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Spacious Lie Groups

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Abstract

The concepts of weakly exponential and spacious groups have been introduced by Hofmann and Mukherjea in [7] where they investigate the density of the image of the exponential function of a Lie group. The question whether being weakly exponential and not being spacious are equivalent has been answered affirmatively by Jaworski in [8]. Weakly exponential groups have also been investigated by Djoković and Thăńg in [4] and by Neeb in [12]. These papers contain classifications of simple weakly exponential Lie groups. The present paper deals with a special form of being spacious which we call completely spacious. A Lie group is completely spacious if it possesses an open subsemigroup S which lies in the complement of the exponential image. The motivation for investigation of this question comes from a problem in topological dynamics. We will give a characterization of completely spacious groups which is very similar to the result of Jaworski. He proves that a semisimple Lie group is spacious if and only if the minimal paraboloic subgroups are disconnected. We will prove that a semisimple Lie group is completely spacious if and only if the minimal parabolic subgroups have infinitely many components.

1. Statement of the problem

We start with the definitions of the basic concepts.

Definition 1.1. A locally compact group G is called *spacious,* if there exists an open subset U such that $U^n \cap U^{n+1} = \emptyset$ for all $n \in \mathbb{N}$. The group is called *weakly exponential* if the union of all one-parameter subgroups is dense in G. A Lie algebra $\mathfrak{g}$ is called *weakly exponential* if there exists a weakly exponential group G with Lie algebra $\mathbf{L}(G) = \mathfrak{g}$. It is called *completely weakly exponential* if the corresponding simply connected Lie group is weakly exponential. Finally we call

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a Lie group *completely spacious* if there exists a nonempty open subsemigroup S with $S \cap \exp \mathfrak{g} = \emptyset$.

The last definition is motivated by a problem in topological dynamics. Let S be a subsemigroup of a topological group G that acts on a manifold M. For a point $p \in M$ we consider the stabilizer group G_p and its orbit $S.p$ under the action of S. A *control set* for the S-action on M is a set D with nonempty interior such that $D \subseteq \overline{S.x}$ for all $x \in D$, cf. [15]. There it is proven that the interior of D consists of those points p for which the stabilizer group G_p intersects $\operatorname{int}(S)$.

If G is a Lie group, then it is natural to ask if there is an infinitesimal counterpart for $\operatorname{int}(S) \cap G_p \neq \emptyset$. Therefore we define for a subsemigroup S of a Lie group G the set

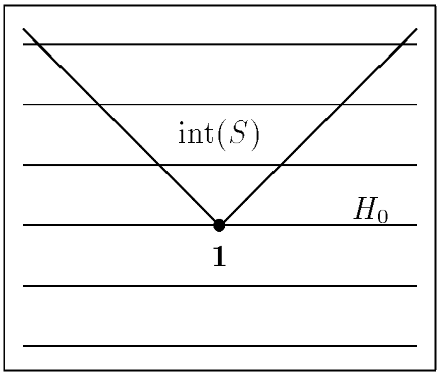
$$\mathcal{L}_\infty(S) = \{X \in \mathfrak{g} \mid (\exists t > 0) \exp(tX) \in \operatorname{int}(S)\}.$$

This notation was introduced by Lawson in [11]. Now let H be a subgroup of G, then $\mathbf{L}(H) \cap \mathcal{L}_\infty(S) \neq \emptyset$ implies $H \cap \operatorname{int}(S) \neq \emptyset$. We would actually like to know for which Lie groups the converse is also true. Thus we must solve the following problem:

Problem 1.2. Suppose that G is a connected Lie group, $H \subseteq G$ a connected Lie subgroup and $S \subseteq G$ a subsemigroup with nonempty interior.
(P) Is it possible that $\operatorname{int}(S) \cap H \neq \emptyset$ and $\exp \mathfrak{h} \cap \operatorname{int}(S) = \emptyset$?
(P$'$) Is it possible that $\mathbf{1} \in \overline{\operatorname{int}(S)}$, $\operatorname{int}(S) \cap H \neq \emptyset$ and $\exp \mathfrak{h} \cap \operatorname{int}(S) = \emptyset$?

The condition that H is connected may be weakened to H having only finitely many connected components, i.e., $|H/H_0| < \infty$, for then $\operatorname{int}(S) \cap H \neq \emptyset$ implies $\operatorname{int}(S) \cap H_0 \neq \emptyset$. Conversely, if we allow H to have infintely many connected components, then the question of Problem (P) must be answered 'Yes!' as the following simple example shows. Let $G = (\mathbb{R}^2, +)$, $H = \mathbb{R} \times \mathbb{Z}$ and $S = \mathbb{R}^+(1,1) + \mathbb{R}^+(-1,1) = \{(x_1, x_2) \mid x_2 \geq |x_1|\}$. Then

$$\operatorname{int}(S) \cap H = \bigcup_{n \in \mathbb{N}} (-n, n) \times \{n\}, \quad \text{but} \quad \exp \mathfrak{h} \cap \operatorname{int}(S) = H_0 \cap \operatorname{int}(S) = \emptyset.$$



This example even satisfies the condition of Problem (P$'$) that $\mathbf{1} \in \overline{\operatorname{int}(S)}$. One realizes moreover that this phenomenon appears quite often. For example, let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{n}$ a hyperplane ideal. Assume that on the group level $G/N \cong \mathbb{R}$. Then pick a pointed generating cone W transversal to $\mathfrak{n}$, i.e., $W \cap \mathfrak{n} = \{0\}$, so $S := \overline{\langle \exp W \rangle}$ is a semigroup satisfying $S \cap N = \{\mathbf{1}\}$. For an arbitrary subalgebra $\mathfrak{h} \subseteq \mathfrak{n}$ and an arbitrary point $X \in \operatorname{int}(W)$ let $H = H_0 \exp(\mathbb{Z}X)$. Then $H \cap \operatorname{int}(S) \neq \emptyset$ but $H_0 \cap \operatorname{int}(S) = \emptyset$.

Since we assume that both H and G are connected, Problem (P) only has to be considered for $H = G$. In this case it can be rephrased as:

> Does G possess an open subsemigroup that does not intersect the exponential image?

Thus we are led to the problem of classifying the completely spacious Lie groups.

Another interesting object that was introduced by Lai to study the image of the exponential function is the index function of a Lie group, see. [10] and also [2], as it may happen that an element $x \in G$ does not lie in $\exp \mathfrak{g}$ but some positive power x^n does.

Definition 1.3. For a Lie group G we define the *index function* $\operatorname{ind}_G: G \to \mathbb{N}$ by $\operatorname{ind}_G(x) = \inf \{n \in \mathbb{N} \mid x^n \in \exp \mathfrak{g}\}$. Moreover we let

$$\operatorname{dom} \operatorname{ind}_G = \{x \in G \mid \operatorname{ind}_G(x) < \infty\}.$$

The following proposition is immediate from the definitions.

Proposition 1.4. *If G is completely spacious, then $\overline{\operatorname{dom} \operatorname{ind}_G} \neq G$. If G is weakly exponential, then $\overline{\operatorname{dom} \operatorname{ind}_G} = G$.* ∎

It may happen that $\operatorname{dom} \operatorname{ind}_G \neq G$ for weakly exponential groups. The universal cover of the group of euclidean motions of the plane, $G = \widetilde{\mathrm{SE}}(2)$ provides an example of a weakly exponential group with $\operatorname{ind}_G(G) = \{1, \infty\}$, so $\operatorname{dom} \operatorname{ind}_G = \exp \mathfrak{g} \neq G$. Let us call a Lie group *nice*, if $\operatorname{dom} \operatorname{ind}_G$ is dense in G. Then the following is obvious:

Lemma 1.5. *If G_1 and G_2 are nice groups, then so is $G_1 \times G_2$.* ∎

We will see in the end that G is nice iff G is not completely spacious. It is clear that a completely spacious group cannot be nice. But the converse still needs to be proved. For example, there is no obvious way to prove the statement of the previous lemma if one replaces 'nice' by 'not completely spacious.'

2. Reductions

Throughout the whole text we will assume that the Lie group G is connected. All results remain true if we relax this to G/G_0 being finite. As a first step, we reduce the problem to the case that G is semisimple. A completely spacious Lie group cannot be solvable because all solvable Lie groups are weakly exponential, hence nice. The reduction arguments in the following two lemmas are nearly a transcript of the proof in [7] that a Lie group is weakly exponential if and only if the quotient modulo its radical is weakly exponential.

Lemma 2.1. *If G is a Lie group, N a closed normal subgroup and G/N completely spacious, then G is completely spacious.*

Proof. Let $\pi: G \to G/N$ be the quotient map. Then $\pi(\operatorname{im}\exp_G) = \operatorname{im}\exp_{G/N}$. Let $S_0 \subseteq G/N$ be an open semigroup with $S_0 \cap \operatorname{im}\exp_{G/N} = \emptyset$, and let $S = \pi^{-1}(S_0)$. Then S is an open subsemigroup of G, $\pi(S) = S_0$ and therefore

$$\pi(S \cap \operatorname{im}\exp_G) \subseteq \pi(S) \cap \pi(\operatorname{im}\exp_G) = S_0 \cap \operatorname{im}\exp_{G/N} = \emptyset.$$

Thus $S \cap \operatorname{im}\exp_G = \emptyset$ follows. ∎

Lemma 2.2. *Let G be a connected Lie group, R its radical, i.e., the largest connected solvable normal subgroup, and $G_s = G/R$. Then G is completely spacious iff G_s is completely spacious.*

Proof. The if-part is proved in Lemma 2.1. Conversely suppose that G_s is not completely spacious and let $S \subseteq G$ be an open subsemigroup. Then $\pi(S) \subseteq G_s$ is an open subsemigroup, hence it intersects $\exp \mathfrak{g}_s$. So we obtain $X \in \mathfrak{g}_s$ with $\exp \mathbb{R}X \cap \pi(S) \neq \emptyset$. Next let $H = \exp(\mathbb{R}X)R \subseteq G$, then $H \cap S \neq \emptyset$ is an open subsemigroup in H. Let $\mathfrak{r}$ denote the radical of $\mathfrak{g}$, then $\mathbf{L}(H) = \mathbb{R}X + \mathfrak{r}$. Since H is connected and solvable, it is weakly exponential. Thus $\exp(\mathbb{R}X + \mathfrak{r})$ must intersect $H \cap S$. Therefore $\exp \mathfrak{g} \cap S \neq \emptyset$ follows. ∎

The reduction in the semisimple case is a little bit more delicate. If G is a semisimple Lie group, then its Lie algebra $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ of simple ideals. If we let $G_i = \langle \exp_G \mathfrak{g}_i \rangle$, then $G = G_1 \cdots G_n$ as a set, and each G_i is a normal subgroup, but G need not be a direct product of the G_i. Nevertheless we will prove that for a *nice* group G_k, the factor group G/G_k is completely spacious if and only if G is completely spacious.

Lemma 2.3. *Suppose G has Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and let $G_2 = \langle \exp_G \mathfrak{g}_2 \rangle$ be the corresponding normal subgroup. If G_2 is nice, then G is completely spacious if and only if G/G_2 is completely spacious.*

Proof. Again the if-part is proved in Lemma 2.1. Conversely, let $S \subseteq G$ be an open semigroup and $\pi: G \to G/G_2$ the quotient map. Then $\pi(S) \cap \pi(\exp_G \mathfrak{g}_1)$ is nonempty, i.e., we find $X_1 \in \mathfrak{g}_1$ and $g \in G_2$ such that $\exp_G(X_1)g \in S$. Since S is open, we have $\exp_G(X_1)gU \subseteq S$ for a sufficiently small identity neighborhood U. Since G_2 is nice, $\operatorname{dom}\operatorname{ind}_{G_2}$ is dense in G_2, i.e., we can find $g' \in gU \cap G_2$, $k \in \mathbb{N}$ and $X_2 \in \mathfrak{g}_2$ such that $(g')^k = \exp_{G_2}(X_2) = \exp_G(X_2) \in \exp_G \mathfrak{g}_2$. Hence

$$S \ni (\exp_G(X_1)g')^k = \exp_G(X_1)^k (g')^k = \exp_G(kX_1)\exp_G(X_2) = \exp_G(kX_1 + X_2).$$

Thus G is not completely spacious. ∎

Next we have to determine some nice groups.

Lemma 2.4. *Suppose that G is semisimple connected and that all Cartan subgroups have finitely many connected components. Then G is nice. In particular, G is* not *completely spacious.*

Proof. We recall that an element $g \in G$ is called regular if the nilspace

$$N(\mathrm{Ad}(g) - \mathrm{id}) = \{X \in \mathfrak{g} \mid (\exists n \in \mathbb{N})(\mathrm{Ad}(g) - \mathrm{id})^n X = 0\}$$

has minimal dimension. The set $\mathrm{Reg}(G)$ of all regular elements is open and dense in G. Pick $g \in \mathrm{Reg}(G)$ and let $\mathfrak{h} = \ker(\mathrm{Ad}(g) - \mathrm{id})$, then $\mathfrak{h}$ is a Cartan algebra and its centralizer $H = C(G, \mathfrak{h})$ is a Cartan subgroup. By our assumption $n := |H/H_0| < \infty$. Since $H_0 = \exp \mathfrak{h} \subseteq \mathrm{im}\exp$ and $g \in H$, we obtain that $g^n \in H_0 \subseteq \exp \mathfrak{g}$, i.e., $g \in \mathrm{dom}\,\mathrm{ind}_G$. Thus $\mathrm{Reg}(G) \subseteq \mathrm{dom}\,\mathrm{ind}_G$. Since $\mathrm{Reg}(G)$ is dense, G must be nice. ∎

Lemma 2.5. *Suppose that G is semisimple with finite center. Then the Cartan subgroups have finitely many components.*

Proof. Let H be a Cartan subgroup and $\mathfrak{h} = \mathbf{L}(H)$. Then there exists an adapted Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that with $\mathfrak{h}_\mathfrak{k} := \mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{h}_\mathfrak{p} := \mathfrak{h} \cap \mathfrak{p}$ we have $\mathfrak{h} = \mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{p}$, cf. [16, Proposition 1.3.1.1]. Now $G = K \exp \mathfrak{p}$ and K is compact because the center of G is finite. Let $H_K = H \cap K$, then by [16, Proposition 1.4.1.2] $H = H_K \exp \mathfrak{h}_\mathfrak{p}$. As K is compact, so must be H_K. Hence it has finitely many connected components. Since $H/H_0 = H_K/(H_K)_0$ this proves that H has only finitely many components. ∎

As an immediate corollary of the previous two lemmas we obtain:

Corollary 2.6. *If G is semisimple with finite center, then G is nice, hence* not *completely spacious.* ∎

3. The simple case

Now we consider the case that G is simple. From Corollary 2.6 we immediately obtain:

Corollary 3.1. *Suppose that G is simple and completely spacious. Then its center contains an infinite cyclic subgroup and its Lie algebra $\mathfrak{g}$ is hermitian.* ∎

Up to isomorphism the simple hermitian Lie algebras are

$$\mathfrak{su}(p,q)\ (p \geq q \geq 1), \quad \mathfrak{sp}(n,\mathbb{R})\ (n \geq 1), \quad \mathfrak{so}(p,2)\ (p \neq 2), \quad \mathfrak{so}^*(2n)\ (n \geq 3),$$

and the exceptional algebras $\mathfrak{e}_{6(-14)}$, $\mathfrak{e}_{7(-25)}$. These can be divided into two different classes, namely the Cayley type algebras and the non-Cayley type algebras. The Cayley type algebras are

$$\mathfrak{su}(n,n), \quad \mathfrak{sp}(n,\mathbb{R}), \quad \mathfrak{so}(p,2), \quad \mathfrak{so}^*(4n), \quad \mathfrak{e}_{7(-25)}.$$

The terminology is taken from [13]. The simple hermitian Lie algebras are symmetric Lie algebras, i.e., one has decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}) \oplus (\mathfrak{q} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{p}),$$

where $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is the Cartan decomposition coming from a Cartan involution θ and $\mathfrak{h}$ and $\mathfrak{q}$ are the $+1$- resp. -1-eigenspaces of an involution τ that commutes with θ. The decompositions are orthogonal with respect to the bilinear form $B_\theta(X,Y)=-\operatorname{trace}(\operatorname{ad}(X),\operatorname{ad}(\theta Y))$. It was shown in [9] that for the corresponding symmetric spaces there exists a generalized Cayley transform, $\mathbf{C}$. For the Cayley type algebras it turns out that the involution τ is actually the square of the associated Cayley transform: $\tau=\mathbf{C}^2$. There are two causality concepts, namely *compactly causal* and *non-compactly causal*, see [6, Definition 3.1.23]. All simple hermitian Lie algebras give rise to compactly causal symmetric spaces. But the Cayley type algebras give also rise to non-compactly causal spaces.

It turns out that for our problem we only have to consider the Cayley type algebras because the non-Cayley type groups are nice.

Theorem 3.2. *The hermitian non-Cayley type algebras are completely weakly exponential, i.e., the corresponding simply connected group is weakly exponential.*

Proof. [12, Theorem IV.6]. ∎

In particular, all other groups with such a Lie algebra are weakly exponential too. On the other hand, it turns out that a Cayley type group with infinite center is completely spacious. The proof requires several lemmas. The first one is an immediate consequence of the proof in [7] that a spacious group G cannot be weakly exponential. It is a simple consequence of the fact that the additive group of reals $(\mathbb{R},+)$ is not spacious.

Lemma 3.3. *If $U\subseteq G$ is an open subset such that $U^n\cap U^{n+1}=\emptyset$ for all $n\in\mathbb{N}$, then $U\cap\operatorname{im}\exp=\emptyset$.* ∎

The next lemma due to Azencott, cf. [1, Lemme III.6, p.66], is crucial because it provides a general semigroup construction. But first we have to fix some terminology. For a semisimple Lie group G let $G=KAN$ be an Iwasawa decomposition, $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$. Let M be the centralizer of A in K. Then $P:=MAN$ is a minimal parabolic subgroup, and all minimal parabolic subgroups are conjugate under innner automorphisms. Let M^* be the normalizer of A in K, then M^*/M is isomorphic to the Weyl group $\mathcal{W}$ of the pair $(\mathfrak{g},\mathfrak{a})$. One has $G=\bigcup_{w\in\mathcal{W}}PwP$, this is the Bruhat decomposition of G. For fixed $w\in\mathcal{W}$ the double coset PwP is called a Bruhat cell. All these cells are disjoint, and there is precisely one Bruhat cell which is open and dense in G. If we denote N_- the nilpotent subgroup opposite to N with respect to the Cartan involution θ, i.e., $N_-=\theta(N)$, then the open Bruhat cell is given by N_-MAN.

Lemma 3.4. *Let G be a semisimple Lie group and MAN a minimal parabolic subgroup. Then there exist a semigroup $S\subseteq G$, an identity neighborhood V and an element $a\in A$ such that $VaM_0\subseteq S\subseteq N_-M_0AN$.* ∎

The following lemma is due to Jaworski.

Lemma 3.5. *Let $\hat{Q}$ be a closed subgroup of $\operatorname{Ad}(K)$, and let $Q=\operatorname{Ad}^{-1}(\hat{Q})$. Then Q has finitely many components if and only if $Z(G)Q_0/Q_0$ is finite.*

Proof. First we observe that $\hat{Q}/\hat{Q}_0$ is finite because $\hat{Q}$ is compact. Since $\hat{Q}_0 = \mathrm{Ad}(Q_0)$, there is a homomorphism $f: Q/Q_0 \to \hat{Q}/\hat{Q}_0$. As $\hat{Q}/\hat{Q}_0$ is finite, $\hat{Q}/\hat{Q}_0 = f(F)$ for a finite subset $F \subseteq Q/Q_0$. From $\ker f = Z(G)Q_0/Q_0$, we obtain $Q/Q_0 = F\,(Z(G)Q_0/Q_0)$, proving the claim. ∎

This lemma applies, of course, to M, thus M/M_0 is finite if and only if $Z(G)M_0/M_0$ is finite. Since the center $Z(G)$ is always contained in K, we have $Z(G) \subseteq M$. If G is semisimple with infinite center, then the intersection $Z(G) \cap M_0$ plays a crucial role. The proof of the following lemma is due to G. Ólafsson.

Lemma 3.6. *If $\mathfrak{g}$ is a simple Cayley type algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ an Iwasawa-decomposition and $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ a minimal parabolic subalgebra, then $\mathfrak{m} \subseteq \mathfrak{k}'$.*

Proof. Let $\mathfrak{c} = \{X \in \mathfrak{q} \cap \mathfrak{k} \mid (\forall Y \in \mathfrak{q} \cap \mathfrak{k})\,[X, Y] = 0\}$. Since $\mathfrak{g}$ is hermitian, we obtain from [13, 1.1, 1.2] that $0 \neq \mathfrak{c} \subseteq \mathfrak{z}(\mathfrak{k})$. Since $\mathfrak{g}$ is simple hermitian, we have $\dim \mathfrak{z}(\mathfrak{k}) = 1$, thus $\mathfrak{z}(\mathfrak{k}) = \mathfrak{c} \subseteq \mathfrak{q} \cap \mathfrak{k}$. As $\mathfrak{k}' \oplus \mathfrak{z}(\mathfrak{k}) = \mathfrak{k} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{k})$ is an orthogonal sum, we deduce from $\mathfrak{z}(\mathfrak{k}) \subseteq \mathfrak{q} \cap \mathfrak{k}$ that $\mathfrak{h} \cap \mathfrak{k} = (\mathfrak{q} \cap \mathfrak{k})^{\perp} \subseteq \mathfrak{z}(\mathfrak{k})^{\perp} = \mathfrak{k}'$.

Since $\mathfrak{g}$ is of Cayley type, it is also non-compactly causal. Thus we can find a nonzero $Y_0 \in \mathfrak{q} \cap \mathfrak{p}$ which is fixed under $e^{\mathrm{ad}\,\mathfrak{k}\cap\mathfrak{h}}$, and moreover its centralizer is $\mathfrak{z}_{\mathfrak{g}}(Y_0) = (\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{q} \cap \mathfrak{p})$, cf. [14, Theorem 2.6]. Now let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace containing Y_0. If $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, then we obtain

$$\mathfrak{m} \subseteq \mathfrak{z}_{\mathfrak{g}}(Y_0) \cap \mathfrak{k} = \big((\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{q} \cap \mathfrak{p})\big) \cap \mathfrak{k} = \mathfrak{h} \cap \mathfrak{k} \subseteq \mathfrak{k}'.$$

Since all minimal parabolic subalgebras are conjugate, this proves the claim. ∎

Corollary 3.7. *If G is a simple Cayley type group, $G = KAN$ an Iwasawa decomposition and MAN a minimal parabolic subgroup, then $M_0 \subseteq K'$ is compact.*

Proof. Since $\mathfrak{m}$ and $\mathfrak{k}'$ are compact Lie algebras, we obtain from the previous lemma that $M_0 = \exp \mathfrak{m} \subseteq \exp \mathfrak{k}' = K'$. But K' is a semisimple group, hence compact. ∎

Lemma 3.8. *If G is a simple Lie group, then M/M_0 is infinite if and only if G is of Cayley type and $Z(G)$ is infinite.*

Proof. If G is a simple Lie group and M/M_0 infinite, then in view of Lemma 3.5 it must have infinite center. Thus it is hermitian. If G is a hermitian non-Cayley type group, then by Theorem 3.2 it is weakly exponential. But then Jaworski's characterization of spacious groups [8, Theorem 12] proves that the minimal parabolic subgroups are connected, thus $M = M_0$. Hence the group G must be of Cayley type. Conversely, if G is a simple Cayley type group with infinite center, then its center is isomorphic to $Z(G) \cong \mathbb{Z} \oplus \mathrm{tor}\, Z(G)$, and $M_0 \subseteq K'$ is compact, hence $M_0 \cap Z(G) \subseteq \mathrm{tor}\, Z(G)$. Let $x \in Z(G) \setminus \mathrm{tor}\, Z(G)$, then $\langle x \rangle \cong \mathbb{Z}$, and

$$\langle x \rangle \cap M_0 = \langle x \rangle \cap Z(G) \cap M_0 \subseteq \langle x \rangle \cap \mathrm{tor}\, Z(G) = \{\mathbf{1}\}.$$

Since $Z(G) \subseteq M$, we deduce $M/M_0 \supseteq \langle x \rangle M_0/M_0 \cong \mathbb{Z}$, i.e., M/M_0 is infinite. ∎

In the sequel a tilde will always be used to denote the corresponding simply connected group.

Lemma 3.9. *If G is semisimple and M/M_0 is infinite, then there exists an element $x \in Z(G) \cap (M \setminus M_0)$ such that $\langle x \rangle \cong \mathbb{Z}$ and $\langle x \rangle \cap M_0 = \{\mathbf{1}\}$.*

Proof. We have $G = \widetilde{G}/D$ and $\widetilde{G} = \prod_i \widetilde{G}_i$. Let $\widetilde{G}_1, \ldots, \widetilde{G}_k$ be the Cayley type factors and $\widetilde{G}_{k+1} \ldots \widetilde{G}_{k+l}$ the hermitian, non-Cayley type factors. Then $Z(\widetilde{G}) \cong A_1 \oplus A_2 \oplus \operatorname{tor} Z(\widetilde{G})$ with $A_1 \cong \mathbb{Z}^k$ and $A_2 \cong \mathbb{Z}^l$. In the proof of the previous lemma we have seen that the hermitian non-Cayley type groups are weakly exponential and that their minimal parabolic subgroups are connected. Thus if $\widetilde{M}$ is a minimal parabolic subgroup of $\widetilde{G}$, then $A_2 \subseteq \widetilde{M}_0 \cap Z(\widetilde{G}) \subseteq A_2 \oplus \operatorname{tor} Z(\widetilde{G})$ follows. In view of Lemma 3.5 we already know that $Z(\widetilde{G})\widetilde{M}_0 D/\widetilde{M}_0 D$ must be infinite. Therefore $A_1 A_2 D/A_2 D$ is infinite. Thus we can find an $x \in A_1$ such that $\langle x \rangle \cap A_2 D = \{\mathbf{1}\}$, hence $\langle x \rangle \cap A_2 \operatorname{tor} Z(\widetilde{G})D = \{\mathbf{1}\}$ whence $\langle x \rangle \cap \widetilde{M}_0 D \subseteq \langle x \rangle \cap A_2 \operatorname{tor} Z(\widetilde{G})D = \{\mathbf{1}\}$. If $\pi: \widetilde{G} \to G$ denotes the quotient map, then $\pi(x)$ has the desired properties because

$$\mathbb{Z} \cong \pi(\langle x \rangle) \subseteq \widetilde{M}D/\widetilde{M}_0 D \cong M/M_0. \qquad \blacksquare$$

Jaworski proves in [8, Lemma 5] that G is spacious if M is disconnected. A modification of his proof allows us to prove the following lemma. In fact, the proof becomes much simpler due to the very special situation we consider here.

Lemma 3.10. *If the group M has infinitely many components, then G is completely spacious.*

Proof. By the previous lemma we can find an element $x \in Z(G) \cap (M \setminus M_0)$ such that $\langle x \rangle \cong \mathbb{Z}$ and $x^n \notin M_0$ for all $n \in \mathbb{N}$. By Lemma 3.4 we can find an element $a \in A$, an identity neighborhood V, and a semigroup S such that $VaM_0 \subseteq S \subseteq N_- M_0 AN$. Let $S_1 = xS$. Then we obtain for all $n, p \in \mathbb{N}$

$$S_1^n \cap S_1^{n+p} = x^n(S^n \cap x^p S^{n+p}) \subseteq x^n(S \cap x^p S) \subseteq x^n(N_- M_0 AN \cap N_- x^p M_0 AN) = \emptyset.$$

The latter is true because the map from $N_- \times M \times A \times N \to G$ is a diffeomorphism from its domain onto its image, the open Bruhat cell in G, cf. [5, p. 407]. Thus $(S_1^k)^n \cap (S_1^k)^{n+1} = \emptyset$ for all $n, k \in \mathbb{N}$. Hence Lemma 3.3 proves that $S_1^k \cap \exp \mathfrak{g}$ must be empty for all $k \in \mathbb{N}$. Therefore $\widetilde{S} := \bigcup_{k \in \mathbb{N}} S_1^k$ is a semigroup with nonempty interior such that $\operatorname{int}(\widetilde{S}) \cap \exp \mathfrak{g} = \emptyset$. $\blacksquare$

Theorem 3.11. *A simple Lie group G is completely spacious if and only if its Lie algebra $\mathfrak{g}$ is of Cayley type and $Z(G)$ is infinite.*

Proof. The if-part is proved in the previous lemma. The only-if-part follows from Theorem 3.2 and Corollary 2.6 which prove that all other simple groups are nice. $\blacksquare$

Corollary 3.12. *A simple Lie group is nice if and only if it is not completely spacious.*

Proof. The statement that a nice group G cannot be completely spacious follows immediately from the definitons. Conversely, if G is simple and not completely spacious, then either its center is finite or G is weakly exponential. In both cases, G is nice. ■

Problem 3.13. Prove the same statement with G semisimple instead of simple. Describe the semisimple groups which are nice.

Finally, combining the reduction arguments with the classification of simple, completely spacious groups, we can now describe the fully reduced group $G_{\rm red}$ that one obtains from a group G after all possible reductions have been carried out. So $G_{\rm red}$ does not contain any nice semisimple normal subgroups.

Lemma 3.14. *If G is a semisimple group and fully reduced, then its Lie algebra is $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, where all $\mathfrak{g}_k$ are of Cayley type, $Z(G_k)$ is infinite and $G = \widetilde{G}/D$ with $D \cap Z(\widetilde{G_i}) \subseteq \operatorname{tor} Z(\widetilde{G_i})$ for all $i = 1, \ldots, k$.*

Proof. Let G be semisimple and fully reduced. Its Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is a direct sum of simple ideals. By Lemma 2.3 we could factor all semisimple subgroups which are nice. As G is fully reduced, there are no nice normal subgroups. Since all simple non-Cayley type Lie groups are nice, this implies that all the $\mathfrak{g}_i$ have to be of Cayley type and that the groups $G_k = \langle \exp \mathfrak{g}_k \rangle$ must have infinte center. Since $G = \widetilde{G}/D$, the latter means that the kernel D can intersect $Z(\widetilde{G}_i)$ only in its torsion part, $D \cap Z(\widetilde{G}_k) \subseteq \operatorname{tor} Z(\widetilde{G}_i)$. ■

4. The final result

First we cite Jaworski's result [8, Theorem 12]:

Theorem 4.1. *For a connected semisimple Lie group G the following statements are equivalent:*

(1) *G is weakly exponential.*
(2) *The Cartan subgroups are connected.*
(3) *The minimal parabolic subgroups are connected.*
(4) *G is not spacious.*

We recall from Lemma 2.2 that a Lie group G is completely spacious if and only if its quotient modulo the radical G/R is completely spacious.

Theorem 4.2. *For a semisimple Lie group G the following statements are equivalent:*

(1) *G is completely spacious.*
(2) *The fully reduced group $G_{\rm red}$ is completely spacious.*
(3) *The minimal parabolic subgroups of $G_{\rm red}$ have infinitely many components.*
(4) *The minimal parabolic subgroups of G have infinitely many components.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from our reduction Lemmas 2.1 and 2.3. The implication (4) $\Rightarrow$ (1) follows by Lemma 3.10. Thus we still have to prove (2) $\Rightarrow$ (3) $\Rightarrow$ (4). The implication (3) $\Rightarrow$ (4) is quite obvious because the minimal parabolic subgroups of G are mapped onto the minimal parabolic subgroups of G_{red} by the quotient maps used to carry out the reduction. As the quotient maps are continuous, each reduction step decreases the number of components. Thus if the minimal parabolic subgroups of G_{red} have infinitely many components, the same has to be true for the minimal parabolic subgroups of G.

Finally we prove (2) $\Rightarrow$ (3). So let G_{red} be fully reduced, then by Lemma 3.14 its Lie algebra $\mathfrak{g}_{\mathrm{red}}$ is the direct sum of simple ideals $\mathfrak{g}_{\mathrm{red}} = \bigoplus_i \mathfrak{g}_i$ each of which is of Cayley type and the corresponding subgroup $G_i \subseteq G_{\mathrm{red}}$ has infinite center. Let $\widetilde{G}_i$ denote the universal covering groups, then $\widetilde{G}_{\mathrm{red}} = \widetilde{G}_1 \times \cdots \times \widetilde{G}_k$ and $G_{\mathrm{red}} = \widetilde{G}_{\mathrm{red}}/D$ for some $D \subseteq Z(\widetilde{G}_{\mathrm{red}}) \cong \mathbb{Z}^k \oplus \operatorname{tor} Z(\widetilde{G}_{\mathrm{red}})$. Moreover $\widetilde{M}_{\mathrm{red}} \supseteq Z(\widetilde{G})$ and $(\widetilde{M}_{\mathrm{red}})_0 \cap Z(\widetilde{G}) \subseteq \operatorname{tor} Z(\widetilde{G})$ by Corollary 3.7. Since $G_i \cong \widetilde{G}_i D/D \cong \widetilde{G}_i/(Z(\widetilde{G}_i) \cap D)$ has infinite center, we deduce that $D \cap Z(\widetilde{G}_i) \subseteq \operatorname{tor} Z(\widetilde{G}_i)$. Hence $\mathbb{Z}^k D/D$ is infinite. Thus we can find $x \in Z(G) \setminus \operatorname{tor}(G)$ such that $\langle x \rangle \cong \mathbb{Z}$ and $\langle x \rangle \cap D = \{\mathbf{1}\}$. Since $(\widetilde{M}_{\mathrm{red}})_0 D \cap Z(\widetilde{G}) = ((\widetilde{M}_{\mathrm{red}})_0 \cap Z(\widetilde{G}))D \subseteq \operatorname{tor} Z(\widetilde{G})D$, and since $\langle x \rangle \cap \operatorname{tor} Z(\widetilde{G})D = \{\mathbf{1}\}$, we deduce that $\langle x \rangle \cap (\widetilde{M}_{\mathrm{red}})_0 D = \{1\}$. Thus if $\pi\colon M_{\mathrm{red}} \to M_{\mathrm{red}}/(M_{\mathrm{red}})_0$ denotes the quotient map, then $\langle \pi(xD) \rangle \cong \mathbb{Z}$, proving that $M_{\mathrm{red}}/(M_{\mathrm{red}})_0$ is infinite. ∎

A characterization by the types of the simple factors does *not* work in general. It may happen that one of the simple factors G_i is completely spacious but G itself is not. This is shown by the following example due to K.-H. Neeb. It is very much in the spirit of Example 2.2 in [7], where it is shown that a weakly exponential group may possess a simple factor which is not weakly exponential.

Example 4.3. Let $G = G_1 \times G_2$ be simply connected, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1 = \mathfrak{su}(2,1)$ and $\mathfrak{g}_2 = \mathfrak{sl}(2,\mathbb{R})$. Let $z_2 \in \widetilde{\mathrm{SL}}(2,\mathbb{R}) = G_2$ be a generator for $Z(G_2)$, so $\langle z_2 \rangle = Z(G_2)$. Let $z_1 \in Z(G_1)$ be the generator of an infinite cyclic subgroup $\langle z_1 \rangle \cong \mathbb{Z}$ and $D = \langle (z_1, z_2) \rangle \subseteq Z(G)$. Then $G_1 \cap D = G_2 \cap D = \{\mathbf{1}\}$, so $G_i D/D \cong G_i$ is simply connected. But our reduction lemma may be applied. Since G_1 is nice and the factor group $G/G_1 \cong \mathrm{PSL}(2,\mathbb{R})$ is not completely spacious, G is not completely spacious. In fact, one can even show that G/D is weakly exponential although the simple normal subgroup $G_2 D/D \cong \widetilde{\mathrm{SL}}(2,\mathbb{R})$ is completely spacious.

Finally we can solve Problem 3.13 using the following lemma the proof of which is due to W. Jaworski.

Lemma 4.4. *If G is a semisimple Lie group, then the minimal parabolic subgroups have finitely many components if and only if the Cartan subgroups have finitely many components.*

Proof. "$\Leftarrow$": Suppose the Cartan subgroups have finitely many components, then Lemma 2.4 shows that G is a nice group, hence not completely spacious. Thus Theorem 4.2 proves that the minimal parabolic subgroups must have finitely many components.

"$\Rightarrow$:" In view of [16, Proposition 1.3.1.1] it suffices to consider Cartan subgroups $H(\mathfrak{h})$ where $\mathfrak{h}$ is stable under the Cartan involution θ, and $\mathfrak{h}_\mathfrak{p} \subseteq \mathfrak{a}$. We must prove that $H_K = H \cap K$ has finitely many components. In view of Lemma 3.5 we only have to prove that for all $z \in Z(G)$ there exists $n \in \mathbb{N}$ such that $z^n \in (H_K)_0$. As M/M_0 is finite, we can find an $n \in \mathbb{N}$ such that $z^n \in M_0$. Thus $z^n = \exp(Z)$ for some $Z \in \mathfrak{m}$. Let $\mathcal{H}_\mathfrak{a}$ denote the set of all θ-stable Cartan algebras $\mathfrak{h}$ containing the Iwasawa-$\mathfrak{a}$, then $\mathfrak{m} = \bigcup_{\mathfrak{h}\in\mathcal{H}_\mathfrak{a}} \mathfrak{h}_\mathfrak{k}$, [8, Lemma 6]. Thus we can find $\widetilde{\mathfrak{h}} \in \mathcal{H}_\mathfrak{a}$ such that $Z \in \widetilde{\mathfrak{h}}_\mathfrak{k}$. Since $\mathfrak{h}_\mathfrak{p} \subseteq \mathfrak{a} = \widetilde{\mathfrak{h}}_\mathfrak{p}$, there exists $k \in K$ such that $\mathrm{Ad}(k)|_{\mathfrak{h}_\mathfrak{p}} = \mathrm{id}_{\mathfrak{h}_\mathfrak{p}}$ and $\mathrm{Ad}(k)\widetilde{\mathfrak{h}}_\mathfrak{k} \subseteq \mathfrak{h}_\mathfrak{k}$, [16, Proposition 1.3.1.3]. Thus $z^n = kz^nk^{-1} = \exp(\mathrm{Ad}(k)Z) \in \exp \mathfrak{h}_\mathfrak{k}$ follows, proving our claim. ∎

Thus we can conclude with the following characterization theorem which is a counterpart of Theorem 4.1.

Theorem 4.5. *For a semisimple Lie group G the following are equivalent:*
(1) *G is nice, i.e.,* $\operatorname{dom}\operatorname{ind}_G$ *is dense in G.*
(2) *The Cartan subgroups have finitely many components.*
(3) *The minimal parabolic subgroups have finitely many components.*
(4) *G is not completely spacious.*
(5) *Every open subsemigroup intersects the exponential image.*
(6) *For every nonempty open subset U we can find $n < m \in \mathbb{N}$ such that $U^n \cap U^m \neq \emptyset$.*

Proof. The implications (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) follow using Proposition 1.4, Theorem 4.2, the previous Lemma, and Lemma 2.4. The equivalence (4) $\Leftrightarrow$ (5) follows by the definition. The semigroup construction in the proof of Lemma 3.10 shows that $\neg(4) \Rightarrow \neg(6)$. Finally Lemma 3.3 proves $\neg(6) \Rightarrow \neg(5)$. ∎

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