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Lie theory for non-Lie groups

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Introduction

The class of locally compact groups admits a strong structure theory. This is due to the fact that—via approximation (projective limits)—important parts of the theory of Lie groups and Lie algebras carry over. This phenomenon becomes particularly striking if one assumes, in addition, that the groups under consideration are connected and of finite dimension. The aim of the present notes is to collect results and to show that Lie theory yields complete information about the rough structure (i.e., the lattice of closed connected subgroups) of locally compact finite-dimensional groups. Moreover, we shall describe the possibilities for locally compact connected non-Lie groups of finite dimension.

We shall only consider Hausdorff groups (and shall, therefore, form quotients only with respect to closed subgroups—except in Example 1.7).

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1. Dimension

First of all, we need a notion of (topological) dimension. Mainly, we shall use the so-called *small inductive dimension*, denoted by dim. In textbooks about dimension theory, small inductive dimension is denoted by 'ind', while 'dim' denotes covering dimension. Since we shall almost exclusively deal with small inductive dimension, the more suggestive notation is preferred here.

Definition 1.1. Let X be a topological space. We say that $\dim X = -1$ if, and only if, X is empty. If X is non-empty, and n is a natural number,

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then we say that $\dim X \leq n$ if, and only if, for every point $x \in X$ and every neighborhood U of x in X there exists a neighborhood V of x such that $V \subseteq U$ and the boundary ∂V satisfies $\dim \partial V \leq n-1$. Finally, let $\dim X$ denote the minimum of all n such that $\dim X \leq n$; if no such n exists, we say that X has infinite dimension.

Obviously, $\dim X$ is a topological invariant. A non-empty space X satisfies $\dim X = 0$ if, and only if, there exists a neighborhood base consisting of closed open sets. Consequently, a T_1 space of dimension 0 is totally disconnected.

See [25] for a study of the properties of dim for separable metric spaces. Although it is quite intuitive, our dimension function does not work well for arbitrary spaces. Other dimension functions, notably covering dimension [38, 3.1.1], have turned out to be better suited for general spaces, while they coincide with dim for separable metric spaces. See [38] for a comprehensive treatment. Note, however, that small inductive dimension coincides with large inductive dimension and covering dimension, if applied to locally compact groups [2], [36], compare [42, 93.5]. The duality theory for compact abelian groups uses covering dimension rather than inductive dimension, cf. [32, pp. 106–111], [15, 3.11]. For this special case, we shall prove the equality in 5.7 below.

We collect some important properties of small inductive dimension.

Theorem 1.2. *For every natural number n, we have that $\dim \mathbb{R}^n = n$.*

Proof. [25, Th. IV 1], or [38, 3.2.7] in combination with [38, 4.5.10]. ∎

Lemma 1.3. *Let X be a non-empty Hausdorff space.*

(a) *For every subspace Y of X, we have that $\dim Y \leq \dim X$.*

(b) *$\dim X \leq n$ if, and only if, every point $x \in X$ has some neighborhood U_x such that $\dim U_x \leq n$.*

(c) *$\dim X \geq n$ if, and only if, there exists a point $x \in X$ with arbitrarily small neighborhoods of dimension at least n.*

(d) *If X is locally homogeneous, then $\dim X = \dim U$ for every neighborhood U in X.*

(e) *If X is locally compact, then $\dim X = 0$ if, and only if, X is totally disconnected.*

(f) *If X is the product of a family of (non-empty) finite discrete spaces, then $\dim X = 0$.*

(g) *If $\dim X = 0$, then $\dim(\mathbb{R}^n \times X) = n$ for every natural number n.*

Proof. An argument by induction on $\dim X$ yields (a), compare [38, 4.1.4]. Assertions (b) and (c) are immediate consequences of the definition, and they imply assertion (d). By (b), it suffices to prove assertion (e) for compact spaces. A compact (Hausdorff) space has at every point a neighborhood base of closed open sets if, and only if, it is totally disconnected [38, 3.1.3]. Assertion (f) follows immediately from (e) since the product of a family of (non-empty) finite discrete spaces is compact and totally disconnected.

Finally, assume that $\dim X = 0$. We proceed by induction on n. If $n = 0$, then $\dim(\mathbb{R}^0 \times X) = \dim X = 0$. So assume that $n > 0$, and that $\dim \mathbb{R}^{n-1} \times X = n-1$. Let U be a neighborhood of (r, x) in $\mathbb{R}^n \times X$. Since

$\dim X = 0$, there exists a closed open set V in X and a ball B around r in $\mathbb{R}^n$ such that $(r, x) \in B \times V \subseteq U$. Since the boundary $\partial(B \times V)$ is contained in $\partial B \times V$, we infer from our induction hypothesis that $\dim U \leq n$. The subspace $\mathbb{R}^n \times \{x\}$ of $\mathbb{R}^n \times X$ has dimension n. This completes the proof of assertion (g).■

Remark 1.4. Assertion 1.3(d) applies, in particular, to topological manifolds, and to topological groups. Note that, if a space X is not locally homogeneous, it may happen that there exists a point $x \in X$ with the property that $\dim U < \dim X$ for every sufficiently small neighborhood U of x. E.g., consider the topological sum of $\mathbb{R}$ and a single point.

Let G be a locally compact group. Finiteness of $\dim G$ allows to obtain analogues of counting arguments, as used in the theory of finite groups. In particular, we have the following:

Theorem 1.5. *Let G be a locally compact group, and assume that H is a closed subgroup of G. Then* $\dim G = \dim G/H + \dim H$.

Proof. This follows from [33, Sect. 5, Cor 2], since by 1.2 and 1.3 inductive dimension has the properties a)–e) that are required in [33, p. 64f]. ■

Remark 1.6. The same conclusion holds if we replace small inductive dimension by large inductive dimension, or by covering dimension, see [37].

Example 1.7. The closedness assumption on H is indispensable in 1.5. E.g., consider the additive group $\mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the factor group $\mathbb{R}/\mathbb{Q}$ has the indiscrete topology. Hence $\dim \mathbb{R}/\mathbb{Q} = 0 = \dim \mathbb{Q}$, but $\dim \mathbb{R} = 1$.

We denote the connected component of G by $G^{\mathbf{1}}$. Using 1.3(e), we obtain:

Theorem 1.8.

(a) *If G is a locally compact group, then* $\dim G = \dim G^{\mathbf{1}}$.

(b) *If H is a closed subgroup of a locally compact group G, and* $\dim H = \dim G < \infty$, *then* $G^{\mathbf{1}} \leq H$.

Proof. Assertion (a) follows from 1.5 and the fact that $G/G^{\mathbf{1}}$ is totally disconnected [15, 7.3]. If $\dim H = \dim G < \infty$ then $\dim G/H = 0$ by 1.5, and assertion (b) follows from the fact that the connected component can only act trivially on the totally disconnected space G/H.

Theorem 1.9. *Let G be a locally compact group. Assume that N is a closed normal subgroup, and C is a closed σ-compact subgroup such that* $\dim(C \cap N) = 0$ *and* $CN = G$. *Then* $\dim G = \dim C + \dim N$.

Proof. Since $G/N = CN/N \cong C/(C \cap N)$ [15, 5.33], the assertion follows from Theorem 1.5. ■

Definition 1.10.

(a) If, in the situation of 1.9, we have in addition that C and N are *connected*, we say that G is an *almost semi-direct product* of C and N.

(b) If, moreover, the subgroup C is normal as well, we say that G is an *almost direct product* of C and N.

Note that a locally compact connected group is generated by any compact neighborhood [15, 5.7], and is therefore σ-compact.

The terminology suggests that every almost (semi-)direct product is a proper (semi-)direct product, 'up to a totally disconnected normal subgroup'. This may be made precise in two different ways. Either the almost (semi-)direct product is obtained *from* a proper (semi-)direct product by forming the quotient modulo a totally disconnected subgroup, or one obtains a proper (semi-)direct product *after* passing to such a quotient. From the first of these viewpoints, our terminology is fully justified. In fact, every almost (semi-)direct product $G = CN$ is isomorphic to the quotient of the proper (semi-)direct product $C \ltimes N$ modulo K, where the action of C on N is given by conjugation in G, and $K = \{(g, g^{-1});\, g \in C \cap N\}$ is isomorphic to the totally disconnected group $C \cap N$.

From the second point of view, our terminology is adequate for almost direct products, but almost semi-direct products are more delicate.

Theorem 1.11. *If G is an almost direct product of (closed connected) subgroups N_1 and N_2, then $N_1 \cap N_2$ is contained in the center of G. The factor group $G/(N_1 \cap N_2)$ is the direct product of $N_1/(N_1 \cap N_2)$ and $N_2/(N_1 \cap N_2)$. Moreover,* $\dim G = \dim (G/(N_1 \cap N_2))$, *and* $\dim N_i = \dim (N_i/(N_1 \cap N_2))$.

Proof. The assertions follow from the fact that the connected group G acts trivially on the totally disconnected normal subgroup $N_1 \cap N_2$, and 1.5. ∎

Example 1.12. For almost semi-direct products $G = CN$, the intersection $C \cap N$ need not be a normal subgroup of G. E.g., let $N = \mathrm{SO}_3\mathbb{R}$, let $C = \mathbb{T}$, the circle group, and let $\gamma: C \to N$ be an embedding. Let a be an element of order 4 in C. Now $(c, x)(d, y) := (cd, (d^{-1})^\gamma x d^\gamma y)$ defines a semi-direct product $G = C \ltimes N$. It is easy to see that $Z = \langle (a, (a^{-1})^\gamma) \rangle$ is contained in the center of G. We infer that $\bar{G} := G/Z$ is an almost semi-direct product of $\bar{C} := ZC/Z$ and $\bar{N} := ZN/Z$, and that $Z(a, 1) = Z(1, a^\gamma)$ belongs to $\bar{C} \cap \bar{N}$, but $Z(1, a)$ is not central in $\bar{G}$. Since $\bar{G}$ is connected and $\bar{C} \cap \bar{N}$ is totally disconnected, normality of $\bar{C} \cap \bar{N}$ would imply that $\bar{C} \cap \bar{N}$ is central.

Lemma 1.13. *Assume that the Hausdorff space X is the countable union of relatively compact neighborhoods U_n such that $\dim U_n = d$ for every n, and let Y be a separable metric space. If $\varphi: X \to Y$ is a continuous injection, then* $\dim X = \dim X^\varphi \leq \dim Y$.

Proof. We adapt the proof from [13]. Small inductive dimension is defined locally, whence $\dim X = \dim \overline{U_n}$ for every n. Since $\overline{U_n}$ is compact, we obtain that $\overline{U_n}$ and $\overline{U_n}^\varphi$ are homeomorphic, and $\dim \overline{U_n} = \dim \overline{U_n}^\varphi$. Now $\dim X^\varphi = \dim \overline{U_n}^\varphi$ by the sum theorem [34, p. 14]. Finally, monotony of dim yields that $\dim X^\varphi \leq \dim Y$. ∎

We obtain the following applications.

Corollary 1.14. *Let G be a locally compact connected group. If G acts on a separable metric space Y, then $\dim(G/G_y) = \dim y^G \leq \dim Y$, where $y \in Y$ is any point, G_y is its stabilizer, and y^G its orbit under the given action. Important special cases are the following.*

(a) *If H is a locally compact group, and $\alpha: G \to H$ is a continuous homomorphism, then $\dim(G/\ker\alpha) = \dim G^\alpha \leq \dim H$.*

(b) *If G acts linearly on $V \cong \mathbb{R}^n$, then $\dim(G/G_v) = \dim vG \leq \dim V = n$, where $v \in V$ is any vector, G_v is its stabilizer, and vG its orbit under the given action.*

Proof. Every locally compact connected group G and every quotient space G/S, where S is a closed subgroup of G, satisfies the assumptions on X in 1.13: in fact, the group G is algebraically generated by every neighborhood of $\mathbb{1}$. Therefore, assertion (b) follows from the fact that the stabilizer G_v is closed in G. Assertion (a) follows from the fact that the image G^α is contained in the connected component $H^{\mathbf{1}}$, which is separable metric by 3.1. ∎

In general, a bijective continuous homomorphism of topological groups need not be a topological isomorphism; e.g., consider the identity with respect to the discrete and some non-discrete group topology. Locally compact connected groups, however, behave well.

Theorem 1.15. *Let G be a locally compact group, and assume that G is σ-compact. Then the following hold:*

(a) *If X is a locally compact space, and $\alpha: (X, G) \to X$ is a continuous transitive action, then the mapping $g \mapsto \alpha(x, g)$ is open for every $x \in X$.*

(b) *If $\mu: G \to H$ is a surjective continuous homomorphism onto a locally compact group H, then μ is in fact a topological isomorphism.*

Proof. Assertion (a) is due to [10], cf. also [23]. Assertion (b) follows by an application of (a) to the regular action of G on $H = G^\mu$. Compare also [15, 3.29]. ∎

Recall that a locally compact group G is σ-compact if it is compactly generated; in particular if $G/G^{\mathbf{1}}$ is compact, or if $G/G^{\mathbf{1}}$ is countable.

2. The Approximation Theorem

If G is a locally compact group such that $G/G^{\mathbf{1}}$ is compact, then there exist arbitrarily small compact normal subgroups such that the factor group is a Lie group. To be precise:

Theorem 2.1. (Approximation Theorem) *Let G be a locally compact group such that $G/G^{\mathbf{1}}$ is compact.*

(a) *For every neighborhood U of $\mathbb{1}$ in G there exists a compact normal subgroup N of G such that $N \subseteq U$ and G/N admits local analytic coordinates that render the group operations analytic.*

(b) *If, moreover,* $\dim G < \infty$, *then there exists a neighborhood* V *of* $\mathbb{1}$ *such that every subgroup* $H \subseteq V$ *satisfies* $\dim H = 0$. *That is, there is a totally disconnected compact normal subgroup* N *such that* G/N *is a Lie group with* $\dim G = \dim G/N$.

Proof. [31, Chap. IV], [11, Th. 9], see also [29, II.10, Th.18]. ∎

Remark 2.2. For locally compact groups in general, one knows that there always exists an open subgroup G such that $G/G^{\mathbf{1}}$ is compact, cf. [11, 3.5].

We obtain a useful criterion.

Theorem 2.3. *A locally compact group* G *is a Lie group if, and only if, every compact subgroup of* G *is a Lie group. If* G *is a locally compact group such that* $G/G^{\mathbf{1}}$ *is compact, then we can say even more: in this case, the group* G *is a Lie group if, and only if, every compact normal subgroup is a Lie group.*

Proof. Closed subgroups of Lie groups are Lie groups; see, e.g., [17, VIII.1]. Conversely, assume that every compact subgroup of G is a Lie group. According to 2.2, there exists an open subgroup H of G such that $H/H^{\mathbf{1}}$ is compact. Let N be a compact normal subgroup of H such that H/N is a Lie group. Then N is a Lie group by our assumption, and has, therefore, no small subgroups. Consequently, there exists a neighborhood U in H such that every subgroup $M \subseteq N \cap U$ is trivial. Let M be a compact normal subgroup of H such that $M \subseteq U$ and H/M is a Lie group. Then $H/(M \cap N)$ is a Lie group as well [11, 1.5], but $M \cap N = \{\mathbb{1}\}$. Thus H is a Lie group, and G is a Lie group as well, since H is open in G. If $G/G^{\mathbf{1}}$ is compact, our proof works for $H = G$, yielding the second part of our assertion. ∎

For the case where $G/G^{\mathbf{1}}$ is compact, the criterion 2.3 can also be deduced from the fact that the class of Lie groups is closed with respect to extensions [26, Th. 7].

Corollary 2.4. *Let* G *be a locally compact group, and assume that* G *is connected and of finite dimension. Then* G *is a Lie group if, and only if, the center of* G *is a Lie group.*

Proof. According to 2.1(b), the question whether or not G is a Lie group is decided in some totally disconnected normal subgroup N. Since G is connected, this subgroup is contained in the center of G. ∎

Compact subgroups play an important rôle in the theory of locally compact groups. They are understood quite well (see also the chapter on compact groups), especially in the connected case.

Theorem 2.5. *Let* G *be a locally compact group such that* $G/G^{\mathbf{1}}$ *is compact.*

(a) *Every compact subgroup of* G *is contained in some maximal compact subgroup of* G.

(b) *The maximal compact subgroups of* G *form a single conjugacy class.*

(c) *There exists some natural number* n *such that the underlying topological space of* G *is homeomorphic to* $\mathbb{R}^n \times C$, *where* C *is one of the maximal compact subgroups of* G.

(d) *In particular, every maximal compact subgroup of a locally compact connected group is connected.*

Proof. [26, §4, Th. 13], cf. also [17, Th. 3.1], and [21]. ∎

Considering, for example, a discrete infinite torsion group, one easily sees that some connectedness assumption is essential for the mere existence of maximal compact subgroups. Note that 2.5, in combination with the solution of D. HILBERT's Fifth Problem, provides another proof for 2.3.

There is, in general, no natural choice of N such that G/N is a Lie group. However, we have:

Theorem 2.6. *Let G be a locally compact connected group of finite dimension. If both N_1 and N_2 are closed normal subgroups such that* $\dim N_i = 0$ *and G/N_i is a Lie group, then G/N_1 is locally isomorphic to G/N_2.*

Proof. The factor group $G/(N_1 \cap N_2)$ is also a Lie group, cf. [11, 1.5]. Now $N_i/(N_1 \cap N_2)$ is a Lie group of dimension 0, and therefore discrete. This implies that $G/(N_1 \cap N_2)$ is a covering group for both G/N_1 and G/N_2. ∎

It is often more convenient to work with compact normal subgroups than with arbitrary closed normal subgroups. The general case may be reduced to the study of quotients with respect to compact kernels.

Theorem 2.7. *Let G be a locally compact connected group of finite dimension. If N is a closed normal subgroup such that* $\dim N = 0$ *and G/N is a Lie group, then there exists a* compact *normal subgroup M of G such that $M \leq N$ and G/M is a Lie group. The natural mapping $\pi: G/M \to G/N$ is a covering.*

Proof. Choose a compact neighborhood U of $\mathbb{1}$ in G. According to 2.1(b), there exists a compact normal subgroup N' such that $N' \subseteq U$ and G/N' is a Lie group. Now $M := N \cap N'$ has the required properties; in fact, G/M is a Lie group by [11, 1.5], and the kernel of the natural mapping $\pi: G/M \to G/N$ is a totally disconnected Lie group, hence discrete. ∎

A main reason why, in general, quotients with respect to compact subgroups behave better than quotients with respect to arbitrary closed subgroups is the following.

Lemma 2.8. *Let G be a topological group, and let H be a compact subgroup of G. Then the natural mapping $\pi: G \to G/H$ is a perfect mapping, i.e., for every compact subset $C \subseteq G/H$ the preimage $C^{\pi^{-1}}$ is also compact.*

Proof. Since H is compact, the natural mapping π is closed [15, 5.18]. Now π is a closed mapping with compact fibers, and therefore perfect [8, XI.5]. ∎

3. Countable bases, metrizability

In several instances, in particular when describing locally compact groups as projective limits of Lie groups, we shall benefit from the following observation.

Theorem 3.1. *Let G be a locally compact connected group of finite dimension. Then the topology of G has a countable neighborhood base. In particular, the topology is separable and metrizable.*

Proof. In view of the Theorem of MALCEV and IWASAWA 2.5(c), it suffices to consider the case where G is compact. According to 5.2, the group G is the semi-direct product of its commutator group G' and some abelian compact connected group A, both of finite dimension. Since G' is a Lie group by 5.1, there remains to show that A has a countable base. The weight of A (i.e., the minimal cardinality of a neighborhood base for A) equals the cardinality of the character group $\hat{A}$, see [15, 24.15]. Since A is connected, we have that $\hat{A}$ is torsion-free [15, 24.25]. Thus $\hat{A}$ is isomorphic to a subgroup of $\mathbb{Q}^r$, where r is the torsion-free rank of $\hat{A}$. From [15, 24.28] we infer that $r = \dim A$ is finite. (Note that covering dimension, as used in [15], coincides with with small inductive dimension for A by 5.7.) Thus $\hat{A}$ is countable, and A possesses a countable neighborhood base. The assertion that G is metrizable follows from the existence of a countable neighborhood base, see [15, 8.3]. ∎

For the conclusion of 3.1, neither the assumption that G is connected nor the assumption of finite dimension can be dispensed with.

Examples 3.2. If D is an uncountable discrete abelian group, then the dual group $\hat{D}$ is a compact group of weight $|D|$ and infinite dimension. For a concrete example, take $D = \mathbb{Z}^{\mathbb{N}}$. Then $\hat{D} = \mathbb{T}^{\mathbb{N}}$ is connected (of infinite dimension). The group D itself is an example of a disconnected group of uncountable weight (and dimension 0).

Corollary 3.3. *Every locally compact connected group of finite dimension is the projective limit of a* sequence *of finite coverings of a Lie group.*

Proof. Assume that G satisfies the assumptions. According to 2.1(b), there exists a totally disconnected compact normal subgroup N of G such that G/N is a Lie group. Since G has a countable base, we find a descending sequence U_n of relatively compact neighborhoods of $\mathbb{1}$ with trivial intersection. Now 2.1(a) asserts the existence of a descending sequence N_n of compact normal subgroups such that $N_n \subseteq U_n$ and $G/(N_n)$ is a Lie group for every n. Since N_n is totally disconnected, we obtain for every $k \leq n$ that $N_n/(N_k)$ is finite, viz. $G/(N_n)$ is a finite covering of $G/(N_k)$. ∎

4. The rough structure

In this section, we introduce the lattice of closed connected subgroups of a locally compact group, with additional binary operations. We show that this structure is preserved under the forming of quotients modulo compact totally disconnected normal subgroups.

For subsets A, B of a topological group G, let $[A, B]$ be the *closed* subgroup that is generated by the set $\{a^{-1}b^{-1}ab;\ a \in A, b \in B\}$. With this notation, we make the following observation.

Lemma 4.1. *Let G be a topological group, and let N be a totally disconnected closed normal subgroup of G. If A,B are connected subgroups of G, then $[A,B]=\{1\!\!1\}$ if, and only if, $[AN/N, BN/N]=\{1\!\!1\}$.*

Proof. This follows directly from the fact that $[A,B]$ is the closure of a continuous image of the connected space $A\times B$, and therefore connected. ∎

Let G be a locally compact group. We are interested in the lattice of closed connected subgroups; i.e., for closed connected subgroups A,B of G, we consider the smallest *closed* (necessarily connected) subgroup $A\vee B$ that contains both A and B, and the biggest *connected* (necessarily closed) subgroup $A\wedge B$ that is contained in both A and B. Note that $A\wedge B=(A\cap B)^{\mathbf{1}}$. Moreover, we are interested in the connected components of the normalizer and the centralizer of B, taken in A (denoted by $\mathrm{N}^{\mathbf{1}}_A B$ and $\mathrm{C}^{\mathbf{1}}_A B$, respectively). Finally, recall that the commutator subgroup $[A,B]$ is necessarily connected, while closedness is enforced by the very definition.

Definition 4.2.

(a) For any locally compact group, let $\mathbf{Struc}(G)$ be the algebra of all closed connected subgroups of G, endowed with the binary operations $\vee$, $\wedge$, $\mathrm{N}^{\mathbf{1}}$, $\mathrm{C}^{\mathbf{1}}$, $[\,,\,]$, as introduced above. We call $\mathbf{Struc}(G)$ the *rough structure* of G.

(b) Let $\mathbf{Comp}(G)$ be the set of *compact* connected subgroups of G, and let $\mathbf{Cpfree}(G)$ be the set of *compact-free* closed connected subgroups of G.

Remarks 4.3.

(a) Note that $\mathbf{Comp}(G)$ and $\mathbf{Cpfree}(G)$ are subsets but, in general, not subalgebras of $\mathbf{Struc}(G)$.

(b) Of course, $\mathbf{Struc}(G)=\mathbf{Struc}(G^{\mathbf{1}})$.

We are going to investigate the effect of continuous group homomorphisms on the rough structure. Our results will justify the vague feeling that the quotient of a locally compact group by a compact totally disconnected normal subgroup has 'roughly the same structure'.

Proposition 4.4. *Let G and H be locally compact groups, and let $\alpha\colon G\to H$ be a continuous homomorphism. For every closed connected subgroup A of G, let $A^{\bar\alpha}$ be the closure of A^α in H.*

(a) The mapping $\bar\alpha$ maps $\mathbf{Struc}(G)$ to $\mathbf{Struc}(H)$, and it maps $\mathbf{Comp}(G)$ to $\mathbf{Comp}(H)$.

(b) For $A\le B\le G$, we have that $A^{\bar\alpha}\le B^{\bar\alpha}$.

(c) For every choice of $A,B\in\mathbf{Struc}(G)$, we have that $A^{\bar\alpha}\vee B^{\bar\alpha}\le(A\vee B)^{\bar\alpha}$ and $A^{\bar\alpha}\wedge B^{\bar\alpha}\ge(A\wedge B)^{\bar\alpha}$.

(d) For every choice of $A,B\in\mathbf{Struc}(G)$, we have that $\left(\mathrm{N}^{\mathbf{1}}_A B\right)^{\bar\alpha}\le\mathrm{N}^{\mathbf{1}}_{A^{\bar\alpha}}B^{\bar\alpha}$, and that $\left(\mathrm{C}^{\mathbf{1}}_A B\right)^{\bar\alpha}\le\mathrm{C}^{\mathbf{1}}_{A^{\bar\alpha}}B^{\bar\alpha}$.

Proof. Assertions (a) and (b) are obvious from the definition of $\bar\alpha$ and the fact that every continuous mapping preserves compactness. From $A,B\le C$ it follows that $A^{\bar\alpha},B^{\bar\alpha}\le C^{\bar\alpha}$. This implies that $A^{\bar\alpha}\vee B^{\bar\alpha}\le(A\vee B)^{\bar\alpha}$. The second part of (c) follows analogously. Assertion (d) follows from the well-known inequalities $(\mathrm{N}_A B)^\alpha \le \mathrm{N}_{A^\alpha} B^\alpha$ and $(\mathrm{C}_A B)^\alpha \le \mathrm{C}_{A^\alpha} B^\alpha$, combined with the fact that continuous images of connected spaces are connected. ∎

Even if α is a quotient morphism with totally disconnected kernel, the mapping $\bar\alpha$ may be far from being injective. E.g., consider a quotient mapping from $\mathbb{R}^2$ onto $\mathbb{T}^2$. In fact, the rough structure $\mathbf{Struc}(\mathbb{R}^2)$ has uncountably many elements, while $\mathbf{Struc}(\mathbb{T}^2)$ is countable. However, we have:

Theorem 4.5. *Let G be a locally compact group, let N be a* compact *totally disconnected normal subgroup, and let π be the natural epimorphism from G onto G/N. Then the following hold:*

(a) *For every $A \in \mathbf{Struc}(G)$, we have that $A^\pi = A^{\bar\pi}$, and that* $\dim A = \dim A^\pi$.

(b) *The mapping π induces an isomorphism $\mathbf{Struc}(G) \cong \mathbf{Struc}(G/N)$.*

(c) *The mapping π induces bijections of $\mathbf{Comp}(G)$ onto $\mathbf{Comp}(G/N)$, and of $\mathbf{Cpfree}(G)$ onto $\mathbf{Cpfree}(G/N)$.*

Proof. According to 4.3, we may assume that G is connected. Hence G is σ-compact, and so is every closed subgroup A of G. In particular, $AB/A \cong B/(A\cap B)$ for every closed subgroup B of $\mathrm{N}_G A$, see [15, 5.33]. Mutatis mutandis, the same assertion holds for the epimorphic images.

(i) Being an epimorphism with compact kernel, the mapping π is closed [15, 5.18]. Moreover, we infer that the restriction of π to A is a closed surjection onto A^π, hence a quotient mapping. Therefore, $\dim A = \dim A^\pi$ by 1.5, and assertion (a) is proved.

(ii) For every $H \in \mathbf{Struc}(G/N)$, let $H^{\pi^\leftarrow}$ be the connected component of the π-preimage $H^{\pi^{-1}}$. Since π is continuous, we infer that $\pi^\leftarrow$ is a mapping from $\mathbf{Struc}(G/N)$ to $\mathbf{Struc}(G)$. For every H in $\mathbf{Struc}(G/N)$, the group $H/(H^{\pi^\leftarrow\pi}) \cong H^{\pi^{-1}}/(H^{\pi^\leftarrow}N)$ is totally disconnected. Hence $H^{\pi^\leftarrow\pi} \ge H^{\mathbf{1}}$, and $H^{\pi^\leftarrow\pi} = H$ since H is connected. For every A in $\mathbf{Struc}(G)$, we have that A is a normal closed subgroup of $AN = A^{\pi\pi^{-1}}$; recall that N centralizes A. The quotient $AN/A \cong N/(N\cap A)$ is totally disconnected. We infer that $A = (AN)^{\mathbf{1}} = A^{\pi\pi^\leftarrow}$.

(iii) The mapping $\pi^\leftarrow$ is monotone. In fact, let $H \le K$ in $\mathbf{Struc}(G/N)$. Then $H^{\pi^\leftarrow}$ is a connected subgroup of $H^{\pi^{-1}} \le K^{\pi^{-1}}$, hence $H^{\pi^\leftarrow} \le K^{\pi^\leftarrow}$. In view of 4.4(b), this shows that π respects the binary operations $\vee$ and $\wedge$.

(iv) From 4.1, we infer that π respects the operations $[\,,]$ and $\mathrm{C}^{\mathbf{1}}$. Arguments similar to those in step (ii) show that π respects the operation $\mathrm{N}^{\mathbf{1}}$ as well; recall that every epimorphism of (discrete) groups maps normalizers to normalizers.

(v) The natural epimorphism π is a proper continuous mapping, see 2.8. Thus π is a bijection of $\mathbf{Comp}(G)$ onto $\mathbf{Comp}(G/N)$. For $A \in \mathbf{Cpfree}(G)$, we obtain that $A \cap N = \{\mathbb{1}\}$, hence $A \cong A^\pi \in \mathbf{Cpfree}(G/N)$. Conversely, assume that $A \in \mathbf{Struc}(G) \setminus \mathbf{Cpfree}(G)$. According to 2.5(d), there exists a connected non-trivial compact subgroup C of A. Now C^π is a non-trivial compact subgroup of A^π. This completes the proof of assertion (c). ∎

5. Compact groups

By the following result, the structure theory of compact connected groups is, essentially, reduced to the theory of compact almost simple Lie groups and the theory of compact abelian groups:

Theorem 5.1.
(a) *Let G be a compact connected group. Then there exist a compact connected abelian group C, a family $(S_i)_{i\in I}$ of almost simple compact Lie groups S_i, and a surjective homomorphism $\eta : C \times \prod_{i\in I} S_i \to G$ with* $\dim\ker\eta = 0$. *The image C^η is the connected component of the center of G, and the commutator group G' equals $(\prod_{i\in I} S_i)^\eta$.*
(b) *Conversely, every group of the form $C \times \prod_{i\in I} S_i$, as in* (a), *is compact; hence also $(C \times \prod_{i\in I} S_i)^\eta$.*

Proof. [28, Th. 1, Th. 2], cf. [5, App. I, no. 3, Prop. 2], [26, Remark after Lemma 2.4], [46, §25]. ∎

Note that, in general, the connected component C^η of the center of G is not a complement, but merely a supplement of the commutator group in G. Since the topology of the commutator group $(\prod_{i\in I} S_i)^\eta$ is well understood, a complement would be fine in order to show the more delicate topological features of G. The following result asserts the existence of a complement (which, in general, is not contained in the center of G).

Theorem 5.2. *Every compact connected group is a semi-direct product of its commutator group and an abelian compact connected group.*

Proof. [20, 2.4]. A generalization to locally compact groups, involving rather technical assumptions, is given in [19, Th. 6]. ∎

For a compact connected group G, let $\eta: C \times \prod_{i\in I} S_i \to G$ be an epimorphism as in 5.1. The possible factors S_i are known from Lie Theory; see, e.g., [35, Ch. 5]. In order to understand the structure of C, one employs the PONTRYAGIN-VAN KAMPEN duality for (locally) compact abelian groups. See [40], [41] for a treatment that stresses the functorial aspects of duality. The dual $\hat{C}$ is a discrete torsion-free abelian group of rank c, and c equals the covering dimension of C if one of the two is finite [32, Th. 34, p. 108], [15, 24.28]. Hence there are embeddings $\mathbb{Z}^{(c)} \to \hat{C}$ and $\hat{C} \to \mathbb{Q} \otimes \hat{C} \cong \mathbb{Q}^{(c)}$. Dualizing again, we obtain a convenient description of the class of compact connected abelian groups:

Theorem 5.3. *Let C be a compact connected abelian group.*
(a) *If C has finite covering dimension c, then there are epimorphisms $\sigma: \hat{\mathbb{Q}}^c \to C$ and $\tau: C \to \mathbb{T}^c$, both with totally disconnected kernel.*
(b) *If C has infinite covering dimension, then there exists a cardinal number c such that there are epimorphisms $\sigma: \hat{\mathbb{Q}}^c \to C$ and $\tau: C \to \mathbb{T}^c$, both with totally disconnected kernel.*

Sometimes, one needs a more detailed description, as supplied by

Remark 5.4. Dualizing the description of $\mathbb{Q}$ as inductive limit of the system $(\frac{1}{n}\mathbb{Z})_{n\in\mathbb{N}}$ (endowed with natural inclusions $\frac{1}{n}\mathbb{Z} \to \frac{1}{nd}\mathbb{Z}$), we obtain that the character group $\hat{\mathbb{Q}}$ is the projective limit of the system $(T_n)_{n\in\mathbb{N}}$, where $T_n = \mathbb{T}$ for each n, with epimorphisms $t \mapsto t^d : T_{nd} \to T_n$. Every one-dimensional compact connected group is an epimorphic image of $\hat{\mathbb{Q}}$.

Within the boundaries that are set up by the fact that locally compact connected abelian groups are divisible [15, 24.25], we are free to prescribe the torsion subgroup of a one-dimensional compact connected group. In fact, let $\mathbb{P}$ be the set of all prime numbers, and let $P \subseteq \mathbb{P}$ be an arbitrary subset. In the multiplicative monoid of natural numbers, let $\mathbb{N}_P$ be the submonoid generated by P (i.e., $\mathbb{N}_P$ consists of all natural numbers whose prime decomposition uses only factors from P). With this notation, we have:

Theorem 5.5. *For every subset $P \subseteq \mathbb{P}$, there exists a compact connected group C with $\dim C = 1$ and the following properties: If $c \in C$ has finite order n, then $n \in \mathbb{N}_P$. Conversely, for every $n \in \mathbb{N}_P$ there exists some $c \in C$ of order n.*

Proof. The limit S_P of the subsystem $(T_n)_{n\in\mathbb{N}_P}$ of the projective system considered in 5.4 has the required property. ∎

See [15, 10.12–10.15] for alternate descriptions of the 'solenoids' S_P.

Examples 5.6. Of course, $S_{\mathbb{P}} = \hat{\mathbb{Q}}$, and $S_\varnothing = \mathbb{T}$. The group $S_{\{p\}}$ is the dual of the group $\bigcup_{n=0}^{\infty} \frac{1}{p^n}\mathbb{Z}$, its torsion group has elements of orders that are not divisible by p.

We conclude this chapter with an observation that relates 5.3 to the inductive dimension function, as used in the rest of this paper.

Theorem 5.7. *Small inductive dimension and covering dimension coincide for compact connected abelian groups.*

Proof. Let A be a compact connected abelian group, and let d denote its covering dimension. The dual group $\hat{A}$ is discrete [15, 23.17] and torsion-free (since A is connected, [15, 24.25]). Assume first that d is finite. According to [32, Th. 34, p. 108], we have the equality $d = \operatorname{rank}\hat{A}$. For a maximal free subgroup F of $\hat{A}$ we infer that $F \cong \mathbb{Z}^d$, and $\hat{A}/F$ is a torsion group. Consequently, the annihilator $F^\perp$ is totally disconnected, and has inductive dimension 0 by 1.8(a). Now $\mathbb{T}^d \cong \hat{F} \cong A/(F^\perp)$, and we conclude from 1.5 and 1.8 that $\dim A = \dim \mathbb{T}^d = d$. If d is infinite, then $\operatorname{rank} A$ is infinite, and we infer that $\dim A$ is infinite as well. ∎

6. The abelian case

In this section, we study connected locally compact abelian groups. Special attention will be given to decompositions of such groups, and their automorphisms.

For the structure theory of locally compact *abelian* groups, Pontrjagin-van Kampen duality is the strongest tool by far. See, e.g. [15, Chap. VI], [3], [32], [39, VI]. For the functorial aspects of duality theory, see [40], [41]. We give some results that are of interest for our special point of view. In particular, we concentrate on the connected case.

Theorem 6.1. (Decomposition Theorem) *Let A be a locally compact connected abelian group. Then there exist closed subgroups R and C of A such that A is the (interior) direct product $R \times C$, and $R \cong \mathbb{R}^a$ for some natural number a, while C is compact and connected. The group C is the maximal compact subgroup of A, hence it is a characteristic subgroup.*

Proof. [15, 9.14], [32, Th. 26]. ∎

The Decomposition Theorem is a special case of the Theorem of Malcev and Iwasawa 2.5. Using the decomposition $A = R \times C$, we shall gain information about the automorphisms of A. The following lemma, which is also of interest for its own sake, will be needed.

Lemma 6.2. *Let a, b be natural numbers. Every* continuous *group homomorphism from $\mathbb{R}^a$ to $\mathbb{R}^b$ is an $\mathbb{R}$-linear mapping.*

Proof. For every $x \in \mathbb{R}^n$ and every integer $z \neq 0$, there exists exactly one element $y \in \mathbb{R}^n$ (namely, $\frac{1}{z}x$) such that $zy = x$. Therefore every additive mapping $\mu: \mathbb{R}^a \to \mathbb{R}^b$ is in fact $\mathbb{Q}$-linear. Continuity of μ implies that μ is even $\mathbb{R}$-linear, since $\mathbb{Q}x$ is dense in $\mathbb{R}x$ for every $x \in \mathbb{R}^a$. ∎

Note that, if $a, b \neq 0$, then there exist many discontinuous $\mathbb{Q}$-linear mappings from $\mathbb{R}^a$ to $\mathbb{R}^b$.

Given a decomposition $A = R \times C$ as in 6.1, the subgroup R is not characteristic in $A = R \times C$, except if $R = A$. In fact, we have the following.

Theorem 6.3. *Let C be a compact group, $R \cong \mathbb{R}^a$, and $A = R \times C$.*

(a) *If $\alpha: R \to C$ is a continuous homomorphism, then $\Gamma_\alpha := \{(x, x^\alpha); x \in R\}$ is a closed subgroup of A, and $\Gamma_\alpha \cong R$. Moreover, the mapping $\mu_\alpha := \big((r,c) \mapsto (r, r^\alpha c)\big)$ is an automorphism of A.*

(b) *If B is a closed subgroup of A such that $B \cong \mathbb{R}^b$, then there exists some continuous homomorphism $\alpha: R \to C$ such that $B \leq \Gamma_\alpha$. In particular, $b \leq a$.*

(c) *If $\mu: \mathbb{R} \to A$ is a continuous homomorphism such that $\mathbb{R}^\mu$ is not closed in A, then $\mathbb{R}^\mu \subset C$.*

Proof. Assertion (a) is straightforward, using the fact that the graph of a continuous function is closed, if the codomain is Hausdorff. Let $B \cong \mathbb{R}^b$ be a closed subgroup of A. We consider the projections $\pi_R: A \to R: (r,c) \mapsto r$ and $\pi_C: A \to C: (r,c) \mapsto c$. Since B is compact-free, the restriction of π_R to B is injective. Hence there exists a retraction $\rho: R \to B$, and for $\alpha := \rho\pi_C$ we infer that $B \leq \Gamma_\alpha$. This proves (b). Let $\mu: \mathbb{R} \to A$ be a continuous homomorphism, and assume that $\mathbb{R}^\mu \not\subseteq C$. Then $\mathbb{R}^{\mu\pi_R}$ is a non-trivial subgroup of $R \cong \mathbb{R}^a$. We infer that $\mu\pi_R$ is an $\mathbb{R}$-linear mapping, and that there exists a section $\sigma: R \to \mathbb{R}$. Now $\mathbb{R}^\mu \leq \Gamma_{\sigma\mu\pi_C}$ is closed in A. This proves assertion (c). ∎

Examples 6.4.

(a) Dense one-parameter subgroups (or dense analytical subgroups) are familiar from Lie theory; most prominent, perhaps, is the 'dense wind' $\mathbb{R} \to \mathbb{T}^2$. In the realm of Lie groups, the closure of a non-closed analytical subgroup has larger dimension than the subgroup itself.

(b) The dual of the monomorphism $\mathbb{Q} \to \mathbb{R}$, where $\mathbb{Q}$ carries the discrete topology, yields a monomorphism $\mathbb{R} \to \hat{\mathbb{Q}}$ with dense image. Note that $\dim \hat{\mathbb{Q}} = 1 = \dim \mathbb{R}$. Equidimensional immersions are typical for non-Lie groups; see [22], and 9.6 below.

Next, we study automorphism groups of abelian locally compact connected groups. We endow $\mathrm{Aut}(A)$ with the coarsest Hausdorff topology that makes $\mathrm{Aut}(A)$ a topological (not necessarily locally compact) transformation group on A (see [1], [15, §26]). With respect to this topology, $\mathrm{Aut}(A)$ and $\mathrm{Aut}(\hat{A})$ are isomorphic as topological groups [15, 26.9]. This has the following immediate consequences [15, 26.8, 26.10]:

Theorem 6.5.

(a) *The group of automorphisms of a compact abelian group is totally disconnected.*

(b) *Let G be a connected group, and assume that N is a compact abelian normal subgroup of G. Then N lies in the center of G.*

6.6. Let R and C be arbitrary topological groups, but assume that C is abelian*. It will be convenient to use additive notation. Let α be an endomorphism of the direct sum $R \oplus C$, and assume that α leaves C invariant. Since $(r+c)^\alpha = r^\alpha + c^\alpha$, we can write α as the (pointwise) sum of the restrictions $\alpha|_R$ and $\alpha|_C$. Since $C^\alpha \le C$, the restriction $\alpha|_C$ may be considered as an endomorphism of C. The restriction $\alpha|_R$ may be decomposed as the sum of the co-restrictions $\alpha|_R^R$ and $\alpha|_R^C$, i.e., we write $r^\alpha = r^{\alpha|_R^R} + r^{\alpha|_R^C}$, where $r^{\alpha|_R^R} \in R$ and $r^{\alpha|_R^C} \in C$. It is very convenient to use the matrix description

$$(r,c)^\alpha = \left(r^{\alpha|_R^R}, r^{\alpha|_R^C} + c^{\alpha|_C}\right) = (r,c)\begin{pmatrix} \alpha|_R^R & \alpha|_R^C \\ 0 & \alpha|_C \end{pmatrix} \quad .$$

In fact, an easy computation shows that the usual matrix product describes the composition of endomorphisms of $R \oplus C$, namely

$$(r,c)^{\alpha\beta} = (r,c)\begin{pmatrix} \alpha|_R^R\beta|_R^R & \alpha|_R^R\beta|_R^C + \alpha|_R^C\beta|_C \\ 0 & \alpha|_C\beta|_C \end{pmatrix} \quad .$$

The group of all automorphisms of $R \oplus C$ that leave C invariant is obtained as

$$\left\{\begin{pmatrix} \rho & g \\ 0 & \gamma \end{pmatrix}; \rho \in \mathrm{Aut}(R), \gamma \in \mathrm{Aut}(C), g \in \mathrm{Hom}(R,C)\right\} \quad .$$

---

* If C is not abelian, the following remarks remain valid if we consider $\mathrm{Hom}(R,Z)$ instead of $\mathrm{Hom}(R,C)$, where Z is the center of C.

Obviously, we have that

$$\left\{ \begin{pmatrix} \mathrm{id}_R & g \\ 0 & \mathrm{id}_C \end{pmatrix} ; g \in \mathrm{Hom}(R, C) \right\}$$

is a normal subgroup, and that

$$\left\{ \begin{pmatrix} \rho & 0 \\ 0 & \mathrm{id}_C \end{pmatrix} ; \rho \in \mathrm{Aut}(R) \right\} \text{ and } \left\{ \begin{pmatrix} \mathrm{id}_R & 0 \\ 0 & \gamma \end{pmatrix} ; \gamma \in \mathrm{Aut}(C) \right\}$$

are subgroups that centralize each other. That is, the group of those automorphisms of $R \oplus C$ that leave C invariant can be written as a semi-direct product $\mathrm{Aut}(R) \ltimes \mathrm{Hom}(R, C) \rtimes \mathrm{Aut}(C)$. Note that parentheses are not necessary.

Theorem 6.7. *Let A be a locally compact connected abelian group, and write $A = RC$, where $R \cong \mathbb{R}^a$ and C is compact and connected.*

(a) *The group of automorphisms of A is isomorphic to the semi-direct product $\mathrm{Aut}(C) \ltimes \mathrm{Hom}(\mathbb{R}^a, C) \rtimes \mathrm{GL}_a\mathbb{R}$, the connected component $\mathrm{Aut}(A)^{\mathbf{1}}$ is isomorphic to $\mathrm{Hom}(\hat{C}, \mathbb{R}^a) \rtimes \mathrm{GL}_a\mathbb{R}$.*

(b) *If $\dim C = c < \infty$, then $\mathrm{Aut}(A)^{\mathbf{1}}$ is a linear Lie group; in fact, there is a monomorphism $\iota\colon \mathrm{Aut}(A) \to \mathrm{GL}_c\mathbb{Q} \ltimes \mathrm{Hom}(\mathbb{Q}^c, \mathbb{R}^a) \rtimes \mathrm{GL}_a\mathbb{R}$, where $\mathbb{Q}$ and $\mathrm{GL}_c\mathbb{Q}$ carry the discrete topologies.*

Proof. The group $\mathrm{Aut}(A)$ leaves invariant the (unique) maximal compact subgroup C of A. Together with the remarks in **6.6**, this gives the first part of the assertion. From $\mathrm{Hom}(R, C) \cong \mathrm{Hom}(\hat{C}, \mathbb{R}^a) \leq \mathrm{Hom}(\mathbb{Q} \otimes \hat{C}, \mathbb{R}^a)$ we infer that there exists a monomorphism from $\mathrm{Aut}(A)$ to the group $L := \mathrm{Aut}(C) \ltimes \mathrm{Hom}(\mathbb{Q}^c, \mathbb{R}^a) \rtimes \mathrm{GL}_a\mathbb{R}$. Now assume that $\dim C < \infty$. According to [15, 24.28], $\dim(\mathbb{Q} \otimes \hat{C}) = \mathrm{rank}\, \hat{C} = \dim C$. Hence L is a (linear) Lie group. Since L has no small subgroups, the same holds for $\mathrm{Aut}(A)$. Hence $\mathrm{Aut}(A)^{\mathbf{1}}$ is a (connected) Lie group [31, Ch. III, 4.4], and the restriction of ι to $\mathrm{Aut}(A)^{\mathbf{1}}$ is analytic, see [17, VII, Th. 4.2] or [45, Sect. 2.11]. ∎

Corollary 6.8. *Let G be a locally compact connected group, and assume that A is a closed connected normal abelian subgroup of G. If $\dim A < \infty$, then $G/\mathrm{C}_G A$ is an analytic subgroup of $\mathbb{R}^{ca} \rtimes \mathrm{GL}_a\mathbb{R}$, where C is a compact group of dimension c, and $A \cong \mathbb{R}^a \times C$.*

An important application is the following.

Theorem 6.9. *Let G be a compact group, and assume that a is a natural number, and that C is a compact connected abelian group. For every continuous homomorphism $\mu\colon G \to \mathrm{Aut}(\mathbb{R}^a \times C)$, the following hold:*

(a) *Both $\mathbb{R}^a$ and C are invariant under G^μ.*

(b) *There exists a positive definite symmetric bilinear form on $\mathbb{R}^a$ that is invariant under G^μ. Consequently, μ induces a completely reducible $\mathbb{R}$-linear action of G on $\mathbb{R}^a$.*

(c) *If G^μ is connected, then G^μ acts trivially on C.*

Proof. Assertion (a) follows from 6.7 and the fact that $\mathrm{Hom}(\hat{C}, \mathbb{R})$ is compact-free. The group G^μ induces a compact subgroup of $\mathrm{GL}_a\mathbb{R}$. According to

[15, 22.23], or [35, Chap. 3 §4], there exists a G^μ-invariant positive definite symmetric bilinear form q on $\mathbb{R}^a$. If V is a G^μ-invariant subspace of $\mathbb{R}^a$, then the orthogonal complement with respect to q is G^μ-invariant as well. This completes the proof of assertion (b). The last assertion follows from 6.5(a). ∎

An interesting feature of locally compact connected abelian groups is the fact that the lattice of closed *connected* subgroups is complemented:

Theorem 6.10. *Let A be a locally compact connected abelian group, and assume that B is a closed connected subgroup of A. Then there exists a closed connected subgroup K of A such that $A = BK$ and $\dim(B\cap K) = 0$ (i.e., $B\cap K$ is totally disconnected).*

Proof. It suffices to show the existence of a closed subgroup S such that $BS = A$ and $\dim(B\cap S) = 0$; in fact, connectedness of A implies that $BS^{\mathbf{1}} = A$ (consider the action of A on the totally disconnected homogeneous space $A/(BS^{\mathbf{1}})$).

(i) Assume first that A is compact. Then the dual group $\hat{A}$ is discrete [15, 23.17] and torsion-free (since A is connected, [15, 24.25]). Consequently, $\hat{A}$ embeds in $Q := \mathbb{Q}\otimes\hat{A}$, taken with the discrete topology. Since $\hat{A}$ spans the $\mathbb{Q}$-vector space Q, there exists a basis $E\subset\hat{A}$ for Q. Moreover, we can choose E in such a way that $E\cap B^\perp$ is a basis for the subspace U spanned by $B^\perp$. Now $E\setminus B^\perp$ spans a complement V of U in Q. Writing $L := V\cap\hat{A}$, we infer that $B^\perp\cap L = \{\mathbb{1}\}$. Since $E\subset B^\perp\cup L$, the factor group $Q/(B^\perp L)$ is a torsion group, and so is $\hat{A}/(B^\perp L)$. We conclude that $BL^\perp = A$, and $\dim(B\cap L^\perp) = 0$.

(ii) In the general case, we write $A = R\times C$ and $B = S\times D$ with compact groups C, D, where $R\cong\mathbb{R}^a$ and $S\cong\mathbb{R}^b$. According to 6.3(b), there exists a continuous homomorphism $\alpha\colon R\to C$ such that S is contained in the graph Γ_α, and $A = \Gamma_\alpha\times C$ by 6.3(a). Therefore, we may assume that $S = \mathbb{R}^b\le R = \mathbb{R}^a$. For any subgroup $Z\cong\mathbb{Z}^a$ of $\mathbb{R}^a$ such that $B\cap Z\cong\mathbb{Z}^b$, the group A/Z is compact, and BZ/Z is a compact, hence closed, subgroup. Now (i) applies, and we infer that there exists a closed subgroup S of A such that $A = BS$ and $\dim(B\cap S) = 0$. ∎

Remarks 6.11.

(a) The example of a two-dimensional indecomposable group in [39, Bsp. 68] shows that, in general, a complement for a closed connected subgroup need not exist.

(b) Complements do exist in abelian connected Lie groups; this can be derived from the fact that, in this case, the dual group is isomorphic to $\mathbb{R}^a\times\mathbb{Z}^c$.

(c) If A is a locally compact abelian group, and B is a closed connected subgroup of A such that B is a Lie group (i.e., B is isomorphic to $\mathbb{R}^a\times\mathbb{T}^c$ for suitable cardinal numbers $a<\infty$ and c), then there exists a complement for B in A, see [3, 6.16].

(d) The assertion of 6.10 can also be derived from (b) and 4.5.

7. Notions of simplicity

We are now going to introduce the concepts 'almost simple', 'semi-simple', 'minimal closed connected abelian normal subgroup', 'solvable radical' in the context of locally compact connected groups of finite dimension. See [17, XII.3.1] for a comparison of the concepts of solvability and nilpotency in topological groups and in discrete groups.

A locally compact connected non-abelian group G is called *semi-simple* if it has no non-trivial closed connected abelian normal subgroup; the group G is called *almost simple* if it has no proper non-trivial closed connected normal subgroup.

Let $(G_i)_{i\in I}$ be a family of normal subgroups of a topological group G. Generalizing 1.10(b), we call the group G an *almost direct product* of the groups G_i, if G is generated by $\bigcup_{i\in I} G_i$ and the intersection of G_j with the subgroup generated by $\bigcup_{i\in I\setminus\{j\}} G_i$ is totally disconnected. Examples are given by compact connected groups 5.1, and also by semi-simple groups:

Theorem 7.1. *A locally compact connected group of finite dimension is semi-simple if, and only if, it is the almost direct product of a finite family $(S_i)_{1\le i\le n}$ of almost simple (closed connected) subgroups S_i.*

Proof. This follows from the corresponding theorem on Lie groups [5, III §9 no. 8 Prop. 26] via the Approximation Theorem 2.1(a) and 4.5. ∎

Theorem 7.2. *Let G be a locally compact connected group.*

(a) *If G is almost simple, then every proper closed normal subgroup is contained in the center Z of G, and Z is totally disconnected. In particular, G/Z is a simple Lie group with $\dim G/Z = \dim G < \infty$.*

(b) *If G is semi-simple and of finite dimension, then every closed connected normal subgroup is of the form $S_{i_1}\cdots S_{i_k}$, where the S_{i_j} are some of the almost simple factors from 7.1.*

Proof. Let N be a proper closed normal subgroup of G. The connected component $N^{\mathbf{1}}$ is a proper closed connected normal subgroup of G. If G is almost simple, we infer that $N^{\mathbf{1}} = \{\mathbb{1}\}$. Via conjugation, the connected group G acts trivially on the totally disconnected group N. Therefore N is contained in Z. Applying this reasoning to the case where $N = Z$, we obtain that Z is totally disconnected. The rest of assertion (a) follows from 1.5 and 2.1. Assertion (b) follows from 4.5 and the corresponding theorem on Lie groups [5, I, §6, no. 2, Cor. 1; III, §6, no. 6, Prop. 14]. ∎

Our next observation makes precise the intuition that an almost simple group either has large compact subgroups, or large solvable subgroups.

Theorem 7.3. *Let G be a locally compact connected almost simple group. Then there exist a compact subgroup C and closed connected subgroups T and D of G such that the following hold.*

(a) *The group C is compact and semi-simple, T is a subgroup of dimension at most 1 that centralizes C, and D is solvable.*

(b) *$G = TCD$, and $\dim G \le \dim C + \dim D + 1$.*

(c) *The group D is a simply connected, compact-free linear Lie group.*

(d) *The center Z of G is contained in TC, and TC/Z is a maximal compact subgroup of G/Z, while CZ/Z is the commutator group of TC/Z.*

Proof. The centralizer of the commutator group of a maximal compact subgroup of a simple Lie group has dimension at most 1. The assertions follow immediately from the Iwasawa decomposition for simple real Lie groups [14, VI, 5.3] by an application of 4.5 and 2.1. ■

Theorem 7.4. *Let G be a locally compact group, and assume that A is a closed connected abelian normal subgroup such that $\dim A < \infty$. Then there exists a minimal closed connected abelian normal subgroup $M \leq A$, and $0 < \dim M \leq \dim A$. Moreover:*

(a) *Either the group M is compact, or it is isomorphic with $\mathbb{R}^m$, where $m = \dim M$.*

(b) *If M is compact, then M lies in the center of the connected component $G^{\mathbf{1}}$.*

Proof. The set $\mathcal{A}$ of closed connected abelian normal subgroups of G that are contained in A is partially ordered by inclusion. Since $\dim X = \dim Y$ for $X, Y \in \mathcal{A}$ implies that $X = Y$ by 1.8(b), there are only chains of finite length in $\mathcal{A}$. The maximal compact subgroup C of a minimal element of $\mathcal{A}$ is a closed connected characteristic subgroup of M, hence either $M = C$ or $C = \{\mathbb{1}\}$ by minimality. In the latter case, $M \cong \mathbb{R}^m$ by 6.1. Assertion (b) is immediate from 6.5(b). ■

From 7.4, we infer that the class of locally compact connected groups of finite dimension splits into the class of semi-simple groups, and the class of groups with a minimal closed connected abelian normal subgroup M. The action of G on M via conjugation is well understood:

Theorem 7.5. *Let G be a locally compact group, and assume that there exists a minimal closed connected abelian normal subgroup $M \cong \mathbb{R}^m$.*

(a) *The group G acts (via conjugation) $\mathbb{R}$-linearly and irreducibly on M.*

(b) *The factor group $L = G/\mathrm{C}_G M$ is a linear Lie group, in fact, a closed subgroup of $\mathrm{GL}_m\mathbb{R}$. The commutator group S of L is also closed in $\mathrm{GL}_m\mathbb{R}$, and we have that $L \cong SZ$, where S is either trivial or semi-simple, and Z is the connected component of the center of L. Moreover, Z is isomorphic to a closed connected subgroup of the multiplicative group $\mathbb{C}^*$.*

(c) *For every one-parameter subgroup R of M, we have that $\dim G/\mathrm{C}_G R \leq \dim M$.*

Proof. The action via conjugation yields a continuous homomorphism from G to $\mathrm{GL}_m\mathbb{R}$, cf. 6.2. Every invariant subspace V of $M \cong \mathbb{R}^m$ is a closed connected normal subgroup of G. Minimality of M implies that $V = M$, or V is trivial. This proves assertion (a).

The factor group $G/\mathrm{C}_G M$ is a Lie group [17, VIII.1.1], which acts effectively on $M \cong \mathbb{R}^m$. This action is a continuous homomorphism of Lie groups. From [5, II.6.2, Cor. 1(ii)] we infer that the image L of $G/\mathrm{C}_G M$ in

$\mathrm{GL}_m\mathbb{R}$ is an analytic subgroup. Moreover, we know that L is irreducible on $\mathbb{R}^m$. According to [7], the group L is closed in $\mathrm{GL}_m\mathbb{R}$. Hence we may identify L and $G/\mathrm{C}_G M$, cf. 1.15. The commutator group S of L is closed, see [17, XVIII.4.5].

From [45, 3.16.2] we infer that the radical of the group L is contained in the center Z of L, whence $L = SZ$. According to Schur's Lemma [27, p. 118, p. 257], the centralizer of L in $\mathrm{End}_{\mathbb{R}}(M)$ is a skew field. Since this skew field is also a finite-dimensional algebra over $\mathbb{R}$, we infer that it is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, cf. [9]. (See also [6, 6.7].) Thus Z generates a commutative subfield of $\mathbb{H}$, hence $Z \le \mathbb{C}^*$. This completes the proof of assertion (b).

Assertion (c) follows readily from 1.14(b), since by linearity $\mathrm{C}_G R = \mathrm{C}_G r$ for every non-trivial element r of R. ∎

Theorem 7.6. *In every locally compact connected group G of finite dimension, there exists a maximal closed connected solvable normal subgroup (called the* solvable radical *$\sqrt{G}$ of G). Of course, $G = \sqrt{G}$ iff G is solvable, and $\sqrt{G}$ is non-trivial if G is not semi-simple. The factor group $G/\sqrt{G}$ is semi-simple (or trivial).*

Proof. Obviously, the radical is generated by the union of all closed connected normal solvable subgroups, cf. [26, Th. 15]. ∎

If G is a connected *linear* Lie group, or a *simply connected* Lie group, it is known [17, XVIII.4], [45, 3.18.13] that there exists a *closed* subgroup S of G such that $G = S\sqrt{G}$ and $\dim(S \cap \sqrt{G}) = 0$. Such a (necessarily semi-simple) subgroup is called a *Levi-complement* in G. Even for Lie groups, however, such an S does not exist in general (see [45, Ch. 3, Ex. 47] for an example). Apart from the fact that, in the Lie case, one has at least an *analytic* (possibly non-closed) Levi complement [45, 3.18.13], one also has some information about the general case:

Theorem 7.7.

(a) *Let L be a Lie group, and let S be a semi-simple analytic subgroup of L. Then the closure of S in L is an almost direct product of S and an abelian closed connected subgroup of L.*

(b) *Let G be a locally compact connected group of finite dimension, and let $\sqrt{G}$ be the solvable radical of G. Then there exists a closed subgroup H of G such that $G = H\sqrt{G}$ and $(H \cap \sqrt{G})^{\mathbf{1}} \le \mathrm{C}_G H$.*

Proof. Without loss, we may assume that S is dense in L. The adjoint action of S on the Lie algebra $\mathfrak{l}$ of L is completely reducible, hence there exists a complement $\mathfrak{c}$ of the Lie algebra $\mathfrak{s}$ of S such that $[\mathfrak{s},\mathfrak{c}] \le \mathfrak{c} \cap [\mathfrak{l},\mathfrak{l}]$. According to [17, XVI.2.1], we have that $[\mathfrak{l},\mathfrak{l}] = [\mathfrak{s},\mathfrak{s}] = \mathfrak{s}$. This implies that $[\mathfrak{s},\mathfrak{c}] = 0$, and assertion (a) follows.

If G is a Lie group, then assertion (b) can be obtained from (a). In fact, the closure of the Levi complement S of G is of the form SC, where C is a closed connected subgroup of $\mathrm{C}_G S$, and the connected component of $SC \cap \sqrt{G}$ is contained in the radical C of SC. Applying 4.5, we obtain assertion (b) in general. ∎

Remark 7.8. If G is an algebraic group, then the decomposition in 7.7(b) is the so-called algebraic Levi decomposition into an almost semi-direct product of a reductive group and the unipotent radical, see [35, Ch. 6].

8. On the existence of non-Lie groups of finite dimension

In this section, we construct some examples of non-Lie groups, and solve the problem whether or not a given simple Lie group is the quotient of some almost simple non-Lie group.

Lemma 8.1. *Assume that $(I,>)$ is a directed set, and let $(\pi_{ij}\colon G_i \to G_j)_{i>j}$ be a projective system of locally compact groups. If π_{ij} has compact kernel for all i,j such that $i>j$, then the projective limit is a locally compact group.*

Proof. Let G be the projective limit. For every $i \in I$, the natural mapping $\pi_i\colon G \to G_i$ has compact kernel, since this kernel is the projective limit of compact groups. Hence π_i is a proper mapping by 2.8, and the preimage of a compact neighborhood in G_i is a compact neighborhood in G. ∎

Lemma 8.2. *Assume that $(I,>)$ is a directed set, and let $(\pi_{ij}\colon G_i \to G_j)_{i>j}$ be a projective system of locally compact groups such that $\ker \pi_{ij}$ is finite for all $i>j$. Let G denote the projective limit. If I has a smallest element, then every projection $\pi_i\colon G \to G_i$ has compact totally disconnected kernel.*

Proof. Assume that a is the smallest element of I. For every $i \in I$, let K_i denote the kernel of π_{ia}. The kernel of π_a is the projective limit K of the system $\left(\pi_{ij}|_{K_i}^{K_j}\colon K_i \to K_j\right)_{i>j}$. Since $\ker \pi_i \le \ker \pi_a$, the assertion follows from the fact that K is a closed subgroup of the compact totally disconnected group $\prod_{i\in I} K_i$. ∎

Theorem 8.3. *Let G be a locally compact connected group of finite dimension.*

(a) *If G is not a Lie group, and N is a* compact *totally disconnected normal subgroup such that G/N is a Lie group, then there exists an infinite sequence $\pi_n\colon L_{n+1} \to L_n$ of c_n-fold coverings of connected Lie groups such that $L_0 = G/N$ and $1 < c_n < \infty$ for every n, and G is the projective limit of the system $(L_n)_{n\in\mathbb{N}}$.*

(b) *Conversely, let L be a connected Lie group, and let $\pi_n\colon L_{n+1} \to L_n$ be an infinite sequence of c_n-fold coverings of connected Lie groups such that $L_0 = L$ and $1 < c_n < \infty$ for every n. Then there exists a locally compact connected non-Lie group G with a compact totally disconnected normal subgroup N such that $G/N \cong L$.*

Proof. Assertion (a) is an immediate consequence of 3.3, recall that a covering of a Lie group is a Lie group again.

In the situation of (b), consider the projective system $\pi_n\colon L_{n+1} \to L_n$. By 8.1, the limit is a locally compact group G. The projective limit N of the kernels of the natural mappings $G \to G_i$ is a compact infinite group, and N is totally disconnected by 8.2. Hence G is not a Lie group. ∎

Remarks 8.4.

(a) Our technical assumption in 8.2 that I has a smallest element seems to be adequate for the application in 8.3(b). The example [22, bottom of page 260] of an infinite-dimensional projective limit of a system of one-dimensional groups shows that 8.2 does not hold without some assumption of the sort.

(b) Theorem 8.3(b) could also be derived from [22, 2.2, 3.3]. Roughly speaking, the method of K.H. HOFMANN, T.S. WU and J.S. YANG [22] consists of a dimension-preserving compactification of the center of a given group.

8.5. The fundamental group of a semi-simple compact Lie group is finite; see, e.g., [45, Th. 4.11.6]. This implies that, for a connected Lie group L, the existence of a sequence of coverings as in Theorem 8.3 is equivalent to the existence of a central torus in a maximal compact subgroup of L. The *simple* Lie groups with this property are sometimes called *hermitian groups*, they give rise to non-compact irreducible hermitian symmetric spaces [14, VIII.6.1]. In the terminology of [44], the corresponding simple Lie algebras are the real forms $\mathrm{A}_l^{\mathbb{C},p}$ $(1 \leq p \leq \frac{l+1}{2})$, $\mathrm{B}_l^{\mathbb{R},2}$ $(l \geq 2)$, $\mathrm{C}_l^{\mathbb{R}}$ $(l \geq 3)$, $\mathrm{D}_l^{\mathbb{R},2}$ $(l \geq 4)$, $\mathrm{D}_{2p}^{\mathbb{H}}$ $(p \geq 3)$, $\mathrm{D}_{2p+1}^{\mathbb{H}}$ $(p \geq 2)$, $\mathrm{E}_{6(-14)}$, $\mathrm{E}_{7(-25)}$. In [35], these algebras are denoted as $\mathfrak{su}_{p,l+1-p}$ (including $\mathfrak{su}_{p,p}$), $\mathfrak{so}_{2,l-1}$, $\mathfrak{sp}_{2l}(\mathbb{R})$, $\mathfrak{so}_{2,l-2}$, $\mathfrak{u}_{2p}^*(\mathbb{H})$, $\mathfrak{u}_{2p+1}^*(\mathbb{H})$, $EIII$, and $EVII$, respectively.

Consequently, we know the locally compact almost simple non-Lie groups.

Theorem 8.6. *Let G be a locally compact connected almost simple group. Then G is not a Lie group if, and only if, the center Z of G is totally disconnected but not discrete. In this case, the factor group G/Z is a hermitian group (cf.* **8.5**), *and G is the projective limit of a sequence of finite coverings of G/Z.*

Of course, a similar result holds for semi-simple non-Lie groups: at least one of the almost simple factors in 7.1 is not a Lie group.

9. Arcwise connected subgroups of locally compact groups

In the theory of Lie groups, arcwise connectedness plays an important rôle. In fact, according to a theorem of H. YAMABE [12], the arcwise connected subgroups of a Lie group are in one-to-one correspondence with the subalgebras of the corresponding Lie algebra. Our aim in this section is to extend this to the case of locally compact groups of finite dimension. To this end, we shall refine the topology of the arc component, and show that we obtain a Lie group topology.

Definition 9.1. Let G be a topological group, and let $\mathcal{U}$ be a neighborhood base at $\mathbb{1}$. For $W \in \mathcal{U}$, let $\mathcal{U}_W = \{U \in \mathcal{U};\, U \subseteq W\}$, of course $\mathcal{U}_W$ is again a neighborhood base at $\mathbb{1}$. For every $U \in \mathcal{U}$, we denote by U^{arc} the arc component of $\mathbb{1}$ in U. For $W \in \mathcal{U}$, let $\mathcal{U}_W^{\mathrm{arc}} = \{U^{\mathrm{arc}};\, U \in \mathcal{U}_W\}$.

Easy verification shows that the system $\{Vg;\, V \in \mathcal{U}_W^{\mathrm{arc}}, g \in G\}$ forms a base for a group topology on G. For every $W \in \mathcal{U}$, we obtain the same group topology on G, this topology shall be denoted by $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$. The following proposition clearly implies that the topology $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$ is locally arcwise connected.

Proposition 9.2.

(a) *The topology $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$ is finer than the original topology on G.*

(b) *A function $\alpha\colon [0,1] \to G$ is continuous with respect to the original topology if, and only if, it is continuous with respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$.*

Proof. For $U \in \mathcal{U}$ and $g \in U$, we find $V \in \mathcal{U}$ such that $Vg \subseteq U$. Now $V^{\mathrm{arc}}g \subseteq U$, and we infer that $U \in \mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$. The 'if'-part of assertion (b) follows immediately from (a). So assume that α is continuous with respect to the original topology, let $r \in [0,1]$ and $U \in \mathcal{U}$. By continuity, there is a connected neighborhood I of r such that $I^\alpha \subseteq Ur^\alpha$. Now continuity of α with respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$ follows from the fact that $I^\alpha \subseteq U^{\mathrm{arc}} r^\alpha$. ∎

Corollary 9.3.

(a) *With respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$, the arc component is again arcwise connected. Thus the arc component G^{arc} of G coincides with the arc component of G with respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$.*

(b) *Algebraically, G^{arc} is generated by U^{arc} for every $U \in \mathcal{U}$.*

While G^{arc} is understood to be endowed with the induced original topology, we shall write $G^{\mathrm{loc\,arc}}$ for the topological group G^{arc} with the topology induced from $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$. According to 9.2(a), the inclusion $G^{\mathrm{arc}} \to G$ yields a continuous injection $\iota\colon G^{\mathrm{loc\,arc}} \to G$.

Theorem 9.4. *Assume that G is a locally compact group of finite dimension, and let $\mathcal{U}$ be a neighborhood base at $\mathbb{1}$. Then the following hold:*

(a) *If $W \in \mathcal{U}$ is the direct product of a compact totally disconnected normal subgroup C of G and some local Lie group $\Lambda \subseteq G$, then G^{arc} is algebraically generated by the connected component $\Lambda^{\mathbf{1}}$. In particular, $G^{\mathbf{1}} \le G^{\mathrm{arc}}C$.*

(b) *The factor group G/C is a Lie group, in fact, the natural mapping $\pi\colon G \to G/C$ restricts to a topological isomorphism of Λ onto a neighborhood of $\mathbb{1}$ in G/C.*

(c) *$G^{\mathrm{loc\,arc}}$ is a connected Lie group, and $\iota\pi\colon G^{\mathrm{loc\,arc}} \to (G/C)^{\mathbf{1}}$ is a covering.*

(d) *The arc component G^{arc} is dense in $G^{\mathbf{1}}$.*

(e) *The sets $\mathrm{Hom}(\mathbb{R}, G)$ and $\mathrm{Hom}(\mathbb{R}, G^{\mathrm{arc}})$ coincide. The mapping*

$$\alpha \mapsto \alpha\iota\colon \mathrm{Hom}(\mathbb{R}, G^{\mathrm{loc\,arc}}) \to \mathrm{Hom}(\mathbb{R}, G)$$

is a bijection.

Proof. For every $U \in \mathcal{U}$, the connected component $G^{\mathbf{1}}$ is contained in the subgroup $\langle U \rangle$ that is algebraically generated by U. In particular, $G^{\mathbf{1}} \le \langle W \rangle = C\langle \Lambda \rangle$; recall that C is a normal subgroup of G. The connected component $\Lambda^{\mathbf{1}}$ is arcwise connected, therefore $\Lambda^{\mathbf{1}} = W^{\mathrm{arc}}$. This implies that $\Lambda^{\mathbf{1}}$ is open in $G^{\mathrm{loc\,arc}}$, whence $G^{\mathrm{loc\,arc}} = \langle \Lambda^{\mathbf{1}} \rangle$. This proves assertion (a). From the fact that W is the direct product of C and Λ, we conclude that $\pi|_\Lambda$ is injective. The quotient mapping π is open, hence $\Lambda^\pi = W^\pi$ is open in G/C. Therefore, the group G/C is locally isomorphic to Λ, and (b) is proved. Since $V := (\Lambda^{\mathbf{1}})^\pi$ is open in G/C, we obtain that $(G/C)^{\mathbf{1}}$ is generated by V. Hence $\iota\pi\colon G^{\mathrm{loc\,arc}} \to (G/C)^{\mathbf{1}}$ is surjective, and assertion (c) holds. An application of 4.5(b) to the closure of G^{arc} and the restriction of π to $G^{\mathbf{1}}$ yields assertion (d). Finally, assertion (e) is an immediate consequence of 9.2(b). ∎

Remarks 9.5.

(a) From K. IWASAWA's local product theorem [11, Th. B] we know that in every locally compact group there exists a neighborhood W with the properties that are required in 9.4(a).

(b) In view of 9.4(e), we define the *Lie algebra* of G as $\mathrm{Hom}(\mathbb{R}, G)$, cf. [29, II.11.9, p. 140]. We then have the exponential mapping

$$\exp\colon \mathrm{Hom}(\mathbb{R}, G) \to G\colon \alpha \mapsto 1^\alpha .$$

For every subalgebra $\mathfrak{s}$ of $\mathrm{Hom}(\mathbb{R}, G)$, it seems reasonable to define the corresponding *arcwise connected* subgroup that is generated by $\exp \mathfrak{s}$. This is in contrast with R. LASHOF's definition [30, 4.20], while our definition of the Lie algebra essentially amounts to the same as R. LASHOF's.

(c) A source for further information on G might be the epimorphism

$$\eta\colon G^{\mathrm{loc\,arc}} \times C \to G = G^{\mathrm{arc}} C\colon (x, c) \mapsto x^\iota c.$$

Note that η is a local isomorphism, and therefore a quotient mapping.

(d) As an immediate consequence of the local product theorem, we have that a locally compact group of finite dimension is a Lie group if, and only if, it is locally connected. However, it is not clear *a priori* that $G^{\mathrm{loc\,arc}}$ is locally compact.

We collect some consequences of 9.4.

Theorem 9.6. *Let G be a locally compact connected group of finite dimension.*

(a) *Let M be a compact normal subgroup such that* $\dim M = 0$ *and G/M is a Lie group. For the natural mapping $\pi_M\colon G \to G/M$, we have that $\iota\pi_M\colon G^{\mathrm{loc\,arc}} \to G/M$ is a covering. In particular,* $\dim G = \dim G/M = \dim G^{\mathrm{loc\,arc}}$.

(b) *The group G is a Lie group if, and only if, the composite $\iota\pi_M$ is a finite covering.*

(c) *If H is a connected Lie group, and $\alpha\colon H \to G$ is a continuous homomorphism, then α factors through ι.*

(d) *The group G is a Lie group if, and only if, the morphism ι is surjective.*

Proof. The kernel $K = G^{\mathrm{arc}} \cap M$ of $\iota\pi$ is closed in $G^{\mathrm{loc\,arc}}$ and totally disconnected. Since $G^{\mathrm{loc\,arc}}$ is a Lie group, we infer that K is discrete. Since

$G^{\mathrm{loc\,arc}}$ and G/M are connected Lie groups of the same dimension, we conclude that $\iota\pi$ is surjective, hence assertion (a) holds. If G is a Lie group, then $G = G^{\mathrm{arc}} = G^{\mathrm{loc\,arc}}$. Being totally disconnected, the subgroup M is discrete and compact, hence finite. Thus π_M is a finite covering. Now assume that $\iota\pi_M$ has finite kernel $K = G^{\mathrm{arc}} \cap M$. Let U be a neighborhood of $\mathbb{1}$ in G such that $U \cap K = \{\mathbb{1}\}$. According to 2.1(b), there exists an normal totally disconnected compact subgroup N such that $N \subseteq U$ and G/N is a Lie group. For every such N, we obtain that $\iota\pi_N$ is an isomorphism. If N is non-trivial, we may pass to a neighborhood V of $\mathbb{1}$ in U such that N is not contained in V. Then we find a normal compact subgroup $N' \subseteq N \cap V$, and obtain a proper covering $G/N' \to G/N$, in contradiction to the fact that $\iota\pi_{N'}$ is an isomorphism. This implies that $N = \{\mathbb{1}\}$, and G is a Lie group. Thus assertion (b) is proved. In the situation of (c), it suffices to show that α is continuous with respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$; in fact H^α is arcwise connected, hence contained in G^{arc}. For every $U \in \mathcal{U}$, we find a neighborhood V of $\mathbb{1}$ in H such that $V^\alpha \subseteq U$. Since H is locally arcwise connected, we may assume that V is arcwise connected. This implies that $V^\alpha \subseteq U^{\mathrm{arc}}$, whence α is continuous with respect to $\mathcal{T}_{\mathcal{U}^{\mathrm{loc\,arc}}}$. In order to prove (d), assume first that ι is surjective. Then ι is a homeomorphism by the open mapping theorem [15, 5.29], hence G is a Lie group. The proof of (d) is completed by the observation that every connected Lie group is arcwise connected. ∎

We remark that 9.6(d) is a result of M. GOTO, see [12].

10. Algebraic groups

In this last section, we briefly indicate how certain results from the theory of complex algebraic groups yield results on the rough structure of locally compact groups of finite dimension.

Let G be a locally compact group. If $\dim G < \infty$, and $A, B \in \mathbf{Struc}(G)$ such that $A < B$, then $\dim A < \dim B$ by 1.8(b). Consequently, every chain in $\mathbf{Struc}(G)$ has a maximal and a minimal element. This corresponds to the fact that analytic (arcwise connected) subgroups of a Lie group are in one-to-one correspondence to the subalgebras of the Lie algebra, where the dimension function is obviously injective on every chain. Upper bounds for the dimension of subgroups of a given locally compact group G yield lower bounds for the dimension of separable metric spaces that admit a non-trivial action of G, cf. 1.14. In order to gain information about the maximal elements in $\mathbf{Struc}(G)$, we shall try to employ information from the theory of algebraic groups. The maximal *algebraic* subgroups of a complex algebraic group are understood quite well. E.g., one has the following result, cf. [24, 30.4].

Theorem 10.1. *Let G be a reductive complex algebraic group. Then every maximal algebraic subgroup of G either is parabolic or has reductive Zariski-component.*

Parabolic subgroups are those that contain a Borel subgroup. Every parabolic subgroup is a conjugate of a so called standard parabolic subgroup, and these are easy to describe. In fact, they are in one-to-one correspondence to the subsets of a base for the lattice of roots of G relative to a maximal torus. Cf. [24, 30.1].

The reductive subgroups of reductive complex algebraic groups have been determined: See, e.g., [4].

There arises the question as to what extent these results are applicable in order to describe the maximal closed subgroups of a given locally compact group, or even a Lie group. First of all, we note that an important class of Lie groups consists in fact of algebraic groups, cf. [35, Ch. 3, Th. 5].

Theorem 10.2. *Let G be a connected complex linear Lie group, and assume that G equals its commutator group. Then G admits a unique complex algebraic structure. In particular, every complex semi-simple linear Lie group is complex algebraic.*

While every algebraic subgroup of a complex algebraic group G is closed in the Lie topology, the converse does not hold in general. However, the structure of the *algebraic closure* H^{alg} of a connected analytic subgroup H of G (i.e., the smallest algebraic subgroup that contains H) is to some extent controlled by the structure of H. In particular, the commutator group of H^{alg} equals that of H, cf. [18, VIII.3.1]. This implies the following.

Theorem 10.3. *Let G be a complex semi-simple linear Lie group. Then every maximal closed connected subgroup is algebraic.*

Via complexification, we obtain an estimate for the possible dimensions of maximal closed subgroups of *real* semi-simple Lie groups (and thus of locally compact semi-simple groups).

Theorem 10.4. *Let G be a semi-simple* (*real*) *Lie group. If H is a proper subgroup, then $\dim H \le m_G$, where m_G denotes the maximal* (*complex*) *dimension of proper subgroups of the complexification of G.*

Since, e.g., the parabolic subgroups have no counterpart in compact real forms, the estimate in 10.4 may be quite rough. However, it is attained in the case of split real forms.

Example 10.5. Consider a complex simple Lie group of type G_2. Then a reductive subgroup is either semi-simple of type A_2, $\mathsf{A}_1 \times \mathsf{A}_1$, A_1, or a product of A_1 with a one-dimensional centralizer, or abelian of dimension at most two. The maximal parabolic subgroups are semi-direct products of a Levi factor of type A_1 and a solvable radical of dimension 6. Consequently, if G is a locally compact almost simple group such that the factor group modulo the center is a real form of G_2, then the maximal elements in $\mathbf{Struc}(G)$ have dimension at most 9. Note that, if G is the compact real form, then every subgroup is reductive, and the maximal elements in $\mathbf{Struc}(G)$ have dimension at most 8. Since $\dim G = 14$, we infer that if G acts non-trivially on a separable metric space X, then $\dim X \ge 5$, and $\dim X \ge 6$ if G is compact.

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