}$ and $\mathfrak{h}_{\mathrm{Re}}$. Note that $\Delta^+\subset\mathfrak{h}^*_{\mathrm{Re}}$. For a root $\alpha\in\Delta^+$, let $\alpha^\vee\in\mathfrak{h}_{\mathrm{Re}}$ be the dual root. Let $P\subset\mathfrak{h}^*_{\mathrm{Re}}$ be the lattice of weights for $(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$; recall that $\langle\lambda,\alpha^\vee\rangle\in\mathbb{Z}$ for $\lambda\in P$ and $\alpha\in\Delta^+$.

Let $P^+\subset P$ (resp., $P_+\subset P$) be the set of dominant weights relative to Δ^+ (resp., Δ^+_c). I.e., these are weights $\lambda\in P$ satisfying the condition $\langle\lambda,\alpha^\vee\rangle\geq 0$ for $\alpha\in\Delta^+$ (for $\alpha\in\Delta^+_c$, respectively). In particular, P^+ is contained in P_+. Further, let P_{+-} consists of those weights $\lambda\in P_+$ that satisfy the condition $\langle\lambda+\rho,\beta^\vee\rangle<0$ for each $\beta\in\Delta^+_n$.

Let $G_{\mathbb{C}}$ be the connected simply connected complex Lie group corresponding to $\mathfrak{g}_{\mathbb{C}}$, and let G, K, H, $K_{\mathbb{C}}$, and $H_{\mathbb{C}}$ stand for the (connected) subgroups in $G_{\mathbb{C}}$ corresponding to the subalgebras $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{k}_{\mathbb{C}}$, and $\mathfrak{h}_{\mathbb{C}}$, respectively. Note that the group G is not simply connected: its fundamental group is

[5] Because these are algebras corresponding to Hermitian symmetric spaces.

[6] Recall that a root is said to be *compact* if the corresponding root vector lies in $\mathfrak{k}_{\mathbb{C}}$, and *noncompact*, otherwise.

isomorphic to $\mathbb{Z}$.

2. The Cauchy–Szegö kernel

As main references to this section one may use Vinberg [24] and Olshanski [19], [20]. Also see the introductory paper Faraut [2].

We maintain the notation of section 1. The property $\mathfrak{k} \neq [\mathfrak{k}, \mathfrak{k}]$ implies that in the vector space $i\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ there exist (nontrivial) closed convex G-invariant cones. Among them there is a unique (within multiplication by -1) minimal cone $C_{\min}$. Let $c_{\min} \subset \mathfrak{h}_{\mathrm{Re}}$ denote the closed convex cone spanned by the dual roots $\beta^\vee$, $\beta \in \Delta_n^+$; then $C_{\min} \cap \mathfrak{h}_{\mathrm{Re}}$ is either $c_{\min}$ or $-c_{\min}$ (depending on agreement), and we may choose the first variant.

Put $\Gamma = G \exp C_{\min} \subset G_{\mathbb{C}}$; this is a complex semigroup in $G_{\mathbb{C}}$. Let $C^0_{\min}$ be the interior of the cone $C_{\min} \subset i\mathfrak{g}$; then $\Gamma^0 := G \exp C^0_{\min}$ is the interior of $\Gamma \subset G_{\mathbb{C}}$; hence Γ^0 is a complex variety.

Let $\mathcal{O}(\Gamma^0)$ be the space of holomorphic functions on Γ^0; $\mathcal{O}(\Gamma^0)$ is equipped with the topology of uniform convergence on compact subsets of Γ^0. The *Hardy space* $H^2(G)$ is a certain Hilbert space, continuously embedded into $\mathcal{O}(\Gamma^0)$. It consists of those functions $f \in \mathcal{O}(\Gamma^0)$ for which

$$\|f\|^2 := \sup_{\gamma \in \Gamma^0} \int |f(g\gamma)|^2 dg < \infty.$$

The reproducing kernel for this Hilbert space is called the *Cauchy–Szegö kernel* for the group G (or the semigroup Γ). This is a certain function $K(\gamma_1, \gamma_2)$ on $\Gamma^0 \times \Gamma^0$, holomorphic in γ_1 and antiholomorphic in γ_2.

Note that $K(\gamma_1, \gamma_2)$ is still well–defined when one of the arguments belongs to the boundary $\Gamma \setminus \Gamma^0$ of Γ^0. This allows one to put

$$L(\gamma) = K(\gamma, e), \qquad \gamma \in \Gamma^0,$$

where e denotes the unity of the group $G_{\mathbb{C}}$. The function $L(\gamma)$ is a holomorphic function in Γ^0, invariant under transformations $\gamma \mapsto g\gamma g^{-1}$, where g ranges over G. Note that the kernel $K(\gamma_1, \gamma_2)$ can be recovered from the function $L(\gamma)$:

$$K(\gamma_1, \gamma_2) = L(\gamma_2^{\#} \gamma_1), \tag{2.1}$$

where $\gamma \mapsto \gamma^{\#}$ denotes the involutive antiholomorphic antiautomorphism of the semigroup Γ, defined by

$$(g \exp X)^{\#} = (\exp X) g^{-1} = g^{-1} \exp(\mathrm{Ad}(g) X).$$

The objective of the present paper is to calculate, for certain groups G, the Cauchy–Szegö kernel K (or, that is the same, the corresponding function L in one variable).

Let $c^0_{\min}$ be the interior of the cone $c_{\min} \subset \mathfrak{h}_{\mathrm{Re}}$, let $H^+_{\mathbb{C}} = H \exp c^0_{\min}$, and let $H^+_{\mathrm{Re}} = \exp c^0_{\min}$. Note that $H^+_{\mathbb{C}} = H_{\mathbb{C}} \cap \Gamma^0$.

Lemma 2.1. *The function L is uniquely determined by its restriction to $H^+_{\mathbb{C}}$, even to H^+_{Re}.*

Proof. Since L is a holomorphic function on Γ^0, it is uniquely determined by its restriction to $\exp C^0_{\min} \subset \Gamma^0$. Now let us regard L as a function on the open cone $C^0_{\min}$. Since L is G-invariant and each G-orbit in $C^0_{\min}$ has a nonempty intersection with $c^0_{\min} \subset \mathfrak{h}_{\mathrm{Re}}$ (see Vinberg [24], Theorem 5), L is uniquely determined by its restriction to $c^0_{\min}$, which completes the proof. ■

3. The holomorphic discrete series and the Harish–Chandra correspondence

The main reference for this section is Harish–Chandra [5]. See also Faraut [2].

Recall that the holomorphic discrete series for the group G consists of those irreducible unitary representations of G that are square–integrable and possess a highest weight λ. The possible values for λ are exactly the set P_{+-} defined above in section 1. Given $\lambda \in P_{+-}$, let us denote by T_λ the corresponding representation of G, which is uniquely determined by the weight λ. Each T_λ admits a (unique) holomorphic extension to a representation $\mathcal{T}_\lambda$ of the semigroup Γ, operating in the Hilbert space of T_λ. For $\gamma \in \Gamma^0$, the operator $\mathcal{T}_\lambda(\gamma)$ is contractive (i.e., $\|\mathcal{T}_\lambda(\gamma)\| \le 1$) and depends holomorphically on γ. Moreover, $\mathcal{T}_\lambda(\gamma)$ is of trace class. (About holomorphic extensions of highest weight unitary representations see [19].[7])

Lemma 3.1. *Let* $\operatorname{fdim} T_\lambda$ *denote the formal dimension of* T_λ (*as a square–integrable representation*). *Then*

$$L(\gamma) = \sum_{\lambda \in P_{+-}} \operatorname{fdim} T_\lambda \cdot \operatorname{tr} \mathcal{T}_\lambda(\gamma), \qquad \gamma \in \Gamma^0. \tag{3.1}$$

Note that both $\operatorname{fdim} T_\lambda$ and $L(\gamma)$ depend on the normalization of the Haar measure dg of the group G.

Proof. For each $\lambda \in P_{+-}$, choose an orthonormal basis $\xi_{\lambda 1}, \xi_{\lambda 2}, \dots$ in the Hilbert space of T_λ. Recall that our Hardy space carries the whole holomorphic discrete series. Therefore, it follows from the orthogonality relations that the (normalized) matrix coefficients

$$f_{\lambda ij}(\gamma) := (\operatorname{fdim} T_\lambda)^{1/2} (\mathcal{T}_\lambda(\gamma)\xi_{\lambda i}, \xi_{\lambda j}), \qquad \gamma \in \Gamma^0,$$

where $\lambda \in P_{+-}$ and $i, j = 1, 2, \dots$, constitute an orthonormal basis in the Hardy space.

It follows that

$$L(\gamma) = K(\gamma, e) = \sum_{\lambda, i, j} f_{\lambda ij}(\gamma) \overline{f_{\lambda ij}(e)}$$

[7] For general Lie groups these topics are discussed in Neeb's papers [17], [18].

$$
\begin{aligned}
&= \sum_{\lambda,i} (\operatorname{fdim} T_\lambda)^{1/2} f_{\lambda ii}(\gamma) \\
&= \sum_{\lambda} \operatorname{fdim} T_\lambda \sum_i (\mathcal{T}_\lambda(\gamma)\xi_{\lambda i}, \xi_{\lambda i}) \\
&= \sum_{\lambda} \operatorname{fdim} T_\lambda \cdot \operatorname{tr} \mathcal{T}_\lambda(\gamma). \qquad \blacksquare
\end{aligned}
$$

Formula (3.1) is the starting point for our computation of the function L.

Since $P_{+-} \subset P_+$, for each $\lambda \in P_{+-}$ there exists an irreducible finite–dimensional representation π_λ of the complex group $K_{\mathbb{C}}$ with highest weight λ. For $x \in H_{\mathbb{C}}$ and $\lambda \in P$, let us write x^λ for the value of λ, viewed as a one–dimensional character of the torus $H_{\mathbb{C}}$, at x. That is to say, $x^\lambda = \exp\langle \lambda, \log x\rangle$.

Lemma 3.2. *For each* $\lambda \in P_{+-}$

$$\operatorname{tr} \mathcal{T}_\lambda(x) = \operatorname{tr} \pi_\lambda(x)\theta(x), \qquad x \in H_{\mathbb{C}},$$

where

$$\theta(x) = \prod_{\beta \in \Delta_n^+} (1 - x^{-\beta})^{-1}. \tag{3.2}$$

Proof. Assume first T_λ is an arbitrary irreducible unitary highest weight representation of G (not necessarily square integrable). Consider the corresponding irreducible unitarizable Harish–Chandra $(\mathfrak{g}, K)$-module that is realized in the space of K-finite vectors of T_λ. Then this module is either the generalized Verma module $\mathcal{M}_\lambda$ (induced from an irreducible finite–dimensional module π_λ over a maximal parabolic subalgebra of $\mathfrak{g}$) or the minimal proper quotient $\mathcal{L}_\lambda$ of $\mathcal{M}_\lambda$. In the former case T_λ is called *nondegenerate*, and in the latter case it is called *degenerate*. It is easily verified that for a nondegenerate T_λ, the trace of its holomorhic extension $\mathcal{T}_\lambda$ is given by the above formula. Finally one applies a Harish–Chandra's result [5] stating (in our terms) that if T_λ is square–integrable then it is nondegenerate. $\blacksquare$

Let $w_0 \in W_c$ stand for the (unique) element of maximal length; note that $w_0^2 = 1$. Then

$$w_0(\Delta_c^+) = -\Delta_c^+, \qquad w_0(\Delta_n^+) = \Delta_n^+,$$

whence

$$w_0(\rho_c) = -\rho_c, \qquad w_0(\rho_n) = \rho_n.$$

Define $\varphi : P \to P$ by

$$\varphi(\lambda) = -w_0(\lambda + \rho) - \rho = -w_0(\lambda) - 2\rho_n.$$

Note that

$$-w_0(\lambda + \rho) = \varphi(\lambda) + \rho,$$

so φ is an involutive map of P.

Lemma 3.3. *One has $\varphi(P_{+-}) = P^+$ and $\varphi(P^+) = P_{+-}$, so that φ defines a bijective correspondence $P_{+-} \leftrightarrow P^+$.*

Proof. Remark that P^+ and P_{+-} may be described as follows:

$$P^+ = \{\mu \in P \mid \langle \mu + \rho, \alpha^\vee \rangle > 0 \quad \text{for each } \alpha \in \Delta^+\}$$

$$P_{+-} = \{\lambda \in P \mid \langle \lambda + \rho, \alpha^\vee \rangle > 0 \quad \text{for } \alpha \in \Delta_c^+ \text{ and } \langle \lambda + \rho, \beta^\vee \rangle < 0 \text{ for } \beta \in \Delta_n^+\}$$

Now the claim of the Lemma follows from the fact that

$$-w_0(\Delta_c^+) = \Delta_c^+, \qquad -w_0(\Delta_n^+) = -\Delta_n^+. \qquad \blacksquare$$

For $\mu \in P^+$, let V_μ denote the irreducible finite–dimensional (complex–analytic) representation of the group $G_{\mathbb{C}}$ with highest weight μ, and let $\dim V_\mu$ denote the dimension of V_μ.

Lemma 3.4. *For an appropriate normalization of the Haar measure dg on the group G,*

$$\operatorname{fdim} T_\lambda = \dim V_{\varphi(\lambda)} \qquad \textit{for each } \lambda \in P_{+-}\,. \tag{3.3}$$

Proof. For a proof, see Harish–Chandra [5], paper VI, Theorem 4. $\blacksquare$

It is natural to call the correspondence $P_{+-} \leftrightarrow P^+$ or $\{T_\lambda\} \leftrightarrow \{V_{\varphi(\lambda)}\}$ the *Harish–Chandra correspondence.*

Throughout the paper we assume that the normalization of the Haar measure is chosen so that relation (3.3) does hold.

Lemmas 3.1–3.4 imply the following

Corollary 3.5. *For each $x \in H_{\mathbb{C}}^+$*

$$L(x) = \theta(x) \sum_{\mu \in P^+} \dim V_\mu \cdot \operatorname{tr} \pi_{\varphi(\mu)}(x), \tag{3.4}$$

where $\theta(x)$ is defined by (3.2). $\blacksquare$

Example 3.6. Consider the simplest case $G = \mathrm{SU}(1,1)$. Then each element $x \in H_{\mathbb{C}}^+$ may be viewed as the 2×2 diagonal matrix with the diagonal entries x^{-1}, x, where x now stands for a complex number, $|x| < 1$. Then the series (3.4) turns into

$$L(x) = (1 - x^2)^{-1} \sum_{m=0}^{\infty} (m+1) \frac{x^{m+2}}{1 - x^2} = \frac{x^2}{(1-x)^3(1+x)}. \tag{3.5}$$

An equivalent expression can be found in Gelfand and Gindikin [4] (see also Faraut [2]).

4. An expression for the function L

Let $\varepsilon : W \to \{\pm 1\}$ denote the standard multiplicative function,

$$\varepsilon(w) = (-1)^{\mathrm{length}(w)}.$$

We shall need Weyl's character formulas for the groups $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$:

$$\operatorname{tr} V_\mu(z) = \frac{\sum_{w\in W} \varepsilon(w) w(z)^{\mu+\rho}}{\sum_{w\in W} \varepsilon(w) w(z)^{\rho}}, \qquad z \in H_{\mathbb{C}}, \quad \mu \in P^+, \tag{4.1}$$

$$\operatorname{tr} \pi_\lambda(z) = \frac{\sum_{w'\in W_c} \varepsilon(w') w'(z)^{\lambda+\rho}}{\sum_{w'\in W_c} \varepsilon(w') w'(z)^{\rho}}, \qquad z \in H_{\mathbb{C}}, \quad \lambda \in P_+. \tag{4.2}$$

(It should be noted that there is a small subtlety in formula (4.2), because the group $K_{\mathbb{C}}$ is not semisimple and ρ is distinct from ρ_c, the object associated with the semisimple group $[K_{\mathbb{C}}, K_{\mathbb{C}}]$. But it is not difficult to derive (4.2) from Weyl's formula for the semisimple group $[K_{\mathbb{C}}, K_{\mathbb{C}}]$.)

Let D_G and D_K denote the denominators in (4.1) and (4.2), respectively:

$$D_G(z) = \sum_{w\in W} \varepsilon(w) w(z)^\rho = \prod_{\alpha\in\Delta^+} (z^{\alpha/2} - z^{-\alpha/2}), \tag{4.3}$$

$$D_K(z) = \sum_{w'\in W_c} \varepsilon(w') w'(z)^\rho = z^{\rho_n} \prod_{\alpha\in\Delta_c^+} (z^{\alpha/2} - z^{-\alpha/2}). \tag{4.4}$$

These are regular functions on the torus $H_{\mathbb{C}}$.

(Strictly speaking, in (4.1) and (4.2) one should assume that $D_G(z) \neq 0$ and $D_K(z) \neq 0$, respectively.)

Let $\Omega \subset \mathfrak{h}^*_{\mathrm{Re}}$ be the dual cone of the cone $c_{\min}$, i.e.,

$$\Omega = \{\nu \in \mathfrak{h}^*_{\mathrm{Re}} \mid \langle \nu, \beta^\vee\rangle \geq 0 \qquad \text{for each } \beta \in \Delta_n^+\}. \tag{4.5}$$

Since P is a lattice in the vector space $\mathfrak{h}^*_{\mathrm{Re}}$, the intersection $\Omega \cap P$ is an additive monoid.

Lemma 4.1. *Suppose $z \in H_{\mathbb{C}}$ is such that $z^{-1} \in H^+_{\mathbb{C}}$. Then the sum*

$$F(z) := \sum_{\nu\in\Omega\cap P} z^\nu \tag{4.6}$$

is absolutely convergent.

(One could call $F(z)$ the *characteristic function* of the monoid $\Omega \cap P$.)

Proof. Write $z = ux$ where $u \in H$, $x = \exp(-X)$, and $X \in c^0_{\min}$. Then $|z^\nu| = \exp(-\langle \nu, X\rangle)$. Since the linear functional $\langle \cdot, X\rangle$ is strictly positive on $\Omega \setminus \{0\}$, the claim follows from obvious estimates. ∎

Let $x \mapsto \widehat{x}$ be the involutive map of the torus $H_{\mathbb{C}}$ defined by

$$\widehat{x} = w_0(x)^{-1}.$$

Note that if $x \in H^+_{\mathbb{C}}$ then the function $F(z)$ is well–defined in a neighborhood of the element $\widehat{x}$ in $H_{\mathbb{C}}$.

Theorem 4.2. *Let the group G satisfy the assumption of* section 1 *and let $L(x)$ be the function on $H_{\mathbb{C}}^{+}$, associated with the Cauchy–Szegö kernel of the Hardy space $H^2(G)$. Let $x \in H_{\mathbb{C}}^{+}$ be such that $D_K(x) \neq 0$, and let t range over the subset of elements of $H_{\mathbb{C}}$ satisfying $D_G(t) \neq 0$. Then*

$$L(x) = \frac{\theta(x)}{D_K(x)} \lim_{t \to e} \frac{1}{D_G(t)} \sum_{w \in W} \varepsilon(w) F(w(t)\widehat{x}), \tag{4.7}$$

where F is the function on $\Omega \cap P$, defined in (4.6).

Proof. We shall transform formula (3.4) of Corollary 3.5. From

$$\dim V_\mu = \operatorname{tr} V_\mu(e) = \lim_{t \to e} \operatorname{tr} V_\mu(t),$$

it follows

$$\dim V_\mu = \lim_{t \to e} \frac{1}{D_G(t)} \sum_{w \in W} \varepsilon(w)(w(t))^{\mu+\rho}, \tag{4.8}$$

by virtue of (4.1) and (4.3).

Further,

$$\operatorname{tr} \pi_{\varphi(\mu)}(x) = \frac{1}{D_K(x)} \sum_{w' \in W_c} \varepsilon(w')(w'(x))^{\varphi(\mu)+\rho}, \tag{4.9}$$

by virtue of (4.2) and (4.4).

Let us rewrite the right–hand side of (4.9) by making use of the definition of φ and $\widehat{x}$:

$$\begin{aligned}
(w'(x))^{\varphi(\mu)+\rho} &= (w'(x))^{-w_0(\mu+\rho)} \\
&= ((w_0 w')(x^{-1}))^{\mu+\rho} \\
&= ((w_0 w' w_0)(w_0(x)^{-1}))^{\mu+\rho} \\
&= ((w_0 w' w_0)(\widehat{x}))^{\mu+\rho}.
\end{aligned}$$

Since $\varepsilon(w') = \varepsilon(w_0 w' w_0)$, one may replace $w_0 w' w_0$ by w', and finally we obtain

$$\operatorname{tr} \pi_{\varphi(\mu)}(x) = \frac{1}{D_K(x)} \sum_{w' \in W_c} \varepsilon(w')(w'(\widehat{x}))^{\mu+\rho}. \tag{4.10}$$

By substituting the expressions (4.8) and (4.10) into formula (3.4) of Corollary 3.5 we obtain

$$L(x) = \frac{\theta(x)}{D_K(x)} \lim_{t \to e} \frac{1}{D_G(t)} \sum_{\mu \in P+} \sum_{w \in W} \sum_{w' \in W_c} \varepsilon(w)\varepsilon(w')(w(t)w'(\widehat{x}))^{\mu+\rho}.$$

After a change of variables, $(w, w') \mapsto (w'w, w')$, this turns into

$$L(x) = \frac{\theta(x)}{D_K(x)} \lim_{t \to e} \frac{1}{D_G(t)} \sum_{\mu \in P+} \sum_{w \in W} \sum_{w' \in W_c} \varepsilon(w)(w(t)\widehat{x})^{w'(\mu+\rho)}. \tag{4.11}$$

Let P_{reg} denote the set of regular weights,

$$P_{\mathrm{reg}} = \{\nu \in P \mid \langle \nu, \alpha^\vee \rangle \neq 0 \qquad \text{for each } \alpha \in \Delta^+\}.$$

When μ runs through P^+, the weight $\mu+\rho$ runs through $P^+ \cap P_{\mathrm{reg}}$. Since the sets of the form $w'(P^+ \cap P_{\mathrm{reg}})$ (where $w' \in W_c$) are pairwise disjoint, we have

$$\sum_{w' \in W_c} \sum_{\mu \in P^+} (w(t)\widehat{x})^{w'(\mu+\rho)} = \sum_{\{\nu\}} (w(t)\widehat{x})^\nu,$$

where

$$\{\nu\} := \bigcup_{w' \in W_c} w'(P^+ \cap P_{\mathrm{reg}}).$$

The set $\{\nu\}$ can be written as

$$\{\nu \in P_{\mathrm{reg}} \mid \langle \nu, \beta^\vee \rangle \geq 0 \qquad \text{for each } \beta \in \Delta_n^+\},$$

from which it follows that

$$\{\nu\} = \Omega \cap P_{\mathrm{reg}}.$$

Therefore,

$$L(x) = \frac{\theta(x)}{D_K(x)} \lim_{t \to e} \frac{1}{D_G(t)} \sum_{w \in W} \varepsilon(w) \sum_{\nu \in \Omega \cap P_{\mathrm{reg}}} (w(t)\widehat{x})^\nu.$$

Now we remark that

$$\sum_{w \in W} \varepsilon(w) w(t)^\nu = 0 \qquad \text{if } \nu \in P \setminus P_{\mathrm{reg}}.$$

Hence summation over $\nu \in \Omega \cap P_{\mathrm{reg}}$ can be replaced by that over $\nu \in \Omega \cap P$. By virtue of the definition (4.6) of the function F we obtain the desired formula. ■

The argument of Theorem 4.2 implies the following corollary:

Theorem 4.3. *Let the group G satisfy the assumption of* section 1, *and let $L(x)$ be the function on $H_{\mathbb{C}}^+ = H_{\mathbb{C}} \cap \Gamma^0$, associated with the Cauchy–Szegö kernel of the Hardy space $H^2(G)$. Then $L(x)$ is restriction to $H_{\mathbb{C}}^+$ of a rational function on the complex torus $H_{\mathbb{C}}$.*

Proof. Probably, the first idea which occurs when looking at formula (4.7) of Theorem 4.2 is to show that the 'characteristic function' (4.6) of the additive monoid $\Omega \cap P$ is rational. However, the structure of this monoid seems to be cumbersome, so it is better to use, instead of (4.7), formula (4.11) (an advantage of the latter formula is that the structure of the monoid P^+ is very simple).

It is immediate from the definition of θ, D_G, and D_K (see (3.2), (4.3), and (4.4)) that these are rational functions. Since both w and w' in (4.11) range over finite sets, it suffices to prove that the series

$$\sum_{\mu \in P^+} z^{\mu+\rho}, \qquad \widehat{z} \in H_{\mathbb{C}}^+, \tag{4.12}$$

is a rational function.

Let $\omega_1, \dots, \omega_r$ stand for the fundamental weights of the system Δ^+, where r is the rank of $\mathfrak{g}_{\mathbb{C}}$. Then we have

$$P^+ = \mathbb{Z}_+\omega_1 + \dots + \mathbb{Z}_+\omega_r,$$

whence

$$\sum_{\mu \in P^+} z^{\mu+\rho} = z^\rho \sum_{\mu \in P^+} z^\mu = z^\rho \prod_{i=1}^{r} \frac{1}{1 - z^{\omega_i}},$$

which is a rational function. ■

5. The case of $\mathrm{Sp}(n, \mathbb{R})$

Throughout this section one assumes $G = \mathrm{Sp}(n, \mathbb{R})$ and $G_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C})$. Let

$$\xi = (\xi_1, \dots, \xi_{2n}) \qquad \text{and} \qquad \eta = (\eta_1, \dots, \eta_{2n})$$

be arbitrary vectors in $\mathbb{C}^{2n}$. Let us regard $G_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C})$ as the group of automorphisms of the alternating form

$$\xi_1\eta_{2n} + \dots + \xi_n\eta_{n+1} - \xi_{n+1}\eta_n - \dots - \xi_{2n}\eta_1,$$

and realize the group $G = \mathrm{Sp}(n, \mathbb{R})$ as the intersection $U(n,n) \cap \mathrm{Sp}(n, \mathbb{C})$ where $U(n,n)$ is realized as the group preserving the Hermitian form

$$[\xi, \eta] = -\xi_1\bar{\eta}_1 - \dots - \xi_n\bar{\eta}_n + \xi_{n+1}\bar{\eta}_{n+1} + \dots + \xi_{2n}\bar{\eta}_{2n}. \tag{5.1}$$

The subgroup K, which is isomorphic to $U(n)$, is realized as the subgroup of G that fixes the direct sum decomposition $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$.

Given complex numbers $z_1, \dots, z_{2n}$, let us write $\mathrm{diag}(z_1, \dots, z_{2n})$ for the diagonal matrix with diagonal entries $z_1, \dots, z_{2n}$.

The space $\mathfrak{h}_{\mathrm{Re}}$ consists of the real matrices of the form

$$X = \mathrm{diag}(X_1, \dots, X_n, -X_n, \dots, -X_1);$$

it will be identified with $\mathbb{R}^n$ via the map $X \mapsto (X_1, \dots, X_n) \in \mathbb{R}^n$. The dual space $\mathfrak{h}^*_{\mathrm{Re}}$ may be identified with $\mathfrak{h}_{\mathrm{Re}}$ and hence with $\mathbb{R}^n$, too.

Let $\epsilon_1, \dots, \epsilon_n$ be the canonical basis of $\mathfrak{h}^*_{\mathrm{Re}} = \mathbb{R}^n$. Then

$$\Delta_c^+ = \{\epsilon_i - \epsilon_j \quad (1 \le i < j \le n)\},$$

$$\Delta_n^+ = \{\epsilon_i + \epsilon_j \quad (1 \le i \le j \le n)\},$$

$$\Delta^+ = \{\epsilon_i \pm \epsilon_j \quad (1 \le i < j \le n), \qquad 2\epsilon_i \quad (1 \le i \le n)\},$$

$$\rho = (n, n-1, \dots, 1),$$

$$\rho_c = (\tfrac{n-1}{2}, \tfrac{n-3}{2}, \dots, -\tfrac{n-3}{2}, -\tfrac{n-1}{2}),$$

$$\rho_n = (\tfrac{n+1}{2}, \dots, \tfrac{n+1}{2}).$$

The weight lattice $P \subset \mathfrak{h}^*_{\mathrm{Re}} = \mathbb{R}^n$ is simply the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, and the set $P^+ \subset P$ is identified with the set of partitions of length $\le n$,

$$P^+ = \{\mu \in \mathbb{Z}^n \mid \mu_1 \ge \dots \mu_n \ge 0\}.$$

The Weyl group W_c is isomorphic to the symmetric group $\mathfrak{S}_n$ of degree n; it operates on $\mathfrak{h}_{\mathrm{Re}} = \mathfrak{h}^*_{\mathrm{Re}} = \mathbb{R}^n$ by permuting the basis vectors $\epsilon_1, \dots, \epsilon_n$. The Weyl group W is isomorphic to the hyperoctahedral group $\mathfrak{H}_n$, the semidirect product of $\mathfrak{S}_n$ with the Abelian group $\mathfrak{A}_n = \mathbb{Z}_2^n$; the i-th generator of $\mathfrak{A}_n$ multiplies the i-th basis vector ϵ_i by -1 and fixes all remaining vectors ϵ_j, $j \ne i$.

The cone $c_{\min} \subset \mathfrak{h}_{\mathrm{Re}} = \mathbb{R}^n$ is the 'hyperoctant' $\mathbb{R}^n_+ \subset \mathbb{R}^n$, and the dual cone Ω also coincides with $\mathbb{R}^n_+$.

The complex torus $H_{\mathbb{C}}$ consists of the diagonal matrices of the form

$$\mathrm{diag}(z_1, \dots, z_n, z_n^{-1}, \dots, z_1^{-1})$$

with nonzero complex $z_1, \dots, z_n$. The subset $H^+_{\mathbb{C}} \subset H_{\mathbb{C}}$ is described by the inequalities $|z_1| > 1, \dots, |z_n| > 1$.

The transformation $z \to \widehat{z} = w_0(z)^{-1}$ of the torus $H_{\mathbb{C}}$ takes the form

$$\mathrm{diag}(z_1, \dots, z_n, z_n^{-1}, \dots, z_1^{-1}) \mapsto \mathrm{diag}(z_n^{-1}, \dots, z_1^{-1}, z_1, \dots, z_n).$$

A complex $2n \times 2n$ matrix γ is called *weakly J-contractive* if $[\gamma\xi, \gamma\xi] \le [\xi, \xi]$ for each $\xi \in \mathbb{C}^{2n}$, where $[\cdot, \cdot]$ is the indefinite inner product (5.1). If the above inequality is strict for each $\xi \ne 0$, then γ is called *strictly J-contractive.* The semigroup Γ is formed by all weakly J-contractive matrices in $\mathrm{Sp}(n, \mathbb{C})$, and the interior $\Gamma^0 \subset \Gamma$ consists of strictly J-contractive symplectic matrices.

Note that the eigenvalues of each matrix $\gamma \in \Gamma^0$ are of the form

$$x_1, x_1^{-1}, \dots, x_n, x_n^{-1}, \qquad \text{where } |x_1|, \dots, |x_n| < 1. \tag{5.2}$$

Theorem 5.1. *Let $K(\gamma_1, \gamma_2)$ be the Cauchy–Szegö kernel for the Hardy space $H^2(\mathrm{Sp}(n, \mathbb{R})) \subset \mathcal{O}(\Gamma^0)$, where Γ^0 is the open semigroup of strictly J-contractive matrices in $\mathrm{Sp}(n, \mathbb{C})$. Let $L(\gamma) = K(\gamma, e)$ be the corresponding function in one variable $\gamma \in \Gamma^0$. Given $\gamma \in \Gamma^0$, write its eigenvalues as in* (5.2). *Finally, recall that the Haar measure on the group is normalized so that the relation* (3.3) *holds.*

Then

$$L(\gamma) = \prod_{i=1}^{n} \frac{x_i^{n+1}}{(1 + x_i)(1 - x_i)^{2n+1}}\,. \tag{5.3}$$

(Note that in the case $n = 1$ the expression (5.3) agrees with formula (3.5).)

This result can be derived in several ways. First this was done by Vladimir Ivanov (a student of the Moscow State University) by a direct calculation. Namely, I suggested him to evaluate the sum in the left–hand side of

(5.13) (see below), where the dimension of an irreducible $\mathrm{Sp}(n,\mathbb{C})$-module is expressed by classical Weyl's formula. When he obtained formula (5.13) I realized that this is in fact a corollary of the well–known character identity (5.12).

I give below two proofs of the theorem: the first proof is based on Theorem 4.2, while the second proof is simply reduction to the identity (5.12).

First Proof. Remark that the right–hand side of (5.3) is a holomorphic function on Γ^0, invariant under conjugation by elements of the group G. Then the argument of Lemma 2.1 shows that it suffices to check (5.3) for diagonal matrices $\gamma \in H_\mathbb{C} \cap \Gamma^0 = H_\mathbb{C}^+$. Hence one may assume

$$\gamma = x = \mathrm{diag}(x_n^{-1}, \dots, x_1^{-1}, x_1, \dots, x_n), \qquad |x_1|, \dots, |x_n| < 1,$$

whence

$$\widehat{x} = \mathrm{diag}(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}).$$

Further, by (3.2),

$$\theta(x) = \prod_{i \le j}(1 - x_i x_j)^{-1} = \prod_{i<j}(1 - x_i x_j)^{-1} \prod_i (1 - x_i^2)^{-1} \tag{5.4}$$

and, by (4.4),

$$\begin{aligned} D_K(x) &= (x_1 \dots x_n)^{-(n+1)/2} \prod_{i<j}(x_j^{-1/2} x_i^{1/2} - x_j^{1/2} x_i^{-1/2}) \\ &= (x_1 \dots x_n)^{-n} \prod_{i<j}(x_i - x_j), \end{aligned}$$

whence

$$\frac{\theta(x)}{D_K(x)} = \frac{(x_1 \dots x_n)^n}{\prod_{i<j}(x_i - x_j) \prod_{i<j}(1 - x_i x_j) \prod_i (1 - x_i^2)}. \tag{5.5}$$

Let

$$t = \mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \in H_\mathbb{C}.$$

Then, by (4.3),

$$\begin{aligned} D_G(t) &= \prod_{i<j}(t_i^{1/2} t_j^{-1/2} - t_i^{-1/2} t_j^{1/2}) \prod_{i \le j}(t_i^{1/2} t_j^{1/2} - t_i^{-1/2} t_j^{-1/2}) \\ &= \prod_{i<j}(t_i + t_i^{-1} - t_j - t_j^{-1}) \prod_i (t_i - t_i^{-1}) \end{aligned} \tag{5.6}$$

As the monoid $\Omega \cap P$ coincides with $\mathbb{Z}_+^n \subset \mathbb{R}^n$, its 'characteristic function' $F(z)$ is given by

$$F(\mathrm{diag}(z_1, \dots, z_n, z_n^{-1}, \dots, z_1^{-1})) = \prod_i (1 - z_i)^{-1}. \tag{5.7}$$

For $w \in W = \mathfrak{H}_n$ and $t \in H_{\mathbb{C}}$ as above, let $w(t)_i$ be the i-th diagonal entry of the matrix $w(t)$, $i = 1, \dots, n$. By Theorem 4.2 and formulas (5.5)–(5.7),

$$\begin{aligned} L(x) = {} & \frac{(x_1 \dots x_n)^n}{\prod_{i<j}(x_i - x_j)\prod_{i<j}(1 - x_i x_j)\prod_i (1 - x_i^2)} \\ & \cdot \lim_{t \to e} \frac{1}{\prod_{i<j}(t_i + t_i^{-1} - t_j - t_j^{-1})\prod_i (t_i - t_i^{-1})} \\ & \cdot \sum_{w \in \mathfrak{H}_n} \varepsilon(w) \prod_i \frac{1}{1 - w(t)_i x_i} \, . \end{aligned} \tag{5.8}$$

To evaluate the latter sum, let us write it as

$$\sum_{s \in \mathfrak{S}_n} \varepsilon(s) \sum_{a \in \mathfrak{A}_n} \varepsilon(a) \prod_i \frac{1}{1 - (a(s(t)))_i x_i} \, . \tag{5.9}$$

First, let us fix $s \in \mathfrak{S}_n$ and denote

$$u_1 = s(t)_1, \dots, u_n = s(t)_n \, ,$$

so that $(u_1, \dots, u_n)$ is a permutation of $(t_1, \dots, t_n)$, determined by s. Then the interior sum in (5.9) equals

$$\begin{aligned} & \prod_i \left(\frac{1}{1 - u_i x_i} - \frac{1}{1 - u_i^{-1} x_i} \right) \\ & \quad = \prod_i x_i \prod_i (u_i - u_i^{-1}) \prod_i \frac{1}{(1 - u_i x_i)(1 - u_i^{-1} x_i)} \\ & \quad = \prod_i x_i \prod_i (t_i - t_i^{-1}) \prod_i \frac{1}{(1 - u_i x_i)(1 - u_i^{-1} x_i)} \, . \end{aligned}$$

Let us substitute this expression into (5.9) and recall that $u_1, \dots, u_n$ are a permutation of $t_1, \dots, t_n$. It follows that alternating over $s \in \mathfrak{S}_n$ gives an $n \times n$ determinant, so that (5.9) turns into

$$\prod_i x_i \prod_i (t_i - t_i^{-1}) \det \left[\frac{1}{(1 - t_j x_i)(1 - t_j^{-1} x_i)} \right]_{1 \le i,j \le n} . \tag{5.10}$$

The determinant in (5.10) can be easily computed by reduction to the well–known Cauchy determinant (see Littlewood [16], Chap. XI, 11.7, Lemma III). The result is as follows

$$\begin{aligned} & \det \left[\frac{1}{(1 - t_j x_i)(1 - t_j^{-1} x_i)} \right]_{1 \le i,j \le n} \\ & \quad = \frac{\prod_{i<j}(t_i + t_i^{-1} - t_j - t_j^{-1})\prod_{i<j}(x_i - x_j)\prod_{i<j}(1 - x_i x_j)}{\prod_i \prod_j (1 - t_j x_i)(1 - t_j^{-1} x_i)} . \end{aligned} \tag{5.11}$$

Let us substitute this into (5.10) and next substitute the resulting expression into (5.8), instead of the sum. Then we obtain, after cancellations,

$$L(x) = \frac{(x_1 \dots x_n)^{n+1}}{\prod_i (1-x_i^2)} \lim_{t_1,\dots,t_n \to 1} \prod_i \prod_j \frac{1}{(1-t_j x_i)(1-t_j^{-1} x_i)}$$
$$= \frac{(x_1 \dots x_n)^{n+1}}{\prod_i (1-x_i^2)(1-x_i)^{2n}},$$

which coincides with (5.3). ■

Another way to establish formula (5.3) is to derive it from the following well–known character identity (which is one of the the so–called Littlewood formulas):

Lemma 5.2. *Let $M = (M_1 \geq M_2 \geq \dots \geq M_n \geq 0)$ range over the set of partitions of length $\leq n$; let $s_M(x_1,\dots,x_M)$ be the Schur function in n variables, indexed by M; and let $sp(n)_M(t_1^{\pm 1},\dots,t_n^{\pm 1})$ be the character of the irreducible finite–dimensional* $\mathrm{Sp}(n,\mathbb{C})$*-module V_M that is indexed by M (one regards the character as a function on the complex torus $H_{\mathbb{C}} \subset \mathrm{Sp}(n,\mathbb{C})$ of diagonal matrices* $\mathrm{diag}(t_1,\dots,t_n,t_n^{-1},\dots,t_1^{-1})$*). Then*

$$\sum_M sp(n)_M(t_1^{\pm 1},\dots,t_n^{\pm 1}) s_M(x_1,\dots,x_n) = \frac{\prod_{i<j}(1-x_i x_j)}{\prod_i \prod_j (1-t_j x_i)(1-t_j^{-1} x_i)} . \tag{5.12}$$

Proof. See Koike and Terada [12], Lemma 1.5.1, or Sundaram [21]. Another argument, due to M. Ishikawa and M. Wakayama (personal communication) is presented below in section 8. Finally, note that the identity (5.12) also can be derived from the explicit weight correspondence in Howe's duality between the compact group $\mathrm{Sp}(k)$ and noncompact group $\mathrm{SO}^*(2n)$, for the special case $k=n$. (See Howe [9] and Enright, Howe and Wallach [1].) ■

Corollary 5.3. *Let M, s_M, and V_M be as in* Lemma 5.2. *Then*

$$\sum_M \dim V_M \, s_M(x_1,\dots,x_n) = \frac{\prod_{i<j}(1-x_i x_j)}{\prod_i (1-x_i)^{2n}} . \tag{5.13}$$

Proof. Take $t_1 = \dots = t_n = 1$ in (5.12). ■

Second Proof of Theorem 5.1. Look at formula (3.4) of Corollary 3.5. We shall see that its right–hand side coincides (up to certain simple factors) with the left–hand side of (5.13), so that formula (5.3) is equivalent to formula (5.13).

Indeed, one may view each dominant weight $\mu \in P^+$ as a partition M of length $\leq n$: the correspondence $\mu \leftrightarrow M$ is given by

$$\mu = \mathrm{diag}(M_1,\dots,M_n,-M_n,\dots,-M_1) \in \mathfrak{h}^*_{\mathrm{Re}}.$$

In this notation,

$$\varphi(\mu) = \operatorname{diag}(-M_n, \dots, -M_1, M_1, \dots, M_n) + (n+1)\operatorname{diag}(-1, \dots, -1, 1, \dots, 1).$$

Let as above

$$\gamma = x = \operatorname{diag}(x_n^{-1}, \dots, x_1^{-1}, x_1, \dots, x_n).$$

Then

$$\operatorname{tr} \pi_{\varphi(\mu)}(x) = (x_1 \dots x_n)^{n+1} s_M(x_1, \dots, x_n).$$

Therefore, formula (3.4) turns into

$$L(x) = \theta(x)(x_1 \dots x_n)^{n+1} \sum_M \dim V_M \, s_M(x_1, \dots, x_n).$$

By using the expression (5.4) for $\theta(x)$ we obtain

$$L(x) = \frac{(x_1 \dots x_n)^{n+1}}{\prod_{i<j}(1 - x_i x_j) \prod_i (1 - x_i^2)} \sum_M \dim V_M s_M(x_1, \dots, x_n).$$

It follows that (5.3) and (5.13) are equivalent. ■

6. The case of $\widetilde{G} = Mp(n, \mathbb{R})$

Consider the metaplectic group $\widetilde{G} = Mp(n, \mathbb{R})$, which is the two–sheeted covering over the symplectic group $G = \mathrm{Sp}(n, \mathbb{R})$. Let $\widetilde{\Gamma}$ and $\widetilde{\Gamma}^0$ be the two–sheeted coverings over Γ and Γ^0, respectively (recall that Γ is the semigroup of J-contractive matrices in $\mathrm{Sp}(n, \mathbb{C})$). The Hardy space $H^2(\widetilde{G}) \subset \mathcal{O}(\widetilde{\Gamma}^0)$ can be defined in exactly the same manner as for the group G (or other linear groups; the lack of a global complexification for $\widetilde{G}$ plays no role here). Let $\widetilde{K}(\widetilde{\gamma}_1, \widetilde{\gamma}_2)$ be the Cauchy–Szegö kernel on $\widetilde{\Gamma}^0 \times \widetilde{\Gamma}^0$ corresponding to that Hardy space, and let $\widetilde{L}(\widetilde{\gamma}) = \widetilde{K}(\widetilde{\gamma}, e)$ be the associated function in one variable $\widetilde{\gamma} \in \widetilde{\Gamma}^0$.

The following theorem was obtained after I learned about the idea of K. Koufany and B. Ørsted to consider the 'odd' part of the Hardy space on the metaplectic group.

Theorem 6.1. *Let $\widetilde{\gamma} \in \widetilde{\Gamma}^0$, let γ be the image of $\widetilde{\gamma}$ in Γ^0, and let $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ be the eigenvalues of γ, where one assumes that $|x_1|, \dots, |x_n| < 1$. Then*

$$\widetilde{L}(\widetilde{\gamma}) = L(\gamma) + L(\gamma) \frac{\prod_{i=1}^n (1 + x_i)}{2^n (x_1 \dots x_n)^{1/2}}, \tag{6.1}$$

where $L(\gamma)$ denotes the function computed in Theorem 5.1.

Note that the function $(x_1 \cdots x_n)^{1/2}$ is well–defined on the covering $\widetilde{\Gamma}^0$. Note also that formula (6.1) may be written as follows

$$\widetilde{L}(\widetilde{\gamma}) = \frac{(x_1 \cdots x_n)^{n+1}}{\prod_{i=1}^n (1+x_i)(1-x_i)^{2n+1}} + \frac{(x_1 \cdots x_n)^{n+1/2}}{2^n \prod_{i=1}^n (1-x_i)^{2n+1}}. \tag{6.2}$$

When $n=1$, the latter expression turns into

$$\widetilde{L}(\widetilde{\gamma}) = \frac{x^2}{(1+x)(1-x)^3} + \frac{x^{3/2}}{2(1-x)^3} \qquad (n=1, \quad x = x_1, \ |x|<1),$$

which can be verified by summing the series

$$\sum_{m \geq 0} (m+1) \frac{x^{m+2}}{1-x^2} + \sum_{m \geq 0} (m+1/2) \frac{x^{m+3/2}}{1-x^2}.$$

Proof. We shall slightly modify the basic notation of section 1 and the preliminary results of sections 3 and 4; then we shall argue as in the first proof of Theorem 5.1.

Let $\widetilde{K} \subset \widetilde{G}$ be the inverse image of the maximal compact subgroup $K \subset G$; when K is identified with $U(n)$, the group $\widetilde{K}$ is identified with the double covering over $U(n)$ making the function $\det(\cdot)^{1/2}$ single–valued. Let $\widetilde{H} \subset \widetilde{K}$ be the corresponding covering over $H \subset K$, and let $\widetilde{H}_{\mathbb{C}} \supset \widetilde{H}$ be the complex torus complexifying the compact torus $\widetilde{H}$. The (additive) group of characters of $\widetilde{H}$ (or $\widetilde{H}_{\mathbb{C}}$) is identified with the lattice

$$\widetilde{P} := P \cup (P + \epsilon) \subset \mathfrak{h}^*_{\mathrm{Re}} = \mathbb{R}^n,$$

where $\epsilon = (\frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{R}^n$.

Let $\widetilde{P}_{+-}$ be the set of those weights $\lambda \in \widetilde{P}$ that satisfy two conditions: first, $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_+$ for each $\alpha \in \Delta_c^+$ and, second, $\langle \lambda + \rho, \beta^\vee \rangle < 0$ for each $\beta \in \Delta_n^+$. This can be restated as follows:

$$\widetilde{P}_{+-} = \{\lambda \in \widetilde{P} \mid \langle \lambda + \rho, \alpha^\vee \rangle > 0 \ \forall \alpha \in \Delta_c^+ \ ; \ \langle \lambda + \rho, \beta^\vee \rangle < 0 \ \forall \beta \in \Delta_n^+\},$$

from which it becomes clear that the set $\widetilde{P}^+ := \varphi(\widetilde{P}_{+-})$ can be described as follows:

$$\widetilde{P}^+ = \{\mu \in \widetilde{P} \mid \langle \mu + \rho, \alpha^\vee \rangle > 0 \ \forall \alpha \in \Delta^+\}.$$

The holomorphic discrete series $\{T_\lambda\}$ for the metaplectic group $\widetilde{G}$ is parametrized by the weights $\lambda \in \widetilde{P}_{+-}$. Each T_λ can be extended to a holomorphic representation $\mathcal{T}_\lambda$ of the covering semigroup $\widetilde{\Gamma}$, and the analogue of Lemma 3.1 reads as follows:

$$\widetilde{L}(\widetilde{\gamma}) = \sum_{\lambda \in \widetilde{P}_{+-}} \operatorname{fdim} T_\lambda \cdot \operatorname{tr} \mathcal{T}_\lambda(\widetilde{\gamma}), \qquad \widetilde{\gamma} \in \widetilde{\Gamma}^0.$$

It is convenient to split this sum into two parts:

$$\widetilde{L}(\widetilde{\gamma}) = \widetilde{L}_{\text{even}}(\widetilde{\gamma}) + \widetilde{L}_{\text{odd}}(\widetilde{\gamma}),$$

where

$$\widetilde{L}_{\text{even}}(\widetilde{\gamma}) = \sum_{\lambda \in P_{+-}} \operatorname{fdim} T_\lambda \operatorname{tr} \mathcal{T}_\lambda(\widetilde{\gamma}),$$

$$\widetilde{L}_{\text{odd}}(\widetilde{\gamma}) = \sum_{\lambda \in \widetilde{P}_{+-} \setminus P_{+-}} \operatorname{fdim} T_\lambda \ \operatorname{tr} \mathcal{T}_\lambda(\widetilde{\gamma}).$$

Note that this splitting exactly corresponds to the splitting

$$H^2(\widetilde{G}) = H^2_{\text{even}}(\widetilde{G}) \oplus H^2_{\text{odd}}(\widetilde{G}) \subset \mathcal{O}(\widetilde{\Gamma}^0) = \mathcal{O}_{\text{even}}(\widetilde{\Gamma}^0) \oplus \mathcal{O}_{\text{odd}}(\widetilde{\Gamma}^0),$$

where 'even' functions are assumed to be constant on fibers of canonical projection $\widetilde{\Gamma}^0 \to \Gamma^0$ whereas 'odd' functions take opposite values on each fiber. Thus, $\widetilde{L}_{\text{even}}$ and $\widetilde{L}_{\text{odd}}$ determine the Cauchy-Szegö kernels for $H^2_{\text{even}}(\widetilde{G})$ and $H^2_{\text{odd}}(\widetilde{G})$, respectively.

Clearly $\widetilde{L}_{\text{even}}(\widetilde{\gamma}) = L(\gamma)$, so that formula (6.1) is equivalent to the following one:

$$\widetilde{L}_{\text{odd}}(\widetilde{\gamma}) = L(\gamma) \frac{\prod_{i=1}^n (1+x_i)}{2^n (x_1 \cdots x_n)^{1/2}} \ . \tag{6.3}$$

Let μ range over the set

$$\widetilde{P}^+ \setminus P^+ = \{\mu \in P + \epsilon \mid \langle \mu + \rho, \alpha^\vee \rangle > 0 \ \forall \alpha \in \Delta^+\}.$$

We cannot write $\operatorname{fdim} T_{\varphi(\mu)} = \dim V_\mu$ as before, because there is no finite–dimensional representation with highest weight μ (for there is no double covering over the group $\mathrm{Sp}(n, \mathbb{C})$!), so a claim similar to that of Lemma 3.4 literally fails. Nevertheless, by Harish-Chandra's result cited above ([5], paper VI, Theorem 4), the formal dimension $\operatorname{fdim} T_{\varphi(\mu)}$ is still given by the same analytic expression as for weights $\mu \in P^+$:

$$\begin{aligned} \operatorname{fdim} T_{\varphi(\mu)} &= \frac{\prod_{\alpha \in \Delta^+} \langle \mu + \rho, \alpha^\vee \rangle}{\prod_{\alpha \in \Delta^+} \langle \rho, \alpha^\vee \rangle} \\ &= \lim_{t \to e} \frac{\sum_{w \in W} \varepsilon(w) (w(t))^{\mu + \rho}}{D_G(t)}. \end{aligned}$$

As for Lemma 3.2, it needs no change.

Now one can repeat the argument of Theorem 4.2, which leads to the following result:

$$\widetilde{L}_{\text{odd}}(\widetilde{\gamma}) = \frac{\theta(x)}{D_K(x)} \lim_{t \to e} \frac{1}{D_G(t)} \sum_{w \in W} \varepsilon(w) \widetilde{F}_{\text{odd}}(w(t)\widehat{x}),$$

where, for $z = \mathrm{diag}(z_1, \dots, z_n, z_n^{-1}, \dots, z_1^{-1})$,

$$\widetilde{F}_{\mathrm{odd}}(z) = \sum_{\nu \in \Omega \cap (P+\epsilon)} z^\nu = (z_1 \cdots z_n)^{1/2} F(z) = \prod_{i=1}^n \frac{z_i^{1/2}}{1 - z_i}.$$

Next we repeat the argument of the first proof of Theorem 5.1. The only new point is that the sum in (5.8) (which is rewritten in (5.9)) must be replaced by the following expression:

$$\sum_{\nu \in \mathfrak{S}_n} \varepsilon(s) \sum_{a \in \mathfrak{A}_n} \varepsilon(a) \frac{(a(u_i)x_i)^{1/2}}{1 - a(u_i)x_i}, \tag{6.4}$$

where, as before, $(u_1, \dots, u_n)$ stands for the permutation of $(t_1, \dots, t_n)$ determined by $s \in \mathfrak{S}_n$. The interior sum in (6.4) is equal to

$$\begin{aligned}
&\prod_{i=1}^n \left(\frac{u_i^{1/2} x_i^{1/2}}{1 - u_i x_i} - \frac{u_i^{-1/2} x_i^{1/2}}{1 - u_i^{-1} x_i} \right) \\
&\qquad = \prod_i x_i^{1/2}(1 + x_i) \prod_i (u_i^{1/2} - u_i^{-1/2}) \prod_i \frac{1}{(1 - u_i x_i)(1 - u_i^{-1} x_i)} \\
&\qquad = \prod_i x_i^{1/2}(1 + x_i) \prod_i (t_i^{1/2} - t_i^{-1/2}) \prod_i \frac{1}{(1 - u_i x_i)(1 - u_i^{-1} x_i)},
\end{aligned}$$

which differs by the factor

$$\frac{\prod_i (1 + x_i)}{(x_1 \cdots x_n)^{1/2} \prod_i (t_i^{1/2} + t_i^{-1/2})}$$

from its counterpart (5.10) in section 5. Since this factor does not vary under alternation over the t_i's, we see that $\widetilde{L}_{\mathrm{odd}}(\widetilde{\gamma})$ differs from $L(\gamma)$ by the factor

$$\lim_{t_1 \to 1, \dots, t_n \to 1} \frac{\prod_i (1 + x_i)}{(x_1 \cdots x_n)^{1/2} \prod_i (t_i^{1/2} + t_i^{-1/2})} = \frac{\prod_i (1 + x_i)}{2^n (x_1 \cdots x_n)^{1/2}},$$

which completes the proof. ∎

7. The case of $G = \mathrm{SU}(p, q)$

In this section we fix numbers $p, q \in \{1, 2, 3, \dots\}$ and we put $n = p + q$. Equip the space $\mathbb{C}^n = \mathbb{C}^{p+q}$ with the indefinite inner product

$$[\xi, \eta] = -\xi_1 \bar{\eta}_1 - \xi_2 \bar{\eta}_2 - \dots - \xi_p \bar{\eta}_p + \xi_{p+1} \bar{\eta}_{p+1} + \dots + \xi_n \bar{\eta}_n, \quad \xi, \eta \in \mathbb{C}^n,$$

and realize the group $G := \mathrm{SU}(p, q)$ as the group of unimodular complex matrices preserving this inner product:

$$G = \mathrm{SU}(p, q) = \{ g \in \mathrm{SL}(n, \mathbb{C}) \mid [g\xi, g\eta] = [\xi, \eta],\ \xi, \eta \in \mathbb{C}^n \}.$$

The group $G_{\mathbb{C}}$ is $\mathrm{SL}(n,\mathbb{C})$. The group K is the subgroup of G that preserves the direct sum decomposition $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^q$; K is isomorphic to $S(U(p)\times U(q))$.

Let $\epsilon_1,\dots,\epsilon_n$ be the canonical basis of $\mathbb{R}^n$ and put $\epsilon=(1,\dots,1)$.

Let $\mathbb{R}^n_0$ be the hyperplane in $\mathbb{R}^n$, orthogonal to ϵ:

$$\mathbb{R}^n_0 = \{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_1+\ldots+x_n=0\}.$$

We have

$$\mathfrak{h}_{\mathrm{Re}} = \{x=\operatorname{diag}(x_1,\dots,x_n) \mid (x_1,\dots,x_n)\in\mathbb{R}^n_0\},$$

so that $\mathfrak{h}_{\mathrm{Re}}$ may be identified with $\mathbb{R}^n_0\subset\mathbb{R}^n$. The dual space $\mathfrak{h}^*_{\mathrm{Re}}$ will be identified with the factor space $\mathbb{R}^n/\mathbb{R}\epsilon$; for brevity, elements of $\mathfrak{h}^*_{\mathrm{Re}}$ often will be written simply as vectors of $\mathbb{R}^n$. In this notation we have

$$\begin{aligned}
\Delta^+_c &= \{\epsilon_i-\epsilon_j \mid 1\le i<j\le p \text{ or } p+1\le i<j\le n\}\\
\Delta^+_n &= \{\epsilon_i-\epsilon_j \mid 1\le i\le p,\ p+1\le j\le n\}\\
\Delta^+ &= \{\epsilon_i-\epsilon_j \mid 1\le i<j\le n\}.
\end{aligned}$$

The weight lattice $P\subset\mathfrak{h}^*_{\mathrm{Re}}$ can be identified with $\mathbb{Z}^n/\mathbb{Z}\epsilon$, and its subset $P^+\subset P$ of dominant weights is described as

$$P^+ = \{\mu\in\mathbb{Z}^n/\mathbb{Z}\epsilon \mid \mu_1\ge\mu_2\ge\ldots\ge\mu_n\}, \tag{7.1}$$

and the transformation $\varphi: P^+\to P_{+-}$ takes the form

$$\varphi(\mu) = (-\mu_p-q,\dots,-\mu_1-q,\ -\mu_n+p,\dots,-\mu_{p+1}+p) \qquad (\mathrm{mod}\ \mathbb{Z}\epsilon). \tag{7.2}$$

The Weyl group W is isomorphic to the symmetric group $\mathfrak{S}_n$; its action on $\mathfrak{h}_{\mathrm{Re}}=\mathbb{R}^n_0$ and $\mathfrak{h}^*_{\mathrm{Re}}=\mathbb{R}^n/\mathbb{R}\epsilon$ consists in permuting the coordinates. The Weyl group $W_c\subset W$ is isomorphic to $\mathfrak{S}_p\times\mathfrak{S}_q$, naturally embedded into $\mathfrak{S}_n$.

The cone $c_{\min}\subset\mathfrak{h}_{\mathrm{Re}}=\mathbb{R}^n_0$ is described as follows:

$$c_{\min} = \{x\in\mathbb{R}^n_0 \mid x_1,\dots,x_p\ge 0\ ;\ x_{p+1},\dots,x_n\le 0\}.$$

The complex torus $H_{\mathbb{C}}$ consists of the complex diagonal matrices of the form

$$\operatorname{diag}(z_1,\dots,z_n)\ ,\quad z_1\cdots z_n=1.$$

The subset $H^+_{\mathbb{C}}\subset H_{\mathbb{C}}$ is described by the inequalities

$$|z_1|>1,\dots,|z_p|>1\ ;\ |z_{p+1}|<1,\dots|z_n|<1.$$

The semigroup Γ is formed by matrices in $\mathrm{SL}(n,\mathbb{C})$ which are weakly J-contractive in the sense of being 'contractive with respect to the inner product $[\cdot,\cdot]$.'

The interior $\Gamma^0\subset\Gamma$ is formed by strictly J-contractive matrices.

Given a matrix $\gamma \in \Gamma^0$, its eigenvalues can be written as

$$x_1^{-1}, \ldots, x_p^{-1},\ y_1, \ldots, y_q, \tag{7.3}$$

where

$$|x_1| < 1, \ldots, |x_p| < 1,\ |y_1| < 1, \ldots, |y_q| < 1 \tag{7.4}$$

and

$$x_1 \cdots x_p = y_1 \cdots y_q. \tag{7.5}$$

Theorem 7.1. *Let $K(\gamma_1, \gamma_2)$ be the Cauchy-Szegö kernel for the Hardy space $H^2(\mathrm{SU}(p,q)) \subset \mathcal{O}(\Gamma^0)$, where Γ^0 is the semigroup of strictly J-contractive matrices in $\mathrm{SL}(n, \mathbb{C})$, $n = p + q$. Let $L(\gamma) = K(\gamma, e)$ be the corresponding function in one variable $\gamma \in \Gamma^0$. Given a matrix $\gamma \in \Gamma^0$, write its eigenvalues as above* (see (7.3)–(7.5)) *and put*

$$u = x_1 \cdots x_p = y_1 \cdots y_q. \tag{7.6}$$

(i) *The function L is given by the formula*

$$\begin{aligned} L(\gamma) = {} & \frac{u^n}{\prod_{1\le i\le p}(1-x_i)^n \prod_{1\le j\le q}(1-y_j)^n} \\ & \cdot \sum_{r=1}^{p} \Big(\prod_{k\ne r} \frac{x_k}{x_k - x_r} \Big) \frac{(1-x_r)^n}{\prod_{1\le j\le q}(1 - x_r y_j)}\,. \end{aligned} \tag{7.7}$$

(ii) *This expression also can be written as*

$$\frac{u^n}{\prod_i (1-x_i)^n \prod_j (1-y_j)^n} \cdot \left(1 - \frac{1}{2\pi i} \oint_{|\xi|=1} F(\xi; x, y) \frac{d\xi}{\xi} \right), \tag{7.8}$$

where

$$F(\xi; x, y) = \frac{(\xi - 1)^n}{(\xi - x_1)\cdots(\xi - x_p)(\xi - y_1^{-1})\cdots(\xi - y_q^{-1})}\,. \tag{7.9}$$

(Recall that we have agreed to normalize the Haar measure of the group as in Lemma 3.4.)

The proof is presented below, after Corollary 7.5. It is based on a character identity (Lemma 7.4), which is an analogue of the identity of Lemma 5.2.

Remark 7.2. Since the groups $\mathrm{SU}(p,q)$ and $\mathrm{SU}(q,p)$ are isomorphic, the function $L(\gamma)$ must be invariant under the transformation

$$(x_1, \ldots, x_p;\ y_1, \ldots, y_q) \longmapsto (y_1, \ldots, y_q;\ x_1, \ldots, x_p).$$

This invariance property is not evident from (7.7) but can be easily obtained from the second claim of Theorem 7.1. Indeed, we remark that

$$F(\xi; x, y) = F(\xi^{-1}; y, x)$$

(because of (7.5)) and replace ξ by ξ^{-1} in the integral (7.8). ∎

The expression (7.7) simplifies when $p = 1$:

Corollary 7.3. *Let K be the Cauchy-Szegö kernel for the group* $\mathrm{SU}(1,q)$, $q=1,2,\ldots$, *and let* $L(\gamma)=K(\gamma,e)$. *Write* $u^{-1},y_1,\ldots,y_q$ *for the eigenvalues of a given matrix* $\gamma\in\Gamma^0$ $(u=y_1\cdots y_q)$. *Then*

$$L(\gamma)=\frac{u^{q+1}}{\prod_{1\le j\le q}(1-y_j)^{q+1}(1-uy_j)}\,. \tag{7.10}$$

■

Note that in the case $q=1$ the group $\mathrm{SU}(1,q)=\mathrm{SU}(1,1)$ is isomorphic to $\mathrm{Sp}(1,\mathbb{R})$), and formula (7.10) agrees with formula (5.3) of Theorem 5.1.

Lemma 7.4. *Let* $\Lambda=(\Lambda_1\ge\ldots\ge\Lambda_p\ge0)$ *and* $M=(M_1\ge\ldots\ge M_q\ge0)$ *be arbitrary partitions of length* $\le p$ *and* $\le q$, *respectively; let* $s_\Lambda(x_1,\ldots,x_p)$ *and* $s_\Lambda(y_1,\ldots,y_q)$ *be the corresponding Schur functions in* p *and* q *variables; let* $V_{\Lambda,M}$ *be the irreducible finite-dimensional* $GL(n,\mathbb{C})$*-module (where* $n=p+q$*) with highest weight*

$$(\Lambda_1,\ldots,\Lambda_p,-M_q,\ldots,-M_1), \tag{7.11}$$

and let $gl(n)_{\Lambda,M}(t_1,\ldots,t_n)$ *be the character of* $V_{\Lambda,M}$, *viewed as a function on the maximal torus of* $GL(n,\mathbb{C})$ *formed by diagonal matrices* $\mathrm{diag}(t_1,\ldots,t_n)$. *Then*

$$\begin{aligned}&\sum_{\Lambda,M} gl(n)_{\Lambda,M}(t_1,\ldots,t_n)s_\Lambda(x_1,\ldots,x_p)s_M(y_1,\ldots,y_q)\\ &\qquad=\frac{\prod_{i=1}^p\prod_{j=1}^q(1-x_iy_j)}{\prod_{k=1}^n\prod_{i=1}^p(1-t_kx_i)\prod_{k=1}^n\prod_{j=1}^q(1-t_k^{-1}y_j)}\,.\end{aligned} \tag{7.12}$$

Proof. This character identity can be verified by the same methods as the identity of Lemma 5.2. In particular, one can derive (7.12) from the explicit weight correspondence in Howe's duality between the groups $U(k)$ and $U(p,q)$ (for the special case $k=p+q$), described in Kashiwara and Vergne [11]. In section 8 we will prove the identity (7.12) by Ishikawa–Wakayama's method. ■

Corollary 7.5. *Let* Λ, M *and* $V_{\Lambda,M}$ *be as in* Lemma 7.4, *and let* $\dim V_{\Lambda,M}$ *denote the dimension of* $V_{\Lambda,M}$. *Then*

$$\begin{aligned}&\sum_{\Lambda,M}\dim V_{\Lambda,M}\cdot s_\Lambda(x_1,\ldots,x_p)s_M(y_1,\ldots,y_q)\\ &\qquad=\frac{\prod_{i=1}^p\prod_{j=1}^q(1-x_iy_j)}{\prod_{i=1}^p(1-x_i)^n\prod_{j=1}^q(1-y_j)^n}\,.\end{aligned} \tag{7.13}$$

Proof. Take $t_1=\ldots=t_n=1$ in (7.12). ■

Proof of Theorem 7.1. (i) As in the case of the group $\mathrm{Sp}(n,\mathbb{R})$ (see the beginning of the first proof of Theorem 5.1), application of Lemma 2.1 allows one to assume that γ is a diagonal matrix contained in $H^+_{\mathbb{C}}$. Then we write γ as

$$\gamma = \mathrm{diag}(x_p^{-1}, \dots, x_1^{-1}, y_1, \dots, y_q),$$

where

$$|x_1| < 1, \dots, |x_p| < 1,\ |y_1| < 1, \dots, |y_q| < 1.$$

By Corollary 3.5,

$$L(\gamma) = \theta(\gamma) \sum_{\mu \in P^+} \dim V_\mu \cdot \mathrm{tr}\, \pi_{\varphi(\mu)}(\gamma). \tag{7.14}$$

In the present situation

$$\theta(\gamma) = \prod_{i=1}^{p} \prod_{j=1}^{q} \frac{1}{1 - x_i y_j}$$

and, by virtue of (7.2) and (7.6),

$$\mathrm{tr}\, \pi_{\varphi(\mu)}(\gamma) = u^n s_{(\mu_1, \dots, \mu_p)}(x_1, \dots, x_p) s_{(-\mu_n, \dots, -\mu_{p+1})}(y_1, \dots, y_q).$$

Recall that each dominant weight $\mu = (\mu_1, \dots, \mu_n) \in P^+$ is an element of the quotient lattice $\mathbb{Z}^n/\mathbb{Z}\epsilon$ (not of the lattice $\mathbb{Z}^n$!). It will be convenient for us to fix a representative of μ in $\mathbb{Z}^n$ by putting $\mu_p = 0$. Then μ will be indexed by a couple (Λ, M) of partitions such that $l(\Lambda) \le p - 1$ (hence $\Lambda_p = 0$) and $l(M) \le q$, i.e.,

$$\mu = (\Lambda_1, \dots, \Lambda_{p-1}, 0, -M_q, \dots, -M_1).$$

Thus, (7.14) can be rewritten as follows:

$$\begin{aligned} L(\gamma) =& u^n \Big(\prod_{i=1}^{p} \prod_{j=1}^{q} \frac{1}{1 - x_i y_j} \Big) \\ & \cdot \sum_{\substack{\Lambda, M \\ \Lambda_p = 0}} \dim V_{\Lambda, M} s_\Lambda(x_1, \dots, x_p) s_M(y_1, \dots, y_q). \end{aligned} \tag{7.15}$$

The restriction $\Lambda_p = 0$ does not allow us to apply directly the character identity of Corollary 7.5. However, this difficulty can be surmounted with help of the following claim:

Lemma 7.6. *Let*

$$\Phi(x_1, \dots, x_p) = \sum_{\Lambda} c(\Lambda) s_\Lambda(x_1, \dots, x_p)$$

be a series on Schur functions in p variables, and define

$$\Phi_0(x_1, \dots, x_p) := \sum_{\Lambda,\ \Lambda_p = 0} c(\Lambda) s_\Lambda(x_1, \dots, x_p).$$

Then

$$(7.16)\qquad \Phi_0(x_1,\dots,x_p)=\sum_{r=1}^{p}\Big(\prod_{k\neq r}\frac{x_k}{x_k-x_r}\Big)\Big(\Phi(x_1,\dots,x_p)\big|_{x_r=0}\Big).$$

Proof of the Lemma. By linearity, it suffices to check (7.16) for $\Phi=s_\Lambda$. That is to say, we must prove that

$$(7.17)\qquad \begin{aligned}&\sum_{r=1}^{p}\Big(\prod_{k\neq r}\frac{x_k}{x_k-x_r}\Big)\Big(s_\Lambda(x_1,\dots,x_p)\big|_{x_r=0}\Big)\\ &\qquad=\begin{cases} s_\Lambda(x_1,\dots,x_p), & \text{if } \Lambda_p=0\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Write

$$s_\Lambda(x_1,\dots,x_p)=\frac{A_\Lambda(x)}{V(x)},$$

where

$$(7.18)\qquad \begin{aligned}A_\Lambda(x)&=\det\Big[x_j^{\Lambda_i+p-i}\Big]_{1\le i,j\le p}\\ &=\sum_{(j_1,\dots,j_p)}\varepsilon(j_1,\dots,j_p)\,x_{j_1}^{\Lambda_1+p-1}x_{j_2}^{\Lambda_2+p-2}\cdots x_{j_p}^{\Lambda_p},\end{aligned}$$

summed over all permutations $(j_1,\dots,j_p)$ of $(1,\dots,n)$ (here $\varepsilon(j_1,\dots,j_p)$ stands for the sign of the permutation $(j_1,\dots,j_p)$), and

$$(7.19)\qquad V(x)=A_{(0,\dots,0)}(x)=\prod_{1\le i<j\le p}(x_i-x_j)$$

is the Vandermonde determinant.

Assume first $\Lambda_p>0$. Then (7.18) implies that $s_\Lambda(x_1,\dots,x_p)$ vanishes as $x_r=0$ for a certain r, which confirms (7.16). So one may assume $\Lambda_p=0$. Then (7.18) implies

$$A_\Lambda(x)\big|_{x_r=0}=\sum_{\substack{(j_1,\dots,j_p)\\ j_p=r}}\varepsilon(j_1,\dots,j_p)\,x_{j_1}^{\Lambda_1+p-1}\,x_{j_2}^{\Lambda_2+p-2}\cdots x_{j_p}^{\Lambda_p},$$

whence

$$(7.20)\qquad \sum_{r=1}^{p}\Big(A_\Lambda\big|_{x_r=0}\Big)=A_\Lambda(x).$$

Further,

$$V(x)\big|_{x_r=0}=\Big(\prod_{k\neq r}\frac{x_k}{x_k-x_r}\Big)V(x),\qquad r=1,\dots,p.$$

Along with (7.20) this yields

$$\sum_{r=1}^{r}\Big(\prod_{k\neq r}\frac{x_k}{x_k-x_r}\Big)\Big(s_\Lambda(x_1,\dots,x_p)\big|_{x_r=0}\Big)=\sum_{r=1}^{p}\Big(\prod_{k\neq r}\frac{x_k}{x_k-x_r}\Big)\frac{A_\Lambda(x)\big|_{x_r=0}}{V(x)\big|_{x_r=0}}$$
$$=\frac{\sum_{r=1}^{p}A_\Lambda(x)\big|_{x_r=0}}{V(x)}$$
$$=\frac{A_\Lambda(x)}{V(x)}$$
$$=s_\Lambda(x_1,\dots,x_p),$$

which completes the proof of the Lemma. ∎

Let us return to the proof of the Theorem. By Corollary 7.5 and Lemma 7.6,

$$\begin{aligned}L(\gamma)=&\frac{u^n}{\prod_{i,j}(1-x_iy_j)}\sum_{r=1}^{p}\Big(\prod_{k\neq r}\frac{x^k}{x_k-x_r}\Big)\\ &\cdot\left\{\frac{\prod_{i,j}(1-x_iy_j)}{\prod_i(1-x_i)^n\prod_j(1-y_j)^n)}\right\}_{x_r=0}.\end{aligned}\tag{7.21}$$

Further,

$$\left\{\frac{\prod_{i,j}(1-x_iy_j)}{\prod_i(1-x_i)^n\prod_j(1-y_j)^n)}\right\}_{x_r=0}=\frac{(1-x_r)^n}{\prod_j(1-x_ry_j)}\cdot\frac{\prod_{i,j}(1-x_iy_j)}{\prod_i(1-x_i)^n\prod_j(1-y_j)^n)}\,.$$

Substituting the latter expression into (7.21) we obtain the desired formula (7.7) for $L(\gamma)$. This completes proof of claim (i) of the Theorem.

(ii) Let us transform the sum in formula (7.7):

$$\begin{aligned}&\sum_{r=1}^{p}\prod_{k\neq r}\frac{x_k}{x_k-x_r}\frac{(1-x_r)^n}{\prod_j(1-x_ry_j)}\\ &=(-1)^{p-1}\sum_{r=1}^{p}\frac{(1-x_r)^n}{x_r}\cdot\frac{x_1\dots x_p}{\prod_j(1-x_ry_j)}\prod_{k\neq r}\frac{1}{x_r-x_k}\,.\end{aligned}\tag{7.22}$$

By (7.5),

$$\frac{x_1\dots x_p}{\prod_j(1-x_ry_j)}=\frac{y_1\dots y_q}{\prod_j(1-x_ry_j)}=\prod_j\frac{y_j}{1-x_ry_j}$$
$$=(-1)^q\prod_j\frac{1}{x_r-y_j^{-1}}\,.$$

Therefore, the expression (7.22) is equal to

$$
\begin{aligned}
&(-1)^{n-1}\sum_{r=1}^{p}\frac{(1-x_r)^n}{x_r}\prod_{k\neq r}\frac{1}{x_r-x_k}\prod_j\frac{1}{x_r-y_j^{-1}}\\
&=-\sum_{r=1}^{p}\frac{(x_r-1)^n}{x_r}\prod_{k\neq r}\frac{1}{x_r-x_k}\prod_j\frac{1}{x_r-y_j^{-1}}\\
&=-\sum_{r=1}^{p}\mathrm{Res}_{\xi=x_r}\frac{(\xi-1)^n}{\xi(\xi-x_1)\dots(\xi-x_p)(\xi-y_1^{-1})\dots(\xi-y_q^{-1})}\,,
\end{aligned}
$$

where '$\mathrm{Res}_{\xi=a}$' means 'residue at the point $\xi=a$'.

Remark that the poles of the function

$$\xi\mapsto\frac{(\xi-1)^n}{\xi(\xi-x_1)\dots(\xi-x_p)(\xi-y_1^{-1})\dots(\xi-y_q^{-1})}$$

in the unit disk $|\xi|<1$ occur at the points $0,x_1,\dots,x_p$, because $|y_j^{-1}|>1$, $1\le j\le q$. Next, remark that the residue at $\xi=0$ equals 1, because

$$\frac{(-1)^n}{(-x_1)\dots(-x_p)(-y_1^{-1})\dots(-y_q^{-1})}=\frac{1}{x_1\dots x_p y_1^{-1}\dots y_q^{-1}}=1.$$

Therefore, (7.22) equals

$$1-\frac{1}{2\pi i}\oint_{|\xi|=1}F(\xi;x,y)\frac{d\xi}{\xi}\,, \tag{7.23}$$

where $F(\xi;x,y)$ is defined by (7.9). By substituting (7.23) into (7.7) (instead of the sum) we obtain the desired formula (7.8). This completes the proof of the Theorem. ∎

8. Proof of character identities

In this section we prove Lemma 5.2 and Lemma 7.4 using an elegant method due to M. Ishikawa and M. Wakayama.

Proof of Lemma 5.2. Recall the identity that has to be proved:

$$\sum_M sp(n)_M(t_1^{\pm1},\dots,t_n^{\pm1})s_M(x_1,\dots,x_n)=\frac{\prod_{i<j}(1-x_ix_j)}{\prod_k\prod_i(1-t_kx_i)(1-t_k^{-1}x_i)}\,, \tag{8.1}$$

where M ranges over the set of partitions of length $\le n$, $sp(n)_M$ stands for the character of $\mathrm{Sp}(n,\mathbb{C})$, indexed by M, and s_M is the Schur function.

Also recall that by Weyl's 'first character formula' (see [25], Theorem VII.8.C),

$$sp(n)_M(t_1^{\pm1},\dots,t_n^{\pm1})=\frac{\det\left[\frac{t_k^{l_i}-t_k^{-l_i}}{t_k-t_k^{-1}}\right]_{1\le k,i\le n}}{V(t_1+t_1^{-1},\dots,t_n+t_n^{-1})}, \tag{8.2}$$

where

$$(l_1, \dots, l_n) = (M_1 + n, \dots, M_n + 1) \tag{8.3}$$

and $V(a_1, \dots, a_n)$ stands for the Vandermonde determinant,

$$V(a_1, \dots, a_n) = \prod_{r<s} (a_r - a_s).$$

Consider two matrices, T and S, with n rows and countably many columns that are labelled by the positive integers, written in reverse order $\dots, 2, 1$,

$$T = \begin{pmatrix} \dots & \frac{t_1^3 - t_1^{-3}}{t_1 - t_1^{-1}} & \dots & \frac{t_1^2 - t_1^{-2}}{t_1 - t_1^{-1}} & \dots & \frac{t_1^1 - t_1^{-1}}{t_1 - t_1^{-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \frac{t_n^3 - t_n^{-3}}{t_n - t_n^{-1}} & \dots & \frac{t_n^2 - t_n^{-2}}{t_n - t_n^{-1}} & \dots & \frac{t_n^1 - t_n^{-1}}{t_n - t_n^{-1}} \end{pmatrix}$$

$$S = \begin{pmatrix} \dots & x_1^2 & x_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \dots & x_n^2 & x_n & 1 \end{pmatrix}.$$

Denoting by S' the transposed matrix we will compute the $n \times n$ determinant $\det TS'$ in two ways.

On the one hand, we have

$$\begin{aligned} (TS')_{ki} &= \sum_{r=1}^{\infty} \frac{t_k^r - t_k^{-r}}{t_k - t_k^{-1}} x_i^{r-1} \\ &= \frac{1}{t_k - t_k^{-1}} \left(\frac{t_k}{1 - t_k x_i} - \frac{t_k^{-1}}{1 - t_k^{-1} x_i} \right) \\ &= (1 - t_k x_i)^{-1} (1 - t_k^{-1} x_i)^{-1}. \end{aligned}$$

It follows

$$\begin{aligned} \det TS' &= \det[(1 - t_k x_i)^{-1}(1 - t_k^{-1} x_i)^{-1}]_{1 \le k, i \le n} \\ &= \frac{V(t_1 + t_1^{-1}, \dots, t_n + t_n^{-1}) V(x_1, \dots, x_n) \prod_{i<j}(1 - x_i x_j)}{\prod_i \prod_k (1 - t_k x_i)(1 - t_k^{-1} x_i)} \end{aligned} \tag{8.4}$$

(see (5.11)).

On the other hand, given arbitrary integers $l_1 > \dots > l_n \ge 1$, let us denote by $\det T_{l_1 \dots l_n}$ and $\det S_{l_1 \dots l_n}$ the n-th order minors formed by the columns $l_1, \dots, l_n$ of the matrix T and S, respectively. Then we have

$$\det TS' = \sum_{l_1 > \dots > l_n \ge 1} \det T_{l_1 \dots l_n} \det S_{l_1 \dots l_n}. \tag{8.5}$$

Let $M = (M_1, \dots, M_n)$ be related with $l = (l_1, \dots, l_n)$ by (8.3). Then M is a partition and, by Weyl's character formula (8.2),

$$\det T_{l_1 \dots l_n} = sp(n)_M (t_1^{\pm 1}, \dots, t_n^{\pm 1}) V(t_1 + t_1^{-1}, \dots, t_n + t_n^{-1}),$$

and we also have

$$\det S_{l_1 \dots l_n} = s_M(x_1, \dots, x_n) V(x_1, \dots, x_n).$$

Therefore,

$$\begin{aligned}\det TS' =& V(t_1 + t_1^{-1}, \dots, t_n + t_n^{-1}) V(x_1, \dots, x_n) \\ & \cdot \sum_M sp(n)_M(t_1^{\pm 1}, \dots, t_n^{\pm 1}) s_M(x_1, \dots, x_n).\end{aligned} \tag{8.6}$$

By comparing both expressions for $\det TS'$, (8.4) and (8.6), we arrive to the desired formula (8.1). ■

Now we will apply a similar argument to prove the second character identity.

Proof of Lemma 7.4. Recall the identity in question:

$$\begin{aligned}&\sum_{\Lambda, M} gl_{\Lambda, M}(t_1, \dots, t_n) s_\Lambda(x_1, \dots, x_p) s_M(y_1, \dots, y_q) \\ &= \frac{\prod_{i=1}^p \prod_{j=1}^q (1 - x_i y_j)}{\prod_{k=1}^n \prod_{i=1}^p (1 - t_k x_i) \prod_{k=1}^n \prod_{j=1}^q (1 - t_k^{-1} y_j)},\end{aligned} \tag{8.7}$$

where $n = p + q$; Λ and M are arbitrary partitions of length $\le p$ and $\le q$, respectively; $gl(n)_{\Lambda, M}$ is the character of $GL(n, \mathbb{C})$ corresponding to the dominant weight

$$(\Lambda_1, \dots, \Lambda_p, -M_q, \dots, -M_1);$$

s_Λ and s_M are Schur functions indexed by Λ and M.

Consider two matrices, T and S, whose rows are labelled by $1, \dots, n$ ($n = p + q$) and columns are labelled by all the integers written in reverse order $\dots, 2, 1, 0, -1, -2, \dots,$

$$T = \begin{pmatrix} \dots & t_1^2 & t_1 & 1, & t_1^{-1} & t_1^{-2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & t_n^2 & t_n & 1, & t_n^{-1} & t_n^{-2} & \dots \end{pmatrix}$$

$$S = \begin{pmatrix} \dots & x_1^2 & x_1 & 1, & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & x_p^2 & x_p & 1, & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0, & 1 & y_1 & y_1^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 0 & 0, & 1 & y_q & y_q^2 & \dots \end{pmatrix}$$

where commas in the matrices were used to separate the 0-th and (-1)-th columns.

Let us again compute $\det TS'$ in two ways.

On the one hand, we have for $1 \leq k \leq n$, $1 \leq i \leq p$, $1 \leq j \leq q$,

$$\begin{aligned}(TS')_{ki} &= (1 - t_k x_i)^{-1} = -x_i^{-1}(t_k - x_i^{-1}),\\ (TS')_{k,p+j} &= (t_k - y_j)^{-1},\end{aligned}$$

whence

$$\det TS' = (-1)^p (x_1 \ldots x_p)^{-1} \det[(t_k - z_r)^{-1}]_{1 \leq k,r \leq n}, \tag{8.8}$$

where

$$(z_1, \ldots, z_n) = (x_1^{-1}, \ldots, x_p^{-1}, y_1, \ldots, y_q).$$

The determinant in (8.8) is readily reduced to Cauchy's determinant, and after simple transformations we obtain

$$\begin{aligned}\det TS' =& (-1)^{q(q-1)/2}(t_1 \ldots t_n)^{-q}\\ &\cdot \frac{V(t_1, \ldots, t_n)V(x_1, \ldots, x_p)V(y_1, \ldots, y_q)\prod_i \prod_j (1 - x_i y_j)}{\prod_k \prod_i (1 - t_k x_i) \prod_k \prod_j (1 - t_k^{-1} y_j)}.\end{aligned} \tag{8.9}$$

On the other hand,

$$\det TS' = \sum_{+\infty > l_1 > \ldots > l_n > -\infty} \det T_{l_1 \ldots l_n} \det S_{l_1 \ldots l_n}. \tag{8.10}$$

¿From the form of the matrix S it is clear that $\det S_{l_1 \ldots l_n}$ vanishes unless $l_p \geq 0$ and $l_{p+1} \leq -1$, so that we may assume

$$\begin{aligned}(l_1, \ldots, l_n) = &(\Lambda_1, \ldots, \Lambda_p, -M_q, \ldots, -M_1)\\ &+ (p-1, \ldots, 0, -1, \ldots, -q),\end{aligned}$$

where Λ and M are partitions. Further, we have

$$\det T_{l_1 \ldots l_n} = gl(n)_{\Lambda,M}(t_1 \ldots t_n)V(t_1, \ldots, t_n)(t_1, \ldots, t_n)^{-q},$$

$$\begin{aligned}\det S_{l_1 \ldots l_n} =& s_\Lambda(x_1, \ldots, x_p)s_M(y_1, \ldots, y_q)\\ &\cdot V(x_1, \ldots, x_p)V(y_1, \ldots, y_q)(-1)^{q(q-1)/2},\end{aligned}$$

whence

$$\begin{aligned}&\det TS' = (-1)^{q(q-1)/2}(t_1 \ldots t_n)^{-q}V(t_1, \ldots, t_n)V(x_1, \ldots, x_p)\\ &\cdot V(y_1, \ldots, y_q)\sum_{\Lambda,M} gl_{\Lambda,M}(t_1, \ldots, t_n)s_\Lambda(x_1, \ldots, x_p)s_M(y_1, \ldots, y_q).\end{aligned} \tag{8.11}$$

By comparing (8.9) and (8.11) we obtain the desired identity (8.7). ∎

References

[1] Enright, T. J., R. Howe, and N. Wallach, *A classification of unitary highest weight modules*, in: Representation theory of reductive groups, Birkhäuser, Boston, 1983, 97–143.

[2] Faraut, J., "Hardy spaces in non–commutative harmonic analysis," Notes of a course in Summer School on Harmonic Analysis and Geometry, Tuczno, 1994.

[3] Faraut, J., and A. Korányi, "Analysis on symmetric cones," Oxford Mathematical Monographs, Clarendon Press, Oxford, 1994.

[4] Gelfand, I. M., and S. G. Gindikin, *Complex manifolds whose skeletons are semi–simple Lie groups, and analytic discrete series of representations*, Funkts. Analiz i Pril. **11** (1977), 19–27 (Russian); English translation: Funct. Anal. Appl. **11** (1977), 258–265.

[5] Harish–Chandra, *Representations of semisimple Lie groups*, IV–VI, Amer. J. Math. **77** (1955), 743–777; **78** (1956), 1–41; **78** (1956), 564–628.

[6] Hilgert, J., and K.-H. Neeb, "Lie semigroups and their applications", Springer Lecture Notes in Math. 1552 (1993).

[7] Hilgert, J., G. Ólafsson, *Analytic continuation of representations, the solvable case*, Japanese J. Math. **18** (1992), 213–290.

[8] Hilgert, J., G. Ólafsson, and B. Ørsted, *Hardy spaces on affine symmetric spaces*, J. reine angew. Math. **415** (1991), 189–218.

[9] Howe, R., *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), 539–570.

[10] Inoue, T., *Unitary representations and kernel functions associated with boundaries of a bounded symmetric domain*, Hiroshima Math. J. **10** (1980).

[11] Kashiwara, M., and M. Vergne, *On the Segal–Shale–Weil representation and harmonic polynomials*, Invent. Math. **44** (1978), 1–47.

[12] Koike, K., and I. Terada, *Young–diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n*, J. Algebra **107** (1987), 466–511.

[13] Koufany, K., and B. Ørsted, *Functions spaces on the Ol'shanskiĭ semigroup and the Gel'fand-Gindikin program*, Preprint, Odense University, 1995 (submitted).

[14] Koufany, K., et B. Ørsted, *Espace de Hardy sur le semi–groupe métaplectique*, Odense, 12 septembre 1995, à paraitre dans Comptes Rendus Acad. Sci. Paris.

[15] Koufany, K., and B. Ørsted, *The Hardy space on the Oscillator semigroup*, Preprint, Odense University, 1995.

[16] Littlewood, D. E., "The theory of group characters and matrix representations of groups," 2nd edition, Clarendon Press, Oxford, 1958.

[17] Neeb, K.-H., *Holomorphic representation theory* I, Math. Annalen **301** (1995), 155–181.

[18] —, *Holomorphic representation theory* II, Acta Math. **173** (1994), 103–133.

[19] Ol'shanskiĭ, G. I., *Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series*, Funkts. Analiz. i Pril. **15**, (1981), 53–66 (Russian); English translation.: Funct. Anal. Appl. **15** (1982), 275–285.

[20] —, *Complex Lie semigroups, Hardy spaces, and the Gelfand-Gindikin program.* In: "Topics in group theory and homological algebra," Yaroslavl University Press, 1982, 85–98 (Russian). English translation: Differential Geometry and its Applications, **1** (1991), 297–308.

[21] Sundaram, S., in: D. Stanton, Editor, "Invariant Theory and Young Tableaux," IMA Volumes in Mathematics and its Applications, **19**, Springer–Verlag, 1990.

[22] Stein, E. M., and G. Weiss, "Introduction to Fourier analysis on Euclidean spaces," Princeton University Press, 1971.

[23] Vergne, M., and G. Rossi, *Analytic continuation of the holomorphic discrete series of a semi–simple Lie group*, Acta Math. **136** (1976), 1–59.

[24] Vinberg, E. B., *Invariant cones and orderings in Lie groups*, Funkts. Analiz i Pril. **14** (1980), 1–13 (Russian); English translation in Funct. Anal. Appl. **14** (1980).

[25] Weyl, H., "The classical groups. Their invariants and representations," Princeton, 1939.

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