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The classification of Lie algebras with invariant cones

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Introduction

We say that a finite dimensional real Lie algebra $\mathfrak{g}$ contains invariant cones if there exists a pointed generating invariant closed convex cone $W \subseteq \mathfrak{g}$ which is invariant under the group of inner automorphisms. In this paper we obtain a classification of all Lie algebras which contain invariant cones.

This classification is the final stage of a development started in [3] and ranging over [4], [5], [6], [22] and [12]. The work of Hilgert and Hofmann on invariant cones in general Lie algebras which is completely documented in Chapter III of [6] provided the basic structure theory for Lie algebras with invariant cones.

We say that a subalgebra $\mathfrak{a}$ of the Lie algebra $\mathfrak{g}$ is *compactly embedded* if the closure of the group generated by $e^{\mathrm{ad}\,\mathfrak{a}}$ is compact. One central fact in [6] is that a Lie algebra with invariant cones always contains a compactly embedded Cartan algebra $\mathfrak{t}$. Associated to such a Cartan algebra $\mathfrak{t}$ is a uniquely determined maximal compactly embedded subalgebra $\mathfrak{k}$ containing $\mathfrak{t}$ ([6, A.2.40]). It was shown in [6] that a necessary condition for a Lie algebra $\mathfrak{g}$ to contain invariant cones is that $\mathfrak{g}$ is *quasihermitean*, i.e., that the centralizer of the center $\mathfrak{z}(\mathfrak{k})$ of $\mathfrak{k}$ is not bigger than $\mathfrak{k}$, and that $\mathfrak{g}$ has *cone potential*, a property which can be formulated in terms of the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{t}$. The work of Spindler [22] contains a universal construction of a class of Lie algebras containing all those with cone potential. On the basis of this construction and the results from [6] we have given in [12] a characterization of the Lie algebras with invariant cones as those with the following properties:

(1) $\mathfrak{g}$ is not compact semisimple,

(2) all simple ideals contained in $\mathfrak{g}$ are either compact or hermitean, and

(3) $\mathfrak{g}$ has *strong cone potential*, another property which can be defined in terms of the root decomposition and which implies cone potential.

Unfortunately this characterization does not give an explicit description of all Lie algebras with invariant cones. Every finite dimensional Lie algebra $\mathfrak{g}$

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with Levi decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ decomposes into a direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{s}_1$, where $\mathfrak{s}_1$ is a maximal semisimple ideal of $\mathfrak{g}$ and the Levi complements of $\mathfrak{g}_1$ act effectively on the radical of $\mathfrak{g}$. If $\mathfrak{g}$ contains invariant cones, then for $\mathfrak{g}_1$ the results in [12] provide a homomorphism $q\colon \mathfrak{g} \to \mathfrak{h}_n \rtimes \mathfrak{sp}(n, \mathbb{R})$ with central kernel, where $\mathfrak{h}_n$ is the $(2n+1)$-dimensional Heisenberg algebra, such that q maps the nilradical $\mathfrak{n}$ onto $\mathfrak{h}_n$ and $\mathfrak{g}$ contains a reductive subalgebra $\mathfrak{l}$ with $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$ such that $\mathfrak{l}$ is mapped into $\mathfrak{sp}(n, \mathbb{R})$. In this paper we obtain an explicit description all Lie algebras with invariant cones which rests on these ideas.

The key ingredient in our classification is an observation relating the results in [14] to the classification problem. To state it, let $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$ be a semidirect decomposition, where $\mathfrak{n}$ is the nilradical and $\mathfrak{l}$ is reductive. Then $V := [\mathfrak{l}, \mathfrak{n}]$ is an $\mathfrak{l}$-module which carries several invariant symplectic structures. The main point is that there exists a symplectic structure Ω on V such that the cone $W_V := \{X \in \mathfrak{g} \colon (\forall v \in V)\Omega(X.v, v) \geq 0\}$ has interior points.

In general we say that a symplectic vector space (V, Ω) on which a reductive Lie algebra $\mathfrak{g}$ acts via a homomorphism $\mathfrak{g} \to \mathfrak{sp}(V, \Omega)$ is of *convex type* if

(1) the cone W_V has interior points, and

(2) the center acts by semisimple mappings with purely imaginary eigenvalues.

The first three sections of this paper consist of a classification of all symplectic modules of convex type. After some reductions one can use the closely related results of contraction representations from [14] which provide an explicit classification in the case where $\mathfrak{g}$ is simple hermitean and V is complex simple. In Section IV we also explain how these representations are related to strongly equivariant embeddings of bounded hermitean domains into the Siegel space which in turn can be characterized by so called (H_1)-homomorphisms on the Lie algebra level (cf. [21], [20]).

In Section V we use these results to obtain a classification of all Lie algebras which contain invariant cones by constructing them from symplectic modules of convex type of reductive quasihermitean Lie algebras. The insights obtained by the point of view of Section III provide also new information on the structure of these Lie algebras which play a crucial role in the theory of highest weight representations of the associated groups and semigroups (cf. [15], [16]) and also in the complex geometry of certain coadjoint orbits which carry Kähler structures (cf. [13]).

The subsection on the relations between Lie algebras with invariant cones and parabolic subalgebras of hermitean Lie algebras is based on an observation of S. Sahi. We thank him for many illuminating discussions on the subject.

We also express our deep gratitude to the referee who made a lot of very valuable suggestions to improve the exposition of the paper and to clarify the arguments. The whole first section of the present version of this paper is a reaction on his comments. It contains some rather general result on semisimple symplectic representations of reductive Lie algebras without the convex type condition.

I. Symplectic representation theory

In this section we consider modules of Lie algebras which carry an invariant symplectic form. We are mainly interested in the special class of symplectic modules of convex type which will be studied in Section II. Our strategy is first to obtain some general results on the structure of symplectic modules that can be specialized to obtain sharper results for modules of convex type. In the following $\mathfrak{g}$ always denotes a finite dimensional real Lie algebra.

Definition I.1. (a) A *symplectic* $\mathfrak{g}$*-module* is a symplectic vector space (V,Ω) on which $\mathfrak{g}$ acts such that the symplectic form Ω is invariant, i.e.,

$$\Omega(X.v,w)+\Omega(v,X.w)=0$$

holds for all $v,w\in V$ and $X\in\mathfrak{g}$. An *isomorphism of symplectic* $\mathfrak{g}$*-modules* (V,Ω) and (V',Ω') is a $\mathfrak{g}$-module isomorphism $\varphi\colon V\to V'$ with

$$\Omega'\big(\varphi(v),\varphi(w)\big)=\Omega(v,w)$$

for all $v,w\in V$.
(b) If (V_j,Ω_j), $j=1,\ldots,n$ are symplectic $\mathfrak{g}$-modules, then the direct sum $V:=\bigoplus_{j=1}^n V_j$ is a symplectic $\mathfrak{g}$-module with respect to

$$\Omega\Big(\sum_{j=1}^n v_j,\sum_{k=1}^n w_k\Big):=\sum_{j=1}^n\Omega(v_j,w_j).$$

(c) A submodule $W\subseteq V$ is called *isotropic* if it is an isotropic subspace of (V,Ω) and *non-degenerate* if the restriction Ω_W of Ω to W is non-degenerate.
(d) We say that a symplectic $\mathfrak{g}$-module (V,Ω) is *indecomposable* if it cannot be written as an orthogonal direct sum of two submodules different from $\{0\}$ and V. ■

Remark I.2. In the following we only consider semisimple modules. We recall that if a Lie algebra $\mathfrak{g}$ has a faithful representation $\rho\colon\mathfrak{g}\to\mathfrak{gl}(V)$ such that V is a semisimple $\mathfrak{g}$-module, then $\mathfrak{g}$ is reductive (cf. [1, Ch. I, §6, No.4]). Therefore it is no loss of generality to assume that the Lie algebra under consideration is reductive. ■

Our first objective is to show that a semisimple symplectic $\mathfrak{g}$-module always decomposes as an orthogonal direct sum of indecomposable summands which are uniquely determined up to isomorphy, and to obtain a description of the indecomposable semisimple modules.

Lemma I.3. *Let (V,Ω) be a symplectic $\mathfrak{g}$-module, $W\subseteq V$ a simple submodule and Ω_W the restriction of Ω to W. Then either W is isotropic or non-degenerate.*

Proof. Let $W_0:=W^\perp\cap W=\{w\in W\colon\Omega(w,W)=\{0\}\}$. Then the invariance of Ω implies that W_0 is a submodule of W. Hence $W_0=\{0\}$ or $W_0=W$ because W is simple. In the first case W is non-degenerate and in the second case W is isotropic. ■

Definition I.4. A simple $\mathfrak{g}$-module W is said to be *of symplectic type* if it carries a $\mathfrak{g}$-invariant symplectic form. ■

The following propositions collects some of the basic properties of symplectic $\mathfrak{g}$-modules.

Proposition I.5. (i) *If W is a $\mathfrak{g}$-module, then*

$$(W \oplus W^*, \Omega_W^*) \quad \textit{with} \quad \Omega_W^*\big((w,\alpha),(w',\alpha')\big) = \alpha(w') - \alpha'(w)$$

is a symplectic $\mathfrak{g}$-module.
(ii) *If (V, Ω) is a symplectic $\mathfrak{g}$-module, $W \subseteq V$ is maximal isotropic and a $\mathfrak{g}$-submodule, then there exists an isomorphism of symplectic $\mathfrak{g}$-modules*

$$\varphi\colon (V, \Omega) \to (W \oplus W^*, \Omega_W^*)$$

with $\varphi|_W = \mathrm{id}_W$ if and only if W has a module complement.
(iii) *If (W, Ω) is a symplectic $\mathfrak{g}$-module, then*

$$(W \oplus W^*, \Omega_W^*) \cong (W, \Omega) \oplus (W, -\Omega).$$

(iv) *If W is a simple $\mathfrak{g}$-module, then either W is of symplectic type and $(W \oplus W^*, \Omega_W^*)$ decomposes into an orthogonal direct sum, or $(W \oplus W^*, \Omega_W^*)$ is indecomposable.*

Proof. (i) That $W \oplus W^*$ is in fact a symplectic $\mathfrak{g}$-module follows from

$$\begin{aligned}\Omega_W^*\big(X.(w,\alpha),(w',\alpha')\big) &= (X.\alpha)(w') - \alpha'(X.w) \\ &= -\alpha(X.w') + (X.\alpha')(w) \\ &= -\Omega_W^*\big((w,\alpha), X.(w',\alpha')\big).\end{aligned}$$

(ii) If φ exists, then $\varphi(W^*) \subseteq V$ is a module complement for W.

Suppose, conversely, that $U \subseteq V$ is a module complement of W. Since W is maximal isotropic, we have $W^\perp = W$ and therefore

$$0 = V^\perp = W^\perp \cap U^\perp = W \cap U^\perp.$$

On the other hand $\dim W = \frac{1}{2}\dim V$ yields $\dim U = \dim U^\perp$, so that we also have the direct module decomposition $V = W \oplus U^\perp$. Let

$$U \to V = W \oplus U^\perp, \quad u \mapsto \big(\gamma(u), \delta(u)\big)$$

denote the corresponding embedding of U. We put

$$U' := \{u - \tfrac{1}{2}\gamma(u) = \big(\tfrac{1}{2}\gamma(u), \delta(u)\big)\colon u \in U\}.$$

Since γ is a homomorphism of $\mathfrak{g}$-modules, U' is a submodule of V. Moreover, $U' \cap W = \{0\}$ because $\delta(u) = 0$ implies $u \in W \cap U = \{0\}$. We conclude that U' is a module complement for W.

We claim that U' is isotropic. In fact, since W is isotropic, we have

$$\begin{aligned}\Omega\big(u-\tfrac{1}{2}\gamma(u),u'-\tfrac{1}{2}\gamma(u')\big) &= \Omega(u,u')-\tfrac{1}{2}\Omega\big(\gamma(u),u'\big)-\tfrac{1}{2}\Omega\big(u,\gamma(u')\big)\\ &= \Omega(u,u')-\tfrac{1}{2}\Omega(u,u')-\tfrac{1}{2}\Omega(u,u')=0.\end{aligned}$$

Now we define $\varphi_1: U' \to W^*$ by $\varphi_1(u)(w) = \Omega(u,w)$ and observe that since W is maximal isotropic, φ_1 is an isomorphism of $\mathfrak{g}$-modules. Hence

$$\varphi: V = W \oplus U' \to W \oplus W^*, \quad (w,u) \mapsto \big(w, \varphi_1(u)\big)$$

is a homomorphism of $\mathfrak{g}$-modules extending $\mathrm{id}|_W$. It remains to show that φ is an isomorphism of symplectic $\mathfrak{g}$-modules. This follows from

$$\begin{aligned}\Omega_W^*\big((w,\varphi_1(u)),(w',\varphi_1(u'))\big) &= \varphi_1(u)(w')-\varphi_1(u')(w)\\ &= \Omega(u,w')-\Omega(u',w)=\Omega(w+u,w'+u').\end{aligned}$$

(iii) Put $(V,\Omega_V) := (W,\Omega)\oplus(W,-\Omega)$. Then $\widetilde{W} := \{(w,w): w \in W\}$ and $U := \{(w,-w): w \in W\}$ are maximal isotropic submodules of V. Therefore (ii) implies that

$$(V,\Omega_V) \cong (\widetilde{W}\oplus\widetilde{W}^*, \Omega^*_{\widetilde{W}}) \cong (W\oplus W^*, \Omega^*_W).$$

(iv) Suppose that $(V,\Omega) := (W\oplus W^*, \Omega_W^*)$ is decomposable. We show that W permits a $\mathfrak{g}$-invariant symplectic structure. Let $V = U \oplus U^\perp$ denote a non-trivial orthogonal decomposition, where U is a non-degenerate submodule of V. Since V is a sum of two simple $\mathfrak{g}$-modules, it follows that U is simple. Since U is non-degenerate, we have $U \cap W = U \cap W^* = \{0\}$. Therefore the projection $p: U \to W$ is an isomorphism of $\mathfrak{g}$-modules and consequently W is of symplectic type.

Conversely, if W is of symplectic type, then (iii) shows that $(W \oplus W^*, \Omega_W^*)$ decomposes as asserted. This completes the proof. ∎

Lemma I.6. *Let V be a $\mathfrak{g}$-module and V^* its dual module. Then the following are equivalent:*

(i) *V carries a symplectic form Ω such that (V,Ω) is a symplectic $\mathfrak{g}$-module.*

(ii) *There exists an isomorphism of $\mathfrak{g}$-modules $\varphi: V \to V^*$ such that $\varphi^* = -\varphi$.*

Proof. To $\varphi \in \mathrm{Hom}(V,V^*)$ we associate the bilinear form $\Omega_\varphi(v,w) = \varphi(v)(w)$ on $V\times V$. The so obtained map $\mathrm{Hom}(V,V^*) \to \mathrm{Bil}(V,\mathbb{R})$ is $\mathfrak{g}$-equivariant with respect to the actions given by $(X.\varphi)(v) = X.\varphi(v) - \varphi(X.v)$ on $\mathrm{Hom}(V,V^*)$ and $(X.\Omega)(v,w) = -\Omega(X.v,w) - \Omega(v,X.w)$ on bilinear forms. Moreover, since $\varphi^*(v)(w) = \varphi(w)(v)$, the condition $\varphi^* = -\varphi$ is equivalent to the skew-symmetry of Ω_φ and φ is an isomorphism if and only if Ω_φ is non-degenerate. We conclude that a skew-symmetric isomorphism of $\mathfrak{g}$-modules $V \to V^*$ exists if and only if V carries a skew-symmetric $\mathfrak{g}$-invariant bilinear form. ∎

Lemma I.7. *Let (V,Ω) be a semisimple symplectic $\mathfrak{g}$-module and $W \subseteq V$ an isotropic simple submodule. Then there exists a simple isotropic submodule $U \subseteq V$ such that $U+W$ is non-degenerate and $U+W \cong W \oplus W^*$ as sympletic $\mathfrak{g}$-modules with respect to the restriction of Ω to $U+W$.*

Proof. Since $W^\perp \neq V$, the semisimplicity of V provides us with a module complement U satisfying $V = W^\perp \oplus U$. Then the pairing

$$W \times U \to \mathbb{R}, (w,u) \mapsto \Omega(w,u)$$

is non-degenerate and $\mathfrak{g}$-invariant so that, as a $\mathfrak{g}$-module, $U \cong W^*$. Since

$$(U+W)^\perp \cap (W+U) = U^\perp \cap W^\perp \cap (W+U) = U^\perp \cap W = \{0\},$$

the submodule $U+W$ is non-degenerate. Moreover $W \subseteq W^\perp$ implies $W \cap U = \{0\}$, hence $\dim(W+U) = 2\dim W$, so that $W \subseteq U+W$ is maximal isotropic. Now Lemma I.5(ii) implies that $U+W \cong W \oplus W^*$ as symplectic $\mathfrak{g}$-modules. We conclude in particular that U can be chosen isotropic. ∎

Proposition I.8. *Let (V,Ω) be a semisimple indecomposable symplectic $\mathfrak{g}$-module which is not simple. Then there exists a simple $\mathfrak{g}$-module W such that $(V,\Omega) \cong (W \oplus W^*, \Omega_W^*)$.*

Proof. Let $W \subseteq V$ be a non-zero submodule of minimal dimension. Then W is a simple $\mathfrak{g}$-module. If W is non-degenerate, then $V = W \oplus W^\perp$ is an orthogonal decomposition. Therefore $V = W$ which in turn contradicts the assumption that V is not simple. Hence W is isotropic and Lemma I.7 applies. So we find a simple isotropic submodule U such that $W+U$ is non-degenerate and isomorphic to $W \oplus W^*$. Since V is indecomposable, we see that $V = W+U$, and this proves the assertion. ∎

Remark I.9. We note that if W and U are isomorphic simple $\mathfrak{g}$-modules not of symplectic type, then the modules $W \oplus W^*$ and $U \oplus U^*$ are isomorphic as symplectic $\mathfrak{g}$-modules. If $W \cong U^*$ via $\varphi\colon W \to U^*$, then we also obtain a symplectic isomorphism

$$\psi\colon W \oplus W^* \to U \oplus U^*, \quad (w,\alpha) \mapsto \big((\varphi^*)^{-1}(\alpha), -\varphi(w)\big)$$

because

$$\langle -\varphi(w), (\varphi^*)^{-1}(\alpha')\rangle - \langle -\varphi(w'), (\varphi^*)^{-1}(\alpha)\rangle = -\langle w, \alpha'\rangle + \langle w', \alpha\rangle = \alpha(w') - \alpha'(w).$$

If, conversely, $W \oplus W^*$ and $U \oplus U^*$ are isomorphic as $\mathfrak{g}$-modules, then either $W \cong U$ or $W \cong U^*$. It follows in particular that the indecomposable symplectic $\mathfrak{g}$-modules $W \oplus W^*$ and $U \oplus U^*$ are isomorphic as $\mathfrak{g}$-modules if and only if $(W \oplus W^*, \Omega_W^*) \cong (U \oplus U^*, \Omega_U^*)$ as symplectic $\mathfrak{g}$-modules.

We will see in the following subsection (cf. Proposition I.16) that if (W,Ω_W) and (U,Ω_U) are simple symplectic $\mathfrak{g}$-modules, then the isomorphy of W and U as $\mathfrak{g}$-modules does not necessarily imply that the symplectic $\mathfrak{g}$-modules (W,Ω_W) and (U,Ω_U) are isomorphic. ∎

Lemma I.10. *Let (V,Ω) be a semisimple symplectic $\mathfrak{g}$-module and W, W' two indecomposable non-degenerate submodules of V that are non-isomorphic as $\mathfrak{g}$-modules. Then W and W' are mutually orthogonal.*

Proof. Let us first assume that W' is a simple module. If W is not orthogonal to W', then Ω induces a $\mathfrak{g}$-invariant pairing $W' \times W \to \mathbb{R}$ which yields an embedding $W' \to W^* \cong W$.

If W is simple, then we obtain a contradiction to our assumption. If W is not simple, then $W \cong U \oplus U^*$ (Proposition I.8), where U is a simple $\mathfrak{g}$-module which is not of symplectic type (Lemma I.5(iii)). The embedding of W' into W now shows that either $W' \cong U$ or $W' \cong U^*$, contradicting the fact that W is of symplectic type.

Finally we assume that $W' = X \oplus X^*$, where X is not symplectic (Proposition I.8). If W is not orthogonal to W', then we may w.l.o.g. assume that X is not orthogonal to U^* (otherwise we exchange X and X^*). Then the pairing between the simple modules X and U^* yields an isomorphism $X \cong (U^*)^* \cong U$. Therefore $W' = X \oplus X^* \cong U \oplus U^* \cong W$ as symplectic $\mathfrak{g}$-modules (Remark I.9). This completes the proof. ■

The first part of the following structure theorem has been suggested by the referee. We note that a complex version of the second part has also appeared in [11].

Theorem I.11. (Structure theorem for semisimple symplectic modules) *Let (V,Ω) be a semisimple symplectic $\mathfrak{g}$-module. Then the following assertions hold:*

(i) *V is an orthogonal direct sum of indecomposable symplectic $\mathfrak{g}$-modules $\bigoplus_{j=1}^m V_j$ and the indecomposable summands are unique up to permutations and isomorphy of $\mathfrak{g}$-modules.*

(ii) *A semisimple indecomposable symplectic $\mathfrak{g}$-module is either simple or isomorphic to a module of the type $(W \oplus W^*, \Omega_W^*)$.*

Proof. (i) If V is indecomposable, there is nothing to show. We proof the assertion by induction over the dimension of V. Suppose that V is not indecomposable. Then V contains a non-degenerate submodule W. Then $V \cong W \oplus W^\perp$ and we obtain the existence of the decomposition into indecomposable modules by applying induction on W and $W^\perp$.

Now let $V = \bigoplus_{j=1}^{n_0} V_j^{n_j} = \bigoplus_{k=1}^{m_0} W_k^{m_k}$ be two decompositions into indecomposable summands, where the modules $V_1, \ldots, V_{n_0}$ are pairwise non-isomorphic as $\mathfrak{g}$-modules and the same holds for the modules $W_1, \ldots, W_{m_0}$. If W_k is not isomorphic to V_1 as a $\mathfrak{g}$-module, then $W_k \perp V_1$ (Lemma I.10). Therefore there exists a uniquely determined index k_0 such that $W_{k_0} \cong V_1$ and we have a non-degenerate pairing between $W_{k_0}^{m_{k_0}}$ and $V_1^{n_1}$. Thus $m_{k_0} = n_1$ holds for dimensional reasons and we conclude from $W_{k_0}^{m_{k_0}} \subseteq \left(\sum_{j=2}^{n_0} V_j^{n_j}\right)^\perp$ that $W_{k_0}^{m_{k_0}} = V_1^{n_1}$. Now the assertion follows by induction applied to the orthogonal complement of this submodule.

(ii) This follows from Lemma I.5 and Proposition I.8. ■

Remark I.12. Let $\mathfrak{g}$ be a real Lie algebra and write $\widehat{\mathfrak{g}}$ for the set of $\mathfrak{g}$-module isomorphy classes of indecomposable finite dimensional symplectic $\mathfrak{g}$-

modules. Then the $\mathfrak{g}$-module isomorphy classes of finite dimensional symplectic $\mathfrak{g}$-modules are uniquely determined by a multiplicity function $m: \widehat{\mathfrak{g}} \to \mathbb{N}_0$ with finite support.

In general a simple symplectic $\mathfrak{g}$-module carries a lot of non-equivalent symplectic forms so that the question whether two general semisimple symplectic $\mathfrak{g}$-modules are isomorphic as $\mathfrak{g}$-modules seems to be rather involved. Actually, in view of Remark I.9, it boils down to the case where $V = W^n$ and W is simple symplectic. In the next subsection we will see how to classify the non-equivalent symplectic structures on a simple symplectic $\mathfrak{g}$-module. It will turn out that the general classification problem is much simpler for the class of symplectic modules of convex type because the convexity condition yields a coupling between the signs of the symplectic structures on the different submodules. In this case we will see that a module of the type W^n permits exactly two non-equivalent symplectic structures leading to modules of convex type (cf. Corollary II.9). ∎

Symplectic structures on simple modules

Let V be a simple real $\mathfrak{g}$-module and $\mathbb{D} := \mathrm{End}_{\mathfrak{g}}(V)$ the algebra of $\mathfrak{g}$-endomorphisms of V. Then $\mathbb{D}$ is a skew-field over the real numbers so that $\mathbb{D} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). We want to determine how many invariant symplectic structures exist on V whenever V is symplectic, i.e., there exists at least one. In the following we write $1, I, J$, and K for the basis elements of the quaternions.

Lemma I.13. *Let $\sigma: d \mapsto d^\sharp$ be an involutive antiautomorphism of $\mathbb{D}$ fixing $\mathbb{R}$. Then σ is $\mathbb{R}$-linear and the following cases occur:*

($\mathbb{R}$) $\mathbb{D} = \mathbb{R}$ *and* $\sigma = \mathrm{id}_{\mathbb{R}}$.

($\mathbb{C}_I$) $\mathbb{D} = \mathbb{C}$ *and* $\sigma(z) = \overline{z}$.

($\mathbb{C}_{II}$) $\mathbb{D} = \mathbb{C}$ *and* $\sigma = \mathrm{id}_{\mathbb{C}}$.

($\mathbb{H}_I$) $\mathbb{D} = \mathbb{H}$ *and* $\sigma(z) = \overline{z}$.

($\mathbb{H}_{II}$) $\mathbb{D} = \mathbb{H}$ *and* $\sigma(z) = a\overline{z}a^{-1}$ *with* $\overline{a} = a^{-1} = -a$.

Proof. Since σ leaves $\mathbb{R} \subseteq \mathbb{D}$ pointwise fixed, we find for $a \in \mathbb{R}$ and $d \in \mathbb{D}$ that

$$\sigma(ad) = \sigma(d)\sigma(a) = \sigma(d)a = a\sigma(d),$$

i.e., σ is $\mathbb{R}$-linear.

We distinguish several cases.

$\mathbb{D} = \mathbb{R}$: Then $\sigma = \mathrm{id}_{\mathbb{R}}$ holds trivially.

$\mathbb{D} = \mathbb{C}$: Then complex conjugation $\sigma(z) = \overline{z}$ is an involutive antiautomorphism and also $\sigma = \mathrm{id}_{\mathbb{C}}$. On the other hand $\mathbb{C}$ is abelian so that σ is a field automorphism, hence there are only these two possibilities.

$\mathbb{D} = \mathbb{H}$: (cf. [19, pp.180,181]) It is clear that conjugation $\sigma(z) = \overline{z}$ is an involutive antiautomorphism of $\mathbb{H}$. Assume that σ is different from conjugation. Since $\mathbb{H}$ is not commutative, $\sigma \neq \mathrm{id}_{\mathbb{H}}$. Moreover $-\sigma: \mathbb{H} \to \mathbb{H}$ is an isomorphism of the real Lie algebra $\mathbb{H}$ with the bracket $[x, y] := xy - yx$. Therefore it preserves the commutator algebra $\mathbb{H}' := \mathrm{span}\{I, J, K\}$ and consequently $\mathbb{H}'$ is invariant under σ. We conclude that σ commutes with conjugation, hence that $\alpha(d) := \sigma(\overline{d})$ defines an involutive automorphism of $\mathbb{H}$ leaving $\mathbb{R}$ pointwise fixed. Put $\mathbb{H}_1 := \{d \in \mathbb{H}: \alpha(d) = d\}$. Then $\mathbb{H}_1$ is a subalgebra of $\mathbb{H}$ containing $\mathbb{R}$. The requirement that $\sigma(d) \neq \overline{d}$ for at least one $d \in \mathbb{H}$ yields $\mathbb{H}_1 \neq \mathbb{H}$. Moreover $\alpha^2 = \mathrm{id}_{\mathbb{H}}$ shows that we have a direct vector space decomposition

$$\mathbb{H} = \mathbb{H}_1 \oplus \{d \in \mathbb{H}': \alpha(d) = -d\} = \mathbb{H}_1 \oplus \{d \in \mathbb{H}': \sigma(d) = d\}.$$

We conclude further from $\sigma \neq \mathrm{id}_{\mathbb{H}}$ that $\mathbb{H}_1 \neq \mathbb{R}$. Therefore $\mathbb{H}_1$ is at least two dimensional. Thus there exists $a \in \mathbb{H}_1 \cap \mathbb{H}'$ with $a^2 = -1$. Then $\mathbb{H}_1 \supseteq \mathrm{span}\{1, a\} \cong \mathbb{C}$ and in this sense $\mathbb{H}_1$ is a complex algebra. This shows that $\mathbb{H}_1 \neq \mathbb{H}$ implies that $\mathbb{H}_1 = \mathrm{span}\{1, a\}$ for dimensional reasons. Finally we conclude that $z \mapsto aza^{-1}$ is involutive, its fixed point set is $\mathbb{H}_1$, and $aza^{-1} = -z$ for z in the orthogonal complement of $\mathbb{H}_1$. Hence $\alpha(z) = aza^{-1}$ beause α has the same eigenspaces for 1 and -1, and therefore $\sigma(z) = a\overline{z}a^{-1}$ for all $z \in \mathbb{H}$.

Conversely, it is trivial that $z \mapsto a\overline{z}a^{-1}$ always defines an involutive antiautomorphism of $\mathbb{H}$. ■

According to the preceding lemma there are five different types of involutive antiautomorphism of skew-fields over $\mathbb{R}$.

Definition I.14. Let (V, Ω) be a symplectic $\mathfrak{g}$-module and $\mathbb{D} = \mathrm{End}_{\mathfrak{g}}(V)$. For $A \in \mathrm{End}_{\mathbb{R}}(V)$ define $A^\sharp \in \mathrm{End}_{\mathbb{R}}(V)$ by

$$\Omega(A.v, w) = \Omega(v, A^\sharp.w)$$

for all $v, w \in V$. Then an easy calculation shows that $\mathbb{D}^\sharp = \mathbb{D}$ so that $d \mapsto d^\sharp$ is an involutive antiautomorphism of the real algebra $\mathbb{D}$ fixing $\mathbb{R}$ pointwise. If V is simple, then $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and we say that V if of type $\mathbb{R}$, $\mathbb{C}_I$, $\mathbb{C}_{II}$, $\mathbb{H}_I$ or $\mathbb{H}_{II}$ if the involution on $\mathbb{D}$ has the corresponding type (cf. Lemma I.13). ■

In the following we write $\mathbb{D}^* := \mathbb{D} \cap \mathrm{Gl}(V)$ for the set of units in the algebra $\mathbb{D} = \mathrm{End}_{\mathfrak{g}}(V)$. Recall that $d \in \mathbb{D} \cap \mathrm{Gl}(V)$ is in fact a unit in $\mathbb{D}$ since the inverse of an intertwining operator intertwines also.

Lemma I.15. *Let (V, Ω) be a symplectic $\mathfrak{g}$-module and $\mathbb{D} = \mathrm{End}_{\mathfrak{g}}(V)$. Then the space of invariant symplectic structures on V is parametrized by $\{d \in \mathbb{D}^*: d^\sharp = d\}$ via $d \mapsto \Omega_d$ and*

$$\Omega_d(v, w) := \Omega(d.v, w).$$

Two symplectic $\mathfrak{g}$-modules (V, Ω_a) and (V, Ω_b) are isomorphic as symplectic $\mathfrak{g}$-modules if and only if there exists $d \in \mathbb{D}$ with $b = d^\sharp ad$.

Proof. Let Ω' be a $\mathfrak{g}$-invariant symplectic structure on V. Then there exists $d \in \mathrm{End}_{\mathbb{R}}(V)$ such that $\Omega'(v, w) = \Omega(d.v, w)$ for all $v, w \in \mathbb{R}$. Then $d \in \mathbb{D}$ since Ω' is invariant, $d^\sharp = d$ follows from the requirement that Ω' is skew-symmetric, and $d \in \mathrm{Gl}(V)$ from the non-degeneracy of Ω'. Conversely, it is clear that each form Ω_d, $d = d^\sharp \in \mathbb{D} \cap \mathrm{Gl}(V)$ is a $\mathfrak{g}$-invariant symplectic form on V.

If Ω and Ω' are two different invariant symplectic forms on V, then the corresponding symplectic $\mathfrak{g}$-modules are equivalent if and only if there exists $d \in \mathbb{D}$ such that $\Omega'(v, w) = \Omega(d.v, d.w)$ for all $v, w \in V$. Applying this to the forms Ω_a and Ω_b, the assertion follows. ■

Proposition I.16. *Let (V,Ω) be a simple symplectic $\mathfrak{g}$-module. Then, according to the type of V, the following assertions hold:*

- ($\mathbb{R}$) *The space of invariant symplectic forms is parametrized by $\mathbb{R}^*$ and there are two equivalence classes of invariant forms.*
- ($\mathbb{C}_I$) *The space of invariant symplectic forms is parametrized by $\mathbb{R}^*$ and there are two equivalence classes of invariant forms.*
- ($\mathbb{C}_{II}$) *The space of invariant symplectic forms is parametrized by $\mathbb{C}^*$ and all such forms are equivalent.*
- ($\mathbb{H}_I$) *The space of invariant symplectic forms is parametrized by $\mathbb{R}^*$ and there are two equivalence classes of invariant forms.*
- ($\mathbb{H}_{II}$) *The space of invariant symplectic forms is parametrized by $a^\perp \setminus \{0\}$ and all such forms are equivalent.*

Proof. We use Lemma I.15 to check the different cases. If $d^\sharp = \overline{d}$ is conjugation, then the space of invariant forms is parametrized by $\mathbb{R}^*$ and $a, b \in \mathbb{R}^*$ lead to equivalent forms Ω_a and Ω_b if and only if there exists $d \in \mathbb{D}$ such that $b = d^\sharp a d = |d|^2 a$. Therefore we have exactly two equivalence classes of forms. They are represented by 1 and -1.

In the case $\mathbb{C}_{II}$ we have $d^\sharp = d$ for all $d \in \mathbb{C}$ so that the space of invariant forms is parametrized by $\mathbb{C}^*$. The numbers a and b lead to equivalent forms if and only if there exists $d \in \mathbb{C}$ with $b = d^\sharp a d = d^2 a$. Thus we have only one equivalence class.

In the case $\mathbb{H}_{II}$, we have $a^\sharp = -a$, and all elements in $a^\perp$ are fixed by $d \mapsto d^\sharp$. Therefore the space of invariant symplectic forms is parametrized by $a^\perp \setminus \{0\}$.

Let $0 \neq x \in \mathbb{H}_0 := \{d \in \mathbb{D}\colon d^\sharp = d\}$. We claim that Ω_x is equivalent to $\Omega = \Omega_1$. Let $A \subseteq \mathbb{H}_0$ be a two dimensional subalgebra containing $\mathbb{R}$ and x. Then $A \cong \mathbb{C}$ so that we find $d \in A$ such that $d^2 = x^{-1}$. Now $d.x = dxd^\sharp = xd^2 = \mathbf{1}$. ∎

Example I.17. (a) If $V = \mathbb{R}^{2n}$ with the standard symplectic form Ω and $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R})$, then the commutant is $\mathbb{D} = \mathbb{R}$. Therefore the $\mathfrak{g}$-invariant symplectic forms are the multiples of Ω and the two different equivalence classes of such forms are represented by Ω and $-\Omega$.

(b) If $V = \mathbb{C}^n$, $\Omega(z,w) = \operatorname{Im}\sum_j z_j\overline{w}_j$, and $\mathfrak{g} = \mathfrak{u}(n)$, then (V,Ω) is a simple symplectic $\mathfrak{g}$-module and since $i\mathbf{1} \in \mathfrak{g}$, it is also simple as a real $\mathfrak{g}$-module. The commutant of $\mathfrak{g}$ is $\mathbb{D} = \mathbb{C}$ and since $(i\mathbf{1})^\sharp = -i\mathbf{1}$, this module is of type $\mathbb{C}_I$.

(c) If $V = \mathbb{C}^{2n}$ and $\Omega_{\mathbb{C}}$ is the standard complex symplectic form, we put $\Omega := \operatorname{Re}\Omega_{\mathbb{C}}$ and $\mathfrak{g} := \mathfrak{sp}(n,\mathbb{C})$. Then V is a real simple $\mathfrak{g}$-module (this follows from (a)) and the commutant of $\mathfrak{g}$ is $\mathbb{D} = \mathbb{C}$, where the involution on $\mathbb{C}$ is the identity. Therefore the space of $\mathfrak{g}$-invariant symplectic forms is $\mathbb{C}^*$ and all such forms are equivalent.

(d) If $V = \mathbb{H}^n$, $\Omega(z,w) = \operatorname{Re}\sum_j z_j J\overline{w}_j$, and $\mathfrak{g} = \mathfrak{so}^*(2n)$, then Ω is a symplectic $\mathfrak{g}$-invariant form on the real vector space V. We will see later (cf. Remark III.14) that V is a simple $\mathfrak{g}$-module over $\mathbb{R}$ and that the commutant of $\mathfrak{g}$ is $\mathbb{H}$.

For $d \in \mathbb{H}'$, $z, w \in \mathbb{H}$ we have

$$\operatorname{Re}(dzJ\overline{w}) = \operatorname{Re}(zJ\overline{w}d) = \operatorname{Re}(zJ\overline{\overline{d}w})$$

showing that $d^\sharp = \overline{d}$ for all $d \in \mathbb{D}$. Therefore V is a symplectic $\mathfrak{g}$-module of type $\mathbb{H}_I$.
(e) If $V = \mathbb{H}^n$, $\Omega_{\mathbb{H}}(z,w) = \mathrm{Im}\sum_j z_j\overline{w}_j$, and $\mathfrak{g} = \mathfrak{sp}(n)$, then $\Omega_{\mathbb{H}}$ is a skew-symmetric $\mathbb{R}$-bilinear $\mathfrak{g}$-invariant form on V. Since the group $\mathrm{Sp}(n) = \exp\big(\mathfrak{sp}(n)\big)$ acts transitively on the unit-sphere in V, the $\mathfrak{g}$-module V is simple as a real module and the commutant of $\mathfrak{g}$ is $\mathbb{H}$. Applying non-zero linear functions on $\mathbb{H}' = \mathrm{span}\{I,J,K\}$ to $\Omega_{\mathbb{H}}$, we obtain a three dimensional space of invariant symplectic forms, hence V is a symplectic $\mathfrak{g}$-module of type $\mathbb{H}_{II}$. ∎

Remark I.18. Suppose that (V,Ω) is an isotypic symplectic $\mathfrak{g}$-module, i.e., $(V,\Omega) \cong (W,\Omega_W)^m$, where W is simple, and that W is of type $\mathbb{R}$,$\mathbb{C}_I$ or $\mathbb{H}_I$. Then $\mathbb{D} = \mathrm{End}_{\mathfrak{g}}(V) \cong \mathbb{M}(m,\mathbb{K})$, where $\mathbb{K} = \mathrm{End}_{\mathfrak{g}}(W)$ is $\mathbb{R}$,$\mathbb{C}$ or $\mathbb{H}$. Since the involution $d \mapsto d^\sharp$ coincides with the natural conjugation on $\mathbb{K}$, it follows easily that the involution $d \mapsto d^\sharp$ on $\mathbb{D} \cong \mathbb{M}(n,\mathbb{K})$ coincides with the natural involution $d \mapsto d^*$ induced by the positive definite hermitean form on $\mathbb{K}^n$ given by $\langle v,w\rangle := \sum_{j=1}^n v_j\overline{w_j}$. So we can identify the set of $\mathfrak{g}$-invariant symplectic forms on V with $\mathrm{Herm}(m,\mathbb{K}) = \{d \in \mathbb{M}(m,\mathbb{K}) : d^* = d\}$, the space of $m\times m$-hermitean matrices over $\mathbb{K}$. The action of $D^* \cong \mathrm{Gl}(m,\mathbb{K})$ on this space whose orbits are the equivalence classes of $\mathfrak{g}$-invariant symplectic structures is given by $d.a = dad^*$. We recall that $\mathrm{Herm}(m,\mathbb{K})$ is a Jordan algebra of rank m (cf. [2]). In this sense the orbits of $\mathrm{Gl}(m,\mathbb{K})$ turn out to be the orbits of the structure group $G(\Omega)$, where $\Omega = \mathrm{int}\,\mathrm{Herm}^+(n,\mathbb{K})$ is the open cone of positive definite hermitean matrices which in this case coincides with the orbit of $\mathbf{1}$ under this action. Now the theory of Jordan algebras implies in particular that there are at most $m+1$ different $\mathbb{D}^*$-orbits in $\mathbb{D}^* \cap \mathrm{Herm}(n,\mathbb{K})$ (cf. [2, Ch. IV]). ∎

II. Symplectic modules of convex type

In this section we specialize the results of the preceding section to symplectic modules of convex type. Here $\mathfrak{g}$ always denotes a real reductive Lie algebra and G a simply connected Lie group with $\mathbf{L}(G) = \mathfrak{g}$.

If C is a closed convex cone in a vector space V, we write $H(C) := C \cap (-C)$ for the *edge of the cone* C and $C^\star := \{\alpha \in V^* : \alpha(C) \subseteq \mathbb{R}^+\}$ for the *dual cone*. The latter notation is also used for arbitrary subsets $C \subseteq V$.

Definition II.1. (a) If (V,Ω) is a symplectic $\mathfrak{g}$-module, then we assign to each $X \in \mathfrak{g}$ the Hamiltonian function defined by

$$\varphi(X)(v) := \tfrac{1}{2}\Omega(X.v,v).$$

Note that $\varphi(X) = 0$ is equivalent to $X.V = \{0\}$. The mapping

$$\Phi: V \to \mathfrak{g}^*, \quad v \mapsto (X \mapsto \varphi(X)(v))$$

is called the *moment map*. One checks easily that it is equivariant with respect to the actions of G on V and the coadjoint action of G on $\mathfrak{g}^*$ which is defined by $\mathrm{Ad}^*(g).\omega := \omega \circ \mathrm{Ad}(g)^{-1}$.

(b) A symplectic $\mathfrak{g}$-module (V,Ω) is said to be *of convex type* if the following two conditions are satisfied:

(i) The closed convex cone $C_V \subseteq \mathfrak{g}^*$ generated by the image $\Phi(V)$ of the moment map is *pointed*, i.e., contains no vector subspace. Note that, according to the duality of convex cones, this condition is equivalent to the condition that the cone $W_V = C_V^\star \subseteq \mathfrak{g}$ mentioned in the introduction is generating.

(ii) The center $\mathfrak{z}(\mathfrak{g})$ acts semisimply on V with purely imaginary eigenvalues. This means that under the homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{sp}(V)$, the image of $\mathfrak{z}(\mathfrak{g})$ is a compactly embedded subalgebra. ■

Lemma II.2. *Let (V,Ω) be a symplectic $\mathfrak{g}$-module of convex type. Then the following assertions hold:*

(i) *Every non-degenerate submodule of V is of convex type.*

(ii) *V is a semisimple $\mathfrak{g}$-module.*

Proof. (i) This is trivial.
(ii) We have to show that V is a sum of simple submodules. To show this, we first decompose V as a direct sum $V = \oplus_{j=1}^k V_j$ of isotypic $\mathfrak{z}(\mathfrak{g})$-modules. Then each V_j is invariant under $\mathfrak{g}$ and two cases occur. If $\mathfrak{z}(\mathfrak{g})$ acts trivially on V_j, then V_j is a sum of simple $\mathfrak{g}$-modules which are exactly the simple $[\mathfrak{g},\mathfrak{g}]$-submodules (Weyl's theorem). If $\mathfrak{z}(\mathfrak{g})$ acts non-trivially, then there exists a complex structure I on V_j and a linear functional α_j on $\mathfrak{z}(\mathfrak{g})$ with $X.v = \alpha_j(X)Iv$ for all $v \in V_j$. Then $I.v = X.v$ for every $X \in \mathfrak{z}(\mathfrak{g})$ with $\alpha_j(X) = 1$ and therefore I commutes with $\mathfrak{g}$. This shows that the simple submodules of V_j are the complex simple $[\mathfrak{g},\mathfrak{g}]$-submodules which generate V_j by Weyl's theorem. ■

In view of Lemma II.2, all the results of Section I on semisimple symplectic $\mathfrak{g}$-modules become available for symplectic $\mathfrak{g}$-modules of convex type. First we show that the indecomposable $\mathfrak{g}$-modules of type $W \oplus W^*$ cannot occur in modules of convex type.

In the following we call a simple $\mathfrak{g}$-module W *non-trivial* if $\mathfrak{g}.W \neq \{0\}$.

Lemma II.3. *Let W be a non-trivial $\mathfrak{g}$-module. Then the symplectic $\mathfrak{g}$-module $(W \oplus W^*, \Omega_W^*)$ cannot be contained in any symplectic $\mathfrak{g}$-module of convex type.*

Proof. Since non-degenerate submodules of modules of convex type are of convex type, it suffices to show that $(W \oplus W^*, \Omega_W^*)$ is not of convex type. For $(w,\alpha) \in W \oplus W^*$ and $X \in \mathfrak{g}$ we have

$$\begin{aligned}\Phi(w,\alpha)(X) &= \tfrac{1}{2}\Omega\big(X.(w,\alpha),(w,\alpha)\big) = \tfrac{1}{2}\Omega\big((X.w, X.\alpha),(w,\alpha)\big)\\ &= \tfrac{1}{2}\left((X.\alpha)(w) - \alpha(X.w)\right) = \tfrac{1}{2}\left(-\alpha(X.w) - \alpha(X.w)\right) = -\alpha(X.w).\end{aligned}$$

Replacing w by $-w$, it follows in particular that $\Phi(W \oplus W^*) = -\Phi(W \oplus W^*)$. If $W \oplus W^*$ is of convex type we therefore conclude that $\Phi(W \oplus W^*) = \{0\}$, hence that $\mathfrak{g}.W = \{0\}$, contradicting the non-triviality of W. ■

Lemma II.4. *If V is a symplectic $\mathfrak{g}$-module of convex type, then every non-trivial simple submodule W is non-degenerate.*

Proof. Let $W \subseteq V$ be a non-trivial simple submodule. We assume that W is isotropic and use Lemma I.7 to find a simple isotropic submodule $U \subseteq V$ such that $U + W \cong W \oplus W^*$ as symplectic $\mathfrak{g}$-modules. This contradicts Lemma II.3. ■

Proposition II.5. *Let V be a symplectic $\mathfrak{g}$-module of convex type,*

$$V_{\rm fix} := \{v \in V : \mathfrak{g}.v = \{0\}\}, \quad \text{and} \quad V_{\rm eff} := \operatorname{span}\{X.v : X \in \mathfrak{g}, v \in V\}.$$

Then $V = V_{\rm fix} \oplus V_{\rm eff}$ is an orthogonal direct sum of symplectic $\mathfrak{g}$-modules, $V_{\rm eff}$ is a symplectic $\mathfrak{g}$-module of convex type, and every submodule of $V_{\rm eff}$ is non-degenerate.

Proof. The fact that $V = V_{\rm fix} \oplus V_{\rm eff}$ is a direct sum of $\mathfrak{g}$-modules follows from the semisimplicity of the $\mathfrak{g}$-module V (Lemma II.2).

Let $W \subseteq V_{\rm eff}$ be a submodule. We claim that W is non-degenerate. Suppose that W is degenerate. Then $W \cap W^\perp$ is a non-zero submodule of W. Let $U \subseteq W \cap W^\perp$ be a minimal non-zero submodule. Then U is a non-trivial simple $\mathfrak{g}$-module so that Lemma II.4 implies that U is non-degenerate, contradicting $U \subseteq W \cap W^\perp \subseteq U^\perp$. This proves that W is non-degenerate.

We conclude in particular that $V_{\rm eff}$ is non-degenerate. Hence $V_{\rm eff}^\perp$ is a non-degenerate submodule of V complementary to $V_{\rm eff}$. Thus $V_{\rm eff}^\perp \subseteq V_{\rm fix}$ and by comparing dimensions we see that $V_{\rm fix} = V_{\rm eff}^\perp$. This completes the proof. ■

Corollary II.6. *Every symplectic $\mathfrak{g}$-module V of convex type decomposes into an orthogonal direct sum of $V_{\rm fix}$ and simple symplectic $\mathfrak{g}$-modules of convex type.*

Proof. This follows from Theorem I.11, Lemma II.4, and Proposition II.5. ■

Lemma II.7. *Let (V, Ω) be a simple symplectic $\mathfrak{g}$-module of convex type. Then V is of type $\mathbb{R}$, $\mathbb{C}_I$ or $\mathbb{H}_I$.*

Proof. The condition that V is of type $\mathbb{R}$, $\mathbb{C}_I$ or $\mathbb{H}_I$ is equivalent to the statement that there exists no complex structure $I \in \operatorname{End}_{\mathfrak{g}}(V)$ satisfying $I^\sharp = I$ (cf. Proposition I.16). Suppose that $I \in \operatorname{End}_{\mathfrak{g}}(V)$ with $I^\sharp = I$. Then

$$\Phi(Iv)(X) = \tfrac{1}{2}\Omega(X.Iv, Iv) = \tfrac{1}{2}\Omega(IX.v, Iv) = \tfrac{1}{2}\Omega(X.v, I^2.v) = -\Phi(v)(X)$$

so that $\Phi(Iv) = -\Phi(v)$. Therefore a non-trivial simple symplectic $\mathfrak{g}$-module of type $\mathbb{C}_{II}$ or $\mathbb{H}_{II}$ cannot be of convex type. ■

Theorem II.8. *Let (V, Ω) be a symplectic $\mathfrak{g}$-module of convex type and $\{V_1, \ldots, V_m\}$ a maximal set of pairwise non-equivalent non-trivial simple submodules. Let $\Omega_j := \Omega\,|_{V_j \times V_j}$, $\Omega_0 := \Omega\,|_{V_{\rm fix} \times V_{\rm fix}}$ and m_j denote the multiplicity of V_j in V. Then the symplectic $\mathfrak{g}$-module (V, Ω) is, as a symplectic $\mathfrak{g}$-module, equivalent the orthogonal direct sum*

$$(V_{\rm fix}, \Omega_0) \oplus \bigoplus_{j=1}^{m} (V_j, \Omega_j)^{m_j}.$$

Proof. According to Corollary II.6, the symplectic $\mathfrak{g}$-module V is the orthogonal direct sum of simple symplectic $\mathfrak{g}$-modules of convex type. Let (W, Ω_W) and (U, Ω_U) be non-trivial simple summands occuring in such a decomposition of (V, Ω). Suppose that $W \cong U$ as $\mathfrak{g}$-modules. If (U, Ω_U) is not equivalent to (W, Ω_W) as a symplectic $\mathfrak{g}$-module, then $(U, \Omega_U) \cong (W, -\Omega_W)$ (Lemma II.7, Proposition I.16). Hence $(W, \Omega_W) \oplus (W, -\Omega_W)$ is isomorphic to a submodule of (V, Ω) and this is impossible if V is of convex type. Therefore any module decomposition $V = V_{\rm fix} \oplus \bigoplus_{j=1}^m V_j^{m_j}$ which is in addition orthogonal with respect to Ω yields an isomorphism

$$(V, \Omega) \cong (V_{\rm fix}, \Omega_0) \oplus \bigoplus_{j=1}^{m} (V_j, \Omega_j)^{m_j}$$

of symplectic $\mathfrak{g}$-modules. ∎

Corollary II.9. *Two symplectic $\mathfrak{g}$-modules (V, Ω) and (V', Ω') of convex type are isomorphic as symplectic $\mathfrak{g}$-modules if and only if $V \cong V'$ as $\mathfrak{g}$-modules and $C_V = C_{V'}$.*

Proof. If (V, Ω) and (V', Ω') are isomorphic as symplectic $\mathfrak{g}$-modules, then they are trivially isomorphic as $\mathfrak{g}$-modules and the cones C_V and $C_{V'}$ coincide.

Suppose conversely that $V \cong V'$ as $\mathfrak{g}$-modules and that $C_V = C_{V'}$. Then V and V' contain the same types of irreducible submodules with the same multiplicities. First $V_{\rm fix}$ and $V'_{\rm fix}$ have the same dimension, so that they are isomorphic as symplectic vector spaces, hence as symplectic $\mathfrak{g}$-modules.

Let V_j be an irreducible submodule of V and $V'_j \subseteq V'$ an isomorphic submodule. We write Ω_j resp. Ω'_j for the induced symplectic stucture on V_j resp. V'_j. Then either $(V'_j, \Omega'_j) \cong (V_j, \Omega_j)$ or $(V'_j, \Omega'_j) \cong (V_j, -\Omega_j)$. Since $\Phi_{V'_j}(V'_j) \cup \Phi_{V_j}(V_j) \subseteq C_{V'} = C_V$ and C_V is a pointed cone, we see that the first possibility holds. Now we apply Theorem II.8 to complete the proof. ∎

The preceding result shows that for modules of convex type it is rather easy to pass from module isomorphy classes to isomorphy classes of symplectic modules. The additional information one needs is the cone C_V which determines the choices of the class of the symplectic structure on each irreducible submodule. The following remark makes this a bit more precise.

Remark II.10. Let (V, Ω) be a symplectic $\mathfrak{g}$-module of convex type. We are interested in a description of the set of all $\mathfrak{g}$-invariant symplectic structures Ω' on V such that (V, Ω') is also of convex type.

Suppose first that $V \cong V_1^{m_1}$ is isotypic as a $\mathfrak{g}$-module. According to Lemma II.7, V_1 permits exactly two classes of $\mathfrak{g}$-invariant symplectic structures represented by (V_1, Ω_1) and $(V_1, -\Omega_1)$, where Ω_1 is the restriction of Ω to V_1. Therefore Theorem II.8 implies that V permits also two classes of invariant symplectic structures Ω' such that (V, Ω') is of convex type. Note that there are much more classes of symplectic structures if we do not impose this condition (cf. Remark I.18).

We conclude with Theorem II.8 that for any $\mathfrak{g}$-invariant symplectic structure Ω' such that (V,Ω') is of convex type, we have

$$(V,\Omega') \cong (V_{\rm fix},\Omega_0) \oplus \bigoplus_{j=1}^m (V_j,\varepsilon_j\Omega_j)^{m_j},$$

where $\varepsilon_j \in \{1,-1\}$. The corresponding cone is given by

$$C'_V = \sum_{j=1}^m \varepsilon_j C_{V_j}.$$

The convex type condition for (V,Ω') means that C'_V is pointed. This yields an additional coupling of the signs $\varepsilon_1,\ldots,\varepsilon_m$.

This coupling condition is rather implicit. Anticipating Theorem III.5 and its consequences, we know that at most one simple non-compact ideal $\mathfrak{g}_j$ acts on the simple module V_j. If V_j and V_k are modules for which $\mathfrak{g}_k = \mathfrak{g}_j$, then their symplectic structures are coupled in the sense that $\varepsilon_j = \varepsilon_k$. The following example illustrates what happens if no such coupling occurs.

Let $V = \mathbb{R}^2$ be the canonical $\mathfrak{sl}(2,\mathbb{R})$-module and consider $V\oplus V$ as module of $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{R})^2$. Then $(V,\Omega)\oplus(V,\Omega)$ and $(V,\Omega)\oplus(V,-\Omega)$ are two different structure of symplectic $\mathfrak{g}$-modules of convex type on $V\oplus V$. Here all choices $(1,1),(1,-1),(-1,1)$, and $(-1,-1)$ for $\varepsilon_1,\varepsilon_2$ are possible and we obtain four classes of pairwise non-equivalent symplectic $\mathfrak{g}$-module structures of convex type on $V\oplus V$.

We believe that the aforementioned coupling condition mentioned in Remark II.10 is not sufficient and we leave it as an open problem to make the coupling condition on the signs more explicit. We will see in Section V how this problem is related to the classification of Lie algebras with invariant cones. ∎

Complexifications of modules of convex type

Lemma II.11. *Let (V,Ω) be a symplectic $\mathfrak{g}$-module, $V_{\mathbb{C}}$ its complexification, $v\mapsto\overline{v}$ complex conjugation on $V_{\mathbb{C}}$, and $\Omega_{\mathbb{C}}$ the complex bilinear extension of Ω to $V_{\mathbb{C}}$. Then the following assertions hold:*

(i) *The prescription $B(v,w) := i\Omega_{\mathbb{C}}(v,\overline{w})$ defines a pseudohermitean form on $V_{\mathbb{C}}$.*

(ii) *$\widetilde{\Omega} := \operatorname{Im} B$ is an invariant symplectic form on the real $\mathfrak{g}$-module $V_{\mathbb{C}}$ with*

$$\widetilde{\Omega}(v_1+iv_2,w_1+iw_2) = \Omega(v_1,w_1)+\Omega(v_2,w_2),$$

i.e., $(V_{\mathbb{C}},\widetilde{\Omega}) \cong (V,\Omega)\oplus(V,\Omega)$ is an orthogonal direct sum of symplectic $\mathfrak{g}$-modules.

Proof. (i) The sesquilinearity is clear. That it is pseudohermitean follows from

$$B(w,v) := i\Omega_{\mathbb{C}}(w,\overline{v}) = i\overline{\Omega_{\mathbb{C}}(\overline{w},v)} = \overline{i\Omega_{\mathbb{C}}(v,\overline{w})} = \overline{B(v,w)}.$$

(ii) For $v = v_1 + iv_2$ and $w = w_1 + iw_2$ we have

$$\begin{aligned}\widetilde{\Omega}(v,w) &= \operatorname{Im} B(v_1+iv_2, w_1+iw_2) = \operatorname{Im} i\Omega_{\mathbb{C}}(v_1+iv_2, w_1-iw_2)\\ &= \operatorname{Re}\Omega_{\mathbb{C}}(v_1+iv_2, w_1-iw_2) = \Omega(v_1,w_1)+\Omega(v_2,w_2).\end{aligned}$$

This proves (ii). ∎

Proposition II.12. *Let (V,Ω) be a symplectic $\mathfrak{g}$-module and $(V_{\mathbb{C}},\widetilde{\Omega})$ its complexification considered as a real $\mathfrak{g}$-module. Then (V,Ω) is of convex type if and only if $(V_{\mathbb{C}},\widetilde{\Omega})$ is of convex type and in this case $C_V = C_{V_{\mathbb{C}}}$.*

Proof. From Lemma II.11(ii) we infer that $\Phi_{V_{\mathbb{C}}}(v_1+iv_2) = \Phi_V(v_1)+\Phi_V(v_2)$. Therefore the sets $\Phi_V(V)$ and $\Phi_{V_{\mathbb{C}}}(V_{\mathbb{C}})$ generate the same cone in $\mathfrak{g}^*$. ∎

Example II.13. Let $V = \mathbb{R}^2$ be the standard $\mathfrak{sl}(2,\mathbb{R})$-module of dimension 2. Then the invariant symplectic form is given by $\Omega(v,w) = \det(v,w)$. Since $G = \mathrm{Sl}(2,\mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{0\}$, the image of the moment mapping is the closure of one coadjoint orbit. This shows that $\Phi(\mathbb{R}^2)$ is the closure of a nilpotent orbit in $\mathfrak{sl}(2,\mathbb{R})^*$. It follows in particular that $\mathbb{R}^2$ is a symplectic module of convex type.

Let $V_{\mathbb{C}} = \mathbb{C}^2$ be the complexification. Then $\Phi_{V_{\mathbb{C}}}(V_{\mathbb{C}}) = \Phi_V(V)+\Phi_V(V)$ is the closed convex invariant cone generated by $\Phi_V(V)$. ∎

Proposition II.14. *Recall the cone*

$$W_V = C_V^\star = \{X \in \mathfrak{g}\colon (\forall \alpha \in C_V)\alpha(X) \geq 0\}.$$

Then the following are equivalent:

(i) $X \in W_V$.

(ii) *The Hamiltonian function defined by $\varphi(X)(v) = \frac{1}{2}\Omega(X.v,v)$ is non-negative on V.*

(iii) *$B(iX.v,v) \leq 0$ holds for all $v \in V_{\mathbb{C}}$.*

Proof. The equivalence of (i) and (ii) follows immediately from the definition of C_V.

Let $v = v_1 + iv_2 \in V_{\mathbb{C}}$ and $X \in \mathfrak{g}$. Then iX is a B-hermitean operator on $V_{\mathbb{C}}$. Therefore $B(iX.v,v)$ is real and consequently

$$-B(iX.v,v) = \operatorname{Im} B(X.v,v) = \widetilde{\Omega}(X.v,v) = 2\Phi_{V_{\mathbb{C}}}(v)(X).$$

which immediately implies the equivalence of (i) and (iii). ∎

Proposition II.14 links the symplectic modules of convex type to those complex modules which carry invariant pseudohermitean forms and for which the cone W_V is generating. Such modules have been studied in detail in [14].

Remark II.15. If W is a complex vector space endowed with a pseudohermitean form B and

$$S=\{s\in \mathrm{Gl}(W)\colon (\forall w\in W) B(s.w,s.w)\leq B(w,w)\}$$

is the *semigroup of B-contractions*, then the group of units of S is the group $U(B)$ of B-unitary mapping and the tangent cone

$$\mathbf{L}(S):=\{X\in \mathfrak{gl}(W)\colon e^{\mathbb{R}^+X}\subseteq S\}$$

can be written as $\mathbf{L}(S)=\mathfrak{u}(B)+iC_W$, where $C_W=\{X\in\mathfrak{gl}(W)\colon(\forall w\in W)B(iX.w,w)\leq 0\}$ (cf. [14]).

In the light of these observations, the cone W_V consists exactly of those elements X in the Lie algebra $\mathfrak{g}$ for which iX generates a one-parameter semigroup of B-contractions on $V_{\mathbb{C}}$. ■

Example II.16. We consider $V=\mathbb{R}^{2n}$ endowed with the usual symplectic structure given by

$$\Omega(v,w)=\langle v,J.w\rangle,$$

where $J=\begin{pmatrix}\mathbf{0}&\mathbf{1}\\ -\mathbf{1}&\mathbf{0}\end{pmatrix}$ and $\mathfrak{g}=\mathfrak{sp}(n,\mathbb{R})$. Then $V_{\mathbb{C}}=\mathbb{C}^{2n}$ and

$$B(v,w)=i\Omega_{\mathbb{C}}(v,\overline{w})=i\langle v,J.w\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the natural scalar product on $\mathbb{C}^{2n}$. For $X\in\mathfrak{g}$ we see that $X\in W_V$ means that

$$\Omega(X.v,v)=\langle X.v,J.v\rangle=-\langle JX.v,v\rangle\geq 0$$

for all $v\in V$ which in turn means that the symmetric operator JX is negative semidefinite on $\mathbb{R}^{2n}$. Recall that $X\in\mathfrak{sp}(n,\mathbb{R})$ means that JX is a symmetric matrix.

The space $\mathfrak{t}$ of all those elements in $\mathfrak{sp}(n,\mathbb{R})$ for which JX is a diagonal matrix is a compactly embedded Cartan subalgebra of $\mathfrak{sp}(n,\mathbb{R})$. ■

Weight spaces in symplectic modules of convex type

In this subsection $\mathfrak{g}$ denotes a reductive real Lie algebra which has a compactly embedded Cartan algebra $\mathfrak{t}$.

Definition II.17. Let V be a $\mathfrak{g}$-module and $V_{\mathbb{C}}$ its complexification. We fix a compactly embedded Cartan algebra $\mathfrak{t}\subseteq\mathfrak{g}$. For $\alpha\in\mathfrak{t}_{\mathbb{C}}^*$ we write

$$V_{\mathbb{C}}^\alpha=\{v\in V_{\mathbb{C}}\colon(\forall X\in\mathfrak{t}_{\mathbb{C}})X.v=\alpha(X)v\}$$

for the *weight space of weight* α. We write $\mathcal{P}_V=\{\alpha\in\mathfrak{t}_{\mathbb{C}}^*\colon V_{\mathbb{C}}^\alpha\neq 0\}$ for the set of all weights of V.

If, in addition, V is a semisimple $\mathfrak{g}$-module, then $\mathfrak{z}(\mathfrak{g})$ acts by diagonalizable operators on $V_{\mathbb{C}}$ and hence $V_{\mathbb{C}}$ is the direct sum of the weight spaces. If, moreover, V is of convex type, then $\mathfrak{z}(\mathfrak{g})$, and therefore $\mathfrak{t}$ acts on V with purely imaginary eigenvalues, hence $\alpha(\mathfrak{t})\subseteq i\mathbb{R}$ holds for every weight $\alpha\in\mathcal{P}_V$. ■

Definition II.18. For this definition we drop the assumption that $\mathfrak{g}$ is reductive. We merely assume that $\mathfrak{t} \subseteq \mathfrak{g}$ is a compactly embedded Cartan subalgebra.

We recall the root space decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C}}^{\alpha}$, of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$, where

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} = \{Y \in \mathfrak{g}_{\mathbb{C}} : (\forall X \in \mathfrak{t}_{\mathbb{C}})[X,Y] = \alpha(X)Y\}$$

and $\Delta = \{\alpha \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\} : \mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}\}$. Let $\mathfrak{k} \supseteq \mathfrak{t}$ denote the unique maximal compactly embedded subalgebra containing $\mathfrak{t}$ ([6, A.2.40]). We say that a root $\alpha \in \Delta$ is *compact* if $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{k}_{\mathbb{C}}$, otherwise we say that α is *non-compact*. We write Δ_k resp. Δ_p for the sets of compact resp. non-compact roots. If $\mathfrak{r} \subseteq \mathfrak{g}$ is the solvable radical, then we also set $\Delta_r := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^{\alpha} \cap \mathfrak{r}_{\mathbb{C}} \neq \{0\}\}$.

We define the *Weyl group* $\mathcal{W}_{\mathfrak{k}} := N_G(\mathfrak{t})/Z_G(\mathfrak{t}) = N_G(\mathfrak{t})/\exp \mathfrak{t}$. A *positive system* of roots is a subset Δ^+ of Δ for which there exists a regular element $X_0 \in i\mathfrak{t}$ such that $\Delta^+ = \{\alpha \in \Delta : \alpha(X_0) > 0\}$.

We say that a positive system Δ^+ is $\mathfrak{k}$*-adapted* if Δ_p^+ is invariant under the Weyl group. Note that this condition is equivalent to the existence of an element $H \in \mathfrak{z}(\mathfrak{k})$ such that $\Delta_p^+ = \{\alpha \in \Delta : i\alpha(H) > 0\}$. We refer to [15, Prop. II.7] for the result that a $\mathfrak{k}$-adapted positive system exists if and only if $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}(\mathfrak{k})) = \mathfrak{k}$. If this condition is satisfied, then $\mathfrak{g}$ is called *quasihermitean.*

For a $\mathfrak{k}$-adapted positive system Δ^+ we define the cones

$$C_{\min} = \operatorname{cone}(\{i[\overline{X}_\alpha, X_\alpha] : \alpha \in \Delta_p^+, X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}\}) \quad \text{and} \quad C_{\max} = (i\Delta_p^+)^\star,$$

where $\operatorname{cone}(E)$ denotes the smallest closed convex cone containing the set E. We say that $\mathfrak{g}$ has *cone potential*, if $[\overline{X}_\alpha, X_\alpha] \neq 0$ holds for all non-zero elements $X_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$, $\alpha \in \Delta_p$ and that $\mathfrak{g}$ has *strong cone potential* if, in addition, there exists a $\mathfrak{k}$-adapted positive system such that the cone $C_{\min}$ is pointed. ∎

If (V,Ω) is an effective module of convex type for the reductive Lie algebra $\mathfrak{g}$, then the cone $C_V \subseteq \mathfrak{g}^*$ is pointed and generating. Therefore the Lie algebra $\mathfrak{g}$ is quasihermitean (cf. [6, Th. III.5.16]) and it follows in particular that it contains a compactly embedded Cartan subalgebra. Thus the assumption that $\mathfrak{g}$ contains a compactly embedded Cartan subalgebra is simply for technical convenience whenever we study only modules of convex type.

Lemma II.19. *Let (V,Ω) be a symplectic $\mathfrak{g}$-module of convex type. Then the following assertions hold:*

(i) *For $\alpha \neq \beta$ in $\mathcal{P}_V$ the weight spaces $V_{\mathbb{C}}^{\alpha}$ and $V_{\mathbb{C}}^{\beta}$ are orthogonal with respect to B.*

(ii) *For $\alpha \in \mathcal{P}_V$, $v \in V_{\mathbb{C}}^{\alpha}$ we have that*

$$\Phi_{V_{\mathbb{C}}}(v) = \tfrac{1}{2}B(v,v)(-i\alpha) = \tfrac{1}{2}\Omega_{\mathbb{C}}(v,\overline{v})\alpha,$$

where we identify $\mathfrak{t}^*$ with the subspace $[\mathfrak{t},\mathfrak{g}]^{\perp}$ in $\mathfrak{g}^*$.

(iii) *For $\lambda \in \mathbb{C}$ and $v \in V_{\mathbb{C}}$ we have $\Phi_{V_{\mathbb{C}}}(\lambda v) = |\lambda|^2 \Phi_{V_{\mathbb{C}}}(v)$.*

Proof. (i) This follows from the invariance of B under $\mathfrak{t}$.
(ii) Let $\beta \in \Delta$ and $X_\beta \in \mathfrak{g}_{\mathbb{C}}^\beta$. Then $X_\beta.v \in V_{\mathbb{C}}^{\alpha+\beta}$ and therefore $B(X_\beta.v, v) = 0$ by (i). Similarly $B(\overline{X}_\beta.v, v) = 0$ holds for $\overline{X}_\beta \in \mathfrak{g}_{\mathbb{C}}^{-\beta}$. Therefore

$$\begin{aligned}\Phi_V(v)(X_\beta + \overline{X}_\beta) &= \frac{1}{2}\operatorname{Im} B\big((X_\beta + \overline{X}_\beta).v, v\big)\\ &= \frac{1}{2}\operatorname{Im} B\big(X_\beta.v, v\big) + \frac{1}{2}\operatorname{Im} B\big(\overline{X}_\beta.v, v\big) = 0.\end{aligned}$$

Let $\mathfrak{g}^{[\beta]} := (\mathfrak{g}_{\mathbb{C}}^\beta \oplus \mathfrak{g}_{\mathbb{C}}^{-\beta}) \cap \mathfrak{g}$. Then the above argument shows that $\Phi_V(v) \in \mathfrak{g}^{[\beta]\perp}$. Since $[\mathfrak{t}, \mathfrak{g}] = \bigoplus_{\alpha\in\Delta} \mathfrak{g}^{[\alpha]}$, we conclude that $\Phi_V(v) \in \mathfrak{t}^*$. To calculate this functional explicitly, we only have to evaluate it on $\mathfrak{t}$. For $X \in \mathfrak{t}$ we obtain

$$\begin{aligned}\Phi_{V_{\mathbb{C}}}(v)(X) &= \frac{1}{2}\operatorname{Im} B(X.v, v) = \frac{1}{2}\operatorname{Im}\big(\alpha(X)B(v,v)\big)\\ &= \frac{1}{2}B(v,v)\operatorname{Im}\alpha(X) = \frac{1}{2}B(v,v)\big(-i\alpha(X)\big).\end{aligned}$$

(iii) This is a simple consequence of the sesquilinearity of B. ∎

Proposition II.20. *If (V, Ω) is a symplectic $\mathfrak{g}$-module of convex type and $0 \neq \alpha \in \mathcal{P}_V$, then two cases arise:*

(i) *B is positive definite on $V_{\mathbb{C}}^\alpha$ and $-i\alpha \in C_V$.*

(ii) *B is negative definite on $V_{\mathbb{C}}^\alpha$ and $i\alpha \in C_V$.*

Proof. First we note that Lemma II.19(i) together with $V_{\mathbb{C}} = \bigoplus_{\alpha\in\mathcal{P}_V} V_{\mathbb{C}}^\alpha$ implies that the restriction of B to $V_{\mathbb{C}}^\alpha$ is non-degenerate. Since C_V is pointed, Lemma II.19(ii) now shows that B is either positive or negative definite on $V_{\mathbb{C}}^\alpha$ depending on whether $-i\alpha$ of $i\alpha$ is contained in C_V. ∎

Corollary II.21. *If (V, Ω) is a symplectic $\mathfrak{g}$-module of convex type and $0 \neq v \in V_{\mathbb{C}}^\alpha$ with $\Phi_{V_{\mathbb{C}}}(v) = 0$, then $\alpha = 0$.*

Proof. In view of Proposition II.20 and Lemma II.19(ii),

$$0 = \Phi_{\mathbb{C}}(v) = \tfrac{1}{2}B(v,v)(-i\alpha)$$

implies that $\alpha = 0$. ∎

We draw some important conlusions in the real setting.

Proposition II.22. *Let $\mathfrak{g}$ be a reductive Lie algebra with compactly embedded Cartan algebra $\mathfrak{t}$ and (V, Ω) a symplectic $\mathfrak{g}$-module of convex type. Then the following assertions hold:*

(i) *$V_{\text{fix}} = \Phi_V^{-1}(0)$.*

(ii) *If $V_{\text{fix}} = \{0\}$, then $0 \notin \mathcal{P}_V$.*

Proof. (i) The inclusion $V_{\rm fix} \subseteq \Phi_V^{-1}(0)$ holds trivially. Suppose conversely that $\Phi_V(v) = 0$. We write $v = \sum_{\alpha\in\mathcal{P}_V} v_\alpha$ with $v_\alpha \in V_{\mathbb{C}}^\alpha$. Then Lemma I.18(i),(ii) imply that for $X \in \operatorname{int} C_V^\star \cap \mathfrak{t}$ we have

$$0 = \Phi_V(v)(X) = \sum_{\alpha\in\mathcal{P}_V} \Phi_{V_{\mathbb{C}}}(v_\alpha)(X).$$

Since $X \in \operatorname{int} C_V^\star$ and all functionals $\Phi_{V_{\mathbb{C}}}(v_\alpha)$ lie in the pointed cone C_V, we conclude that all summands are 0, hence that $v_\alpha \in \Phi_{V_{\mathbb{C}}}^{-1}(0)$. Now Corollary II.21 shows that $v_\alpha = 0$ for $\alpha \neq 0$ and that $v \in V_{\mathbb{C}}^0 \cap V$.

This proves that $\Phi_V^{-1}(0) = V_{\mathbb{C}}^0 \cap V$. On the other hand, the equivariance of Φ_V shows that $\Phi_V^{-1}(0)$ is invariant under G, hence under $\mathfrak{g}$. Since $\mathfrak{t}$ acts trivially on $V_{\mathbb{C}}^0 \cap V$, the same holds for the whole Lie algebra $\mathfrak{g}$ because $\mathfrak{g}$ is the smallest ideal containing $\mathfrak{t}$. This means that $V_{\mathbb{C}}^0 \cap V = V_{\rm fix}$.
(ii) This follows from (i) and Lemma II.19(ii). ∎

Proposition II.23. *Let (V,Ω) be a symplectic $\mathfrak{g}$-module with $V_{\rm fix} = \{0\}$ and where $\mathfrak{z}(\mathfrak{g})$ acts semisimply with purely imaginary eigenvalues. Then the following are equivalent:*

(i) *V is of convex type.*
(ii) *The cone $W_V = C_V^\star$ has interior points.*
(iii) *There exists $X \in \mathfrak{z}(\mathfrak{k})$ such that $\Omega(X.v, v) > 0$ for all $v \in V \setminus \{0\}$.*
(iv) *There exists $X \in \mathfrak{z}(\mathfrak{k})$ such that $\alpha(X) \neq 0$ holds for all $\alpha \in \mathcal{P}_V$ and B is positive definite on $V_{\mathbb{C}}^\alpha$ if and only if $i\alpha(X) < 0$.*

Proof. The equivalence of (i) and (ii) follows from the duality theory of cones which implies that C_V is pointed if and only if its dual cone W_V is generating.
(ii) $\Rightarrow$ (iii): If (ii) holds, then the cone W_V is a generating invariant cone in $\mathfrak{g}$. Therefore there exists an element $X \in \mathfrak{z}(\mathfrak{k}) \cap \operatorname{int} W_V$ (cf. [6, Prop. III.5.5]). Then $\Phi_V(v)(X) > 0$ holds for all $v \in V$ with $\Phi_V(v) \neq 0$. Finally $\Phi_V^{-1}(0) = V_{\rm fix} = \{0\}$ follows from Proposition II.22(i) and this implies (iii).
(iii) $\Rightarrow$ (iv): Using (iii), we choose $X \in \mathfrak{z}(\mathfrak{k})$ such that $\Omega(X.v, v) > 0$ holds for all $v \in V \setminus \{0\}$. Then also

$$\operatorname{Im} B(X.v, v) = \widetilde{\Omega}(X.v, v) > 0$$

for all $v \in V_{\mathbb{C}} \setminus \{0\}$. For $v \in V_{\mathbb{C}}^\alpha$ we find in particular that

$$\operatorname{Im} B(X.v, v) = B(v, v)(-i\alpha)(X) > 0.$$

Hence B is positive definite on $V_{\mathbb{C}}^\alpha$ for $i\alpha(X) < 0$ and negative definite if $i\alpha(X) > 0$.
(iv) $\Rightarrow$ (iii): Let $v \in V$ and write it as $v = \sum_{\alpha\in\mathcal{P}_V} v_\alpha$ with $v_\alpha \in V_{\mathbb{C}}^\alpha$. Then

$$\Omega(X.v, v) = \sum_{\alpha\in\mathcal{P}_V} B(v_\alpha, v_\alpha)(-i\alpha)(X) > 0$$

for $v \neq 0$ follows from (iv).
(iii) $\Rightarrow$ (i): Let $S \subseteq V$ be a sphere for some norm on V. Then $K := \operatorname{conv} \Phi_V(S)$ is a compact convex set and $\langle X, \alpha\rangle > 0$ holds for all $\alpha \in K$. Therefore $C_V = \mathbb{R}^+ K$ is a pointed closed convex cone in $\mathfrak{g}^*$. ∎

Complex structures on simple modules

We have already seen that a symplectic $\mathfrak{g}$-module of convex type breaks up into a symplectic direct sum of simple modules of convex type. Our next objective is to show that the classification problem for these modules can be translated into a classification problem for certain simple complex $\mathfrak{g}$-modules. This problem is much easier to deal with since the complex simple $\mathfrak{g}$-modules can easily be classified by their highest weights (cf. Section III).

Lemma II.24. *Let (V,Ω) be a symplectic $\mathfrak{g}$-module and $I \in \mathrm{End}_{\mathfrak{g}}(V)$ an invariant complex structure with $I^\sharp = -I$, i.e., $I \in \mathrm{Sp}(V,\Omega)$. Then*

$$B(v,w) := \Omega(I.v,w) + i\Omega(v,w)$$

defines a $\mathfrak{g}$-invariant pseudohermitean form on the complex vector space (V,I) such that $\Omega = \mathrm{Im}\, B$.

Proof. For $v,w \in V$ we have

$$B(I.v,w) = \Omega(I^2.v,w) + i\Omega(I.v,w) = -\Omega(v,w) + i\Omega(I.v,w) = iB(v,w)$$

and

$$\begin{aligned} B(w,v) &= \Omega(I.w,v) + i\Omega(w,v) = \Omega(w,-I.v) - i\Omega(v,w) \\ &= \Omega(I.v,w) - i\Omega(v,w) = \overline{B(v,w)}. \end{aligned}$$

Thus B is a pseudohermitean form on V which is easily seen to be invariant under $\mathfrak{g}$. ■

We recall that if (V,Ω) is a non-trivial simple $\mathfrak{g}$-module of convex type, then every invariant complex structure $I \in \mathrm{End}_{\mathfrak{g}}(V)$ is skew-symmetric (cf. Lemma II.7), hence that the construction of the preceding lemma works for $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}, \mathbb{H}$.

Definition II.25. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. A *complex symplectic $\mathfrak{g}$-module* is a triple (V,Ω,I), where (V,Ω) is a symplectic $\mathfrak{g}$-module and $I \in \mathrm{Sp}(V,\Omega) \cap \mathrm{End}_{\mathfrak{g}}(V)$ is a $\mathfrak{g}$-invariant complex structure. We say that (V,Ω,I) is *simple* if it is simple as a complex $\mathfrak{g}$-module.

An isomorphism of two complex symplectic $\mathfrak{g}$-modules is required to be an isomorphism of symplectic $\mathfrak{g}$-modules which is in addition complex linear or antilinear. ■

Theorem II.26. *Let (V,Ω) be a simple $\mathfrak{g}$-module of convex type and $\mathbb{D} = \mathrm{End}_{\mathfrak{g}}(V)$. If $\mathbb{D} = \mathbb{R}$ we put $(\widetilde{V},\widetilde{\Omega},I) := (V_{\mathbb{C}}, \mathrm{Im}\,\Omega_{\mathbb{C}}, i\,\mathrm{id})$ and for $\mathbb{D} = \mathbb{H},\mathbb{C}$ we put $(\widetilde{V},\widetilde{\Omega}) = (V,\Omega,I)$, where I is any invariant complex structure on V. Then this prescription associates to each isomorphy class $[(V,\Omega)]$ of simple symplectic $\mathfrak{g}$-modules of convex type an isomorphy class $[(\widetilde{V},\widetilde{\Omega},I)]$ of simple complex symplectic $\mathfrak{g}$-modules. It yields a bijection between the classes of simple* *symplectic* $\mathfrak{g}$*-modules of convex type and the classes of simple complex symplectic* $\mathfrak{g}$*-modules of convex type.*

Proof. First we show that the assignment is well defined on the level of classes. For $\mathbb{D} = \mathbb{R}$ this follows from Lemma II.10. For $\mathbb{D} = \mathbb{C}$ we have to choose between the two complex structures I and $-I$. Since the identitiy map $(V, I) \to (V, -I)$ is complex antilinear, the two complex symplectic $\mathfrak{g}$-modules (V, Ω, I) and $(V, \Omega, -I)$ are isomorphic.

Now let $\mathbb{D} = \mathbb{H}$. Then V is of type $\mathbb{H}_I$ (Lemma II.7). If I and $\widetilde{I} \in \mathbb{D}$ are two different complex structures, then there exists an element $d \in \mathbb{D}$ with $|d| = 1$ such that $\widetilde{I} = dId^{-1}$. Since $d^\sharp = \overline{d} = d^{-1}$, it follows that $d \in \mathrm{Sp}(V, \Omega)$. Hence the map

$$d: (V, \Omega, I) \to (V, \Omega, \widetilde{I})$$

is a complex linear isomorphism between complex symplectic $\mathfrak{g}$-modules. This shows that our prescription yields a well defined map $[(V, \Omega)] \mapsto [(\widetilde{V}, \widetilde{\Omega}, I)]$ on the level of isomorphy classes.

We claim that this map is injective. So assume that $[(\widetilde{V}, \widetilde{\Omega}_V, I_V)] = [(\widetilde{W}, \widetilde{\Omega}_W, I_W)]$. Let $\mathbb{D} := \mathrm{End}_{\mathfrak{g}}(V)$ and $\mathbb{D}' := \mathrm{End}_{\mathfrak{g}}(W)$. If $\mathbb{D} = \mathbb{R}$, then $\widetilde{V}$ is reducible as a real $\mathfrak{g}$-module and otherwise $\mathbb{D} = \mathrm{End}_{\mathfrak{g}}(\widetilde{V})$. Therefore $\mathbb{D} = \mathbb{D}'$. We distinguish three cases:

$\mathbb{D} = \mathbb{R}$: In the following we write $\Omega_{\mathbb{C}}$ for the complex bilinear extension of a symplectic form Ω on V to $V_{\mathbb{C}}$. Suppose that $[(V_{\mathbb{C}}, \mathrm{Im}\,\Omega_{V,\mathbb{C}})] = [(W_{\mathbb{C}}, \mathrm{Im}\,\Omega_{W,\mathbb{C}})]$. Since $V_{\mathbb{C}} \cong V \oplus V$ as real $\mathfrak{g}$-modules, we observe that $V \cong W$. If $[(V, \Omega_V)] \neq [(W, \Omega_W)]$, then $[(W, \Omega_W)] = [(V, -\Omega_V)]$. In that case we obtain a contradiction to $C_V = C_{V_{\mathbb{C}}} = C_{W_{\mathbb{C}}} = -C_V$.

$\mathbb{D} = \mathbb{C}, \mathbb{H}$: If $[(V, \Omega_V, I_V)] = [(W, \Omega_W, I_W)]$, then $V \cong W$ as real $\mathfrak{g}$-modules. If $[(V, \Omega_V)] \neq [(W, \Omega_W)]$, we have $[(W, \Omega_W)] = [(V, -\Omega_V)]$ and we obtain the same contradiction as above.

This proves that our assignment is injective. Next we show that it is also surjective. Let (W, Ω_W, I) be a simple complex symplectic $\mathfrak{g}$-module of convex type.

If W is simple as a real module, then (W, Ω_W) is a simple $\mathfrak{g}$-module of convex type and since it carries an invariant complex structure, $\mathrm{End}_{\mathfrak{g}}(W) \cong \mathbb{C}$ or $\mathbb{H}$. Therefore $[(W, \Omega, I)]$ is the image of the class $[(W, \Omega)]$.

Suppose that W is not simple as a real module. Let $V \subseteq W$ be a simple non-zero submodule. Then $V + IV$ is a complex submodule of W so that $W = V + IV$. Since W is simple and different from V, we also have $V \cap IV = \{0\}$. Therefore $W \cong V_{\mathbb{C}}$ as complex $\mathfrak{g}$-modules. The fact that W is of convex type implies that V is a non-degenerate submodule (Lemma II.4). We claim that $IV = V^\perp$. Let $B(v, w) := \Omega(Iv, w)$. Then B is a real symmetric bilinear form on V which is invariant under $\mathfrak{g}$ (cf. Lemma II.24). Hence there exists $d \in \mathrm{End}_{\mathfrak{g}}(V)$ with $d^\sharp = -d$ such that $B(v, w) = \Omega_V(d.v, w)$. On the other hand $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{R}$ consists of symmetric elements, so that $B = 0$, consequently $V \perp IV$ and hence $V^\perp = IV$. We conclude that

$$\begin{aligned} \Omega_W(v + Iw, v' + Iw') &= \Omega_W(v, v') + \Omega_W(Iw, Iw') \\ &= \Omega_V(v, v') + \Omega_V(w, w') = \Omega_{V_{\mathbb{C}}}(v + Iw, v' + Iw'). \end{aligned}$$

Thus we also have surjectivity. ■

For the following lemma we note that a compact Lie algebra which has a non-trivial symplectic module of convex type cannot be semisimple because the dual of a semisimple compact Lie algebra contains no non-zero pointed invariant cone ([6, Prop. III.2.2]).

Proposition II.27. *Let $\mathfrak{g}$ be a compact Lie algebra with non-trivial center and V an irreducible symplectic $\mathfrak{g}$-module. Then the following are equivalent:*

(1) *V is of convex type.*

(2) *The center $\mathfrak{z}(\mathfrak{g})$ acts semisimply non-trivially with purely imaginary eigenvalues.*

In this case $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}$ *or* $\mathbb{H}$.

Proof. (1) $\Rightarrow$ (2): This follows directly from the definitions.
(2) $\Rightarrow$ (1): Let $\pi: \mathfrak{g} \to \mathfrak{sp}(V) \subseteq \mathfrak{gl}(V)$ be the Lie algebra homomorphism defining the symplectic module structure on V. If (2) is satisfied, then the group generated by $e^{\pi(\mathfrak{g})}$ has compact closure in $\mathrm{Sp}(V)$. Now the fact that $\mathrm{U}(n) := \mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{Gl}(n, \mathbb{C})$ is maximal compact in $\mathrm{Sp}(n, \mathbb{R}) \subseteq \mathrm{Gl}(2n, \mathbb{R})$ shows that there exists a $\mathfrak{g}$-invariant complex structure I on V such that $\Omega(v, I.v) > 0$ holds for all $v \in V \setminus \{0\}$. Pick $X \in \mathfrak{z}(\mathfrak{g})$ which acts non-trivially on V. Then this mapping is complex linear and by Schur's lemma we may assume that $X.v = I.v$ holds for all $v \in V$. Now Proposition II.23 implies that (V, Ω) is a symplectic module of convex type and that V admits an invariant complex structure so that either $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}$ or $\mathbb{H}$. ■

We will see later that this result does not hold in general for a non-compact Lie algebra. This will be a byproduct of the classification. A typical example is the $\mathrm{Sp}(n, \mathbb{R})$-module $\mathbb{R}^{2n}$ which does not admit any invariant complex structure.

In many applications it is possible to obtain information on modules of reductive Lie algebras by looking at their $\mathfrak{sl}(2, \mathbb{R})$-subalgebras. The following result paves the way to such a reduction in the context of modules of convex type.

Proposition II.28. *Let V be a symplectic module of convex type of $\mathfrak{g}$, $\mathfrak{t} \subseteq \mathfrak{g}$ a compactly embedded Cartan algebra and $\mathfrak{s} \subseteq \mathfrak{g}$ a $\mathfrak{t}$-invariant subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Then V is a symplectic $\mathfrak{s}$-module of convex type, and every non-trivial irreducible $\mathfrak{s}$-submodule of V is non-degenerate, of convex type and 2-dimensional.*

Proof. We may w.l.o.g. assume that $\mathfrak{g}$ acts effectively on V. Then the cone W_V is pointed and generating. Fix a compactly embedded Cartan algebra $\mathfrak{t} \subseteq \mathfrak{g}$. Then there exists a positive $\mathfrak{k}$-adapted positive system of roots Δ^+ such that

$$C_{\min} \subseteq W_V \cap \mathfrak{t} \subseteq C_{\max}$$

(cf. Definition II.18, [12, Prop. III.15]).

We set $\mathfrak{t}_s := \mathfrak{t} \cap \mathfrak{s}$. Since $\mathfrak{s}$ is invariant under $\mathfrak{t}$ and all derivations of $\mathfrak{s}$ are inner, it follows that there exists an element $U \in \mathfrak{s}$ such that $[\mathfrak{t}, U] = \{0\}$.

Hence $\mathfrak{t}_s = \mathbb{R}U \neq \{0\}$ is a compactly embedded Cartan subalgebra of $\mathfrak{s}$. Moreover, since $\mathfrak{s}$ is invariant under $\mathfrak{t}$ and three dimensional, we then have $\mathfrak{s}_{\mathbb{C}}$ as $\mathfrak{t}_{s\mathbb{C}} + \mathfrak{g}_{\mathbb{C}}^{\beta} + \mathfrak{g}_{\mathbb{C}}^{-\beta}$ for a root $\beta \in \Delta_p^+$ and choose a non-zero element $X_\beta \in \mathfrak{g}_{\mathbb{C}}^{\beta}$. Then $X_0 := i[\overline{X}_\beta, X_\beta] \in C_{\min} \subseteq W_V$ and this means in particular that $X_0 \in C_V^\star$. Hence the restriction mapping $\mathfrak{g}^* \to \mathfrak{s}^*$ maps C_V into a pointed invariant cone and thus V is an $\mathfrak{s}$-module of convex type.

Therefore the results from above apply to the $\mathfrak{s}$-module V. We conclude that every non-trivial irreducible submodule E is non-degenerate (Proposition II.5) and therefore a symplectic $\mathfrak{s}$-module of convex type. Thus $W_E \neq \{0\}$ and consequently $\dim E = 2$ by Theorem V.8 in [14] and Remark II.15. ∎

III. The classification of simple modules of convex type

In the preceding section we have analyzed the structure of symplectic $\mathfrak{g}$-modules of convex type. Theorem II.8 and Corollary II.9 provide effective tools to reduce the classification of general modules of convex type to the simple case. On the other hand we have Theorem II.26 which translates the classification problem of the real simple modules of convex type to the problem of classifying simple complex symplectic modules of convex type modulo complex linear or antilinear isomorphisms of symplectic $\mathfrak{g}$-modules. In this section we adopt this point of view that will permit us by using results obtained in [14] to obtain a complete classification of the simple modules of convex type.

Before we turn directly towards the classification we first have to use tensor product decompositions to reduce the problem to the case where the Lie algebra under consideration is simple hermitean.

Tensor products of complex modules as real modules

Lemma III.1. *Let V resp. W be semisimple modules of the reductive $\mathbb{K}$-Lie algebras $\mathfrak{a}$ resp. $\mathfrak{b}$. Then* $\mathrm{End}_{\mathfrak{a}\oplus\mathfrak{b}}(V \otimes_{\mathbb{K}} W) \cong \mathrm{End}_{\mathfrak{a}}(V) \otimes_{\mathbb{K}} \mathrm{End}_{\mathfrak{b}}(W)$.

Proof. Since the map $A \otimes B \mapsto A \otimes B$ induces an $\mathfrak{a} \oplus \mathfrak{b}$-equivariant algebra isomorphism

$$\mathrm{End}_{\mathbb{K}}(V) \otimes_{\mathbb{K}} \mathrm{End}_{\mathbb{K}}(W) \to \mathrm{End}_{\mathbb{K}}(V \otimes_{\mathbb{K}} W),$$

we only have to show that for a semisimple $\mathfrak{a}$-module X and a semisimple $\mathfrak{b}$-module Y we have

$$(X \otimes Y)_{\mathrm{fix}} = X_{\mathrm{fix}} \otimes Y_{\mathrm{fix}},$$

where $X \otimes Y$ is considered as $(\mathfrak{a} \oplus \mathfrak{b})$-module. Then we apply this to $X = \mathrm{End}_{\mathbb{K}}(V)$ and $Y = \mathrm{End}_{\mathbb{K}}(W)$.

Since X and Y decompose as $X = X_{\mathrm{fix}} \oplus X_{\mathrm{eff}}$ and $Y = Y_{\mathrm{fix}} \oplus Y_{\mathrm{eff}}$, it follows that

$$X \otimes Y \cong (X_{\mathrm{fix}} \otimes Y_{\mathrm{fix}}) \oplus (X_{\mathrm{eff}} \otimes Y_{\mathrm{fix}}) \oplus (X \otimes Y_{\mathrm{eff}}).$$

On the other hand, we see that for $x \in X$, $y \in Y$ and $b \in \mathfrak{b}$ that

$$x \otimes (y - b.y) = x \otimes y - b.(x \otimes y) \in (X \otimes Y)_{\rm eff}.$$

We conclude that

$$(X \otimes Y_{\rm eff}) + (X_{\rm eff} \otimes Y) \subseteq (X \otimes Y)_{\rm eff}$$

so that the assertion follows from $X \otimes Y = (X \otimes Y)_{\rm fix} \oplus (X \otimes Y)_{\rm eff}$. ■

Let V be a complex simple $\mathfrak{g}$-module. We consider V as a real $\mathfrak{g}$-module. Then there are three possibilities:
V is of *real type*: V is reducible over $\mathbb{R}$. Then $V = (V^0)_{\mathbb{C}}$ for a real simple $\mathfrak{g}$-module V^0. In this case $\mathrm{End}_{\mathfrak{g}}(V) \cong \mathbb{M}(2, \mathbb{R})$.
V is of *complex type*: V is simple over $\mathbb{R}$ and $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}$.
V is of *quaternionic type*: V is simple over $\mathbb{R}$ and $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{H}$.

Lemma III.2. *Let V resp. W be two complex simple modules for the real Lie algebra $\mathfrak{a}$ resp. $\mathfrak{b}$. We consider $X := V \otimes_{\mathbb{C}} W$ as a real module for the Lie algebra $\mathfrak{g} := \mathfrak{a} \oplus \mathfrak{b}$. Then X is simple as a complex $\mathfrak{g}$-module and the type of X can be determined as follows:*

(1) *If V is of real type, then X has the same type as W.*
(2) *If V is of complex type and W is not of real type, then X is of complex type.*
(3) *If V and W are of quaternionic type, then X is of real type.*

Proof. We distinguish several cases:
If V is of real type with $V = (V^0)_{\mathbb{C}}$, then the map $V^0 \otimes_{\mathbb{R}} W \to V \otimes_{\mathbb{C}} W$ is a module isomorphism. Hence Lemma III.1 implies that

$$\mathrm{End}_{\mathfrak{g}}(V \otimes_{\mathbb{C}} W) \cong \mathrm{End}_{\mathfrak{g}}(V^0 \otimes_{\mathbb{R}} W) \cong \mathrm{End}_{\mathfrak{a}}(V^0) \otimes_{\mathbb{R}} \mathrm{End}_{\mathfrak{b}}(W) \cong \mathrm{End}_{\mathfrak{b}}(W).$$

Since this is a division algebra, we see that $V \otimes_{\mathbb{C}} W$ is simple as a real $\mathfrak{g}$-module and that its type is the type of W.
Suppose that V and W are not of real type. Then the map $V \otimes_{\mathbb{R}} W \to V \otimes_{\mathbb{C}} W$ has a non-trivial kernel. Moreover

$$\mathrm{End}_{\mathfrak{g}}(V \otimes_{\mathbb{R}} W) \cong \mathrm{End}_{\mathfrak{a}}(V) \otimes_{\mathbb{R}} \mathrm{End}_{\mathfrak{b}}(W).$$

If V and W are of complex type, then $\mathrm{End}_{\mathfrak{g}}(V \otimes_{\mathbb{R}} W) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ possesses one idempotent different from 0 and 1. Therefore $V \otimes_{\mathbb{R}} W$ decomposes into a direct sum of two simple modules so that $V \otimes_{\mathbb{C}} W$ must be simple over $\mathbb{R}$ and of complex type.

If V is of complex and W of quaternionic type, then

$$\mathrm{End}_{\mathfrak{g}}(V \otimes_{\mathbb{R}} W) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{M}(2, \mathbb{C})$$

as one easily verifies by viewing $\mathbb{H}$ as a subalgebra of $\mathbb{M}(2, \mathbb{C})$ (cf. [19, p.190]). Therefore $V \otimes_{\mathbb{R}} W$ is a direct sum of two equivalent submodules with commutant $\mathbb{C}$. Thus $V \otimes_{\mathbb{C}} W$ must be simple over $\mathbb{R}$ and of complex type.

We claim that if V and W are of quaternionic type, then

$$\mathrm{End}_{\mathfrak{g}}(V \otimes_{\mathbb{R}} W) \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{M}(4, \mathbb{R}).$$

In fact, using the left and right regular representation of the $\mathbb{R}$-algebra $\mathbb{H}$, we obtain an algebra homomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \to \mathbb{M}(4, \mathbb{R})$ which turns out be surjective since the $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$-module $\mathbb{H}$ is simple with commutant isomorphic to $\mathbb{R}$ (Wedderburn's Theorem). Hence, counting dimensions yields $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{M}(4, \mathbb{R})$ (cf. [19, p.190]).

We conclude that $V \otimes_{\mathbb{R}} W$ is a direct sum of four equivalent simple submodules and counting dimensions shows that $V \otimes_{\mathbb{C}} W$ is reducible over $\mathbb{R}$. ∎

Towards the classification of modules of convex type

Lemma III.3. *Let (V, Ω_V, I_V) resp. (W, Ω_W, I_W) be a complex symplectic $\mathfrak{g}$-module. Then $(V \otimes_{\mathbb{C}} W, \Omega, I)$ is a complex symplectic $\mathfrak{g}$-module, where*

$$\Omega(v \otimes w, v' \otimes w') = \Omega_V(v, v')\Omega_W(I_W.w, w') + \Omega_W(w, w')\Omega_V(I_V.v, v').$$

Proof. According to Lemma II.24, the forms $(v, v') \mapsto \Omega_V(I_V.v, v')$ and $(w, w') \mapsto \Omega_W(I_W.w, w')$ are symmetric so that

$$(v \otimes w, v' \otimes w') \mapsto \Omega_V(v, v')\Omega_W(I_W.w, w') + \Omega_W(w, w')\Omega_V(I_V.v, v')$$

is a well defined skew symmetric map on $V \otimes_{\mathbb{R}} W$. To see that this map factors to the quotient $V \otimes_{\mathbb{C}} W$, we have to show that it vanishes if one of the arguments has the form $v \otimes w + I_V.v \otimes I_W.w$. For the first argument this follows from

$$\begin{aligned} &\Omega_V(v, v')\Omega_W(I_W.w, w') + \Omega_W(w, w')\Omega_V(I_V.v, v') \\ &- \Omega_V(I_V.v, v')\Omega_W(w, w') - \Omega_W(I_W.w, w')\Omega_V(v, v') = 0. \end{aligned}$$

This shows that Ω defines a skew-symmetric real bilinear form on $V \otimes_{\mathbb{C}} W$ which is invariant under the complex structure I. To see that this form is non-degenerate, we note that it is the imaginary part of the pseudo-hermitean form B on $V \otimes_{\mathbb{C}} W$ that arises from the pseudo-hermitean forms B_V resp. B_W on V resp. W (cf. Lemma II.24) by

$$B\big((v \otimes w), (v' \otimes w')\big) := B_V(v, v')B_W(w, w'). \tag{3.1}$$

Since B_V and B_W are non-degenerate, the same follows for B and hence for Ω by choosing orthogonal bases in V resp. W. The $\mathfrak{g}$-invariance of Ω is verified with (3.1) by an easy calculation. ∎

Definition III.4. If (V, Ω_V, I_V) resp. (W, Ω_W, I_W) are complex symplectic $\mathfrak{g}$-modules, then the complex symplectic $\mathfrak{g}$-module $(V \otimes_{\mathbb{C}} W, \Omega, I)$ defined in Lemma III.3 is called the *tensor product* of the complex symplectic $\mathfrak{g}$-modules (V, Ω_V, I_V) and (W, Ω_W, I_W). It is denoted by $(V, \Omega_V, I_V) \otimes (W, \Omega_W, I_W)$. ∎

We recall the following result which is Theorem IV.13 in [14]. Here we state it in the terms relevant for our present purposes.

Theorem III.5. *Let $\mathfrak{g}$ be a reductive Lie algebra and V a simple complex symplectic $\mathfrak{g}$-module such that $W_V \neq \{0\}$ and $\mathfrak{g}$ acts effectively. Then $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_2$ is a compact semisimple Lie algebra, the commutator algebra of $\mathfrak{g}_1$ is either trivial or a hermitean simple ideal, and*

$$(V, \Omega, I) \cong (V_1, \Omega_1, I_1) \otimes (V_2, \Omega_2, I_2)$$

is a tensor product of simple complex symplectic $\mathfrak{g}$-modules, where V_1 resp. V_2 is a $\mathfrak{g}_1$ resp. $\mathfrak{g}_2$-module, $W_{V_1} \neq \{0\}$, and the canonical pseudo-hermitean form B_2 on V_2 is positive definite.

If, conversely, the simple complex symplectic $\mathfrak{g}_1$-module (V_1, Ω_1, I_1) and the simple complex symplectic $\mathfrak{g}_2$-module (V_2, Ω_2, I_2) have these properties, then their tensor product $V := V_1 \otimes_{\mathbb{C}} V_2$ is a simple complex symplectic $\mathfrak{g}$-module and

$$W_V \cap \mathfrak{g}_1 = W_{V_1} \neq \{0\}.$$

Proof. To see that this is the statement of Theorem IV.13 in [14], we only have to note that, in view of Lemma II.24, the concept of a complex symplectic $\mathfrak{g}$-module is the same as that of a complex $\mathfrak{g}$-module with an invariant non-degenerate pseudo-hermitean form. ∎

To link Theorem III.5 to modules of convex type, we need the following lemma.

Lemma III.6. *Let $\mathfrak{g}$ be a non-zero quasihermitean reductive Lie algebra and (V, Ω) an effective simple symplectic $\mathfrak{g}$-module such that $\mathfrak{z}(\mathfrak{g})$ acts semisimply with purely imaginary spectrum. Then the following are equivalent*

(1) *$W_V \neq \{0\}$*

(2) *W_V is generating.*

Proof. (1) $\Rightarrow$ (2): Suppose that $W_V \neq \{0\}$. Since V is an effective $\mathfrak{g}$-module $C_V^\perp = H(W_V) = \{0\}$, i.e., W_V is pointed. Put $\mathfrak{a} := W_V - W_V$. Then $\mathfrak{a}$ is a non-zero ideal of $\mathfrak{g}$ containing the pointed generating invariant cone W_V. Let us consider V as a symplectic $\mathfrak{a}$-module and let $p_\mathfrak{a} : \mathfrak{g}^* \to \mathfrak{a}^*$ denote the restriction map. Then

$$\Phi_\mathfrak{a}(V) = p_\mathfrak{a}\big(\Phi_\mathfrak{g}(V)\big) \subseteq p_\mathfrak{a}(C_V) \cong C_V / H(C_V)$$

lies in a pointed cone, whence (V, Ω) is a symplectic $\mathfrak{a}$-module of convex type.

Moreover the Lie algebra $\mathfrak{a}$ contains a pointed generating invariant cone so that it is quasihermitean (cf. [6, Th. III.5.16]). Let $\mathfrak{k} \subseteq \mathfrak{a}$ be a maximal compactly embedded subalgebra and $Z \in \mathfrak{z}(\mathfrak{k}) \cap \operatorname{int} C_V^\star$. We claim that the function $\varphi(Z)$ on V is positive on $V \backslash \{0\}$. Suppose that $\varphi(Z)(v) = \Phi(v)(Z) = 0$. Since $\Phi(v) \in C_V$, we conclude that $\Phi(v) \in H(C_V) = \mathfrak{a}^\perp$. Therefore $\Phi_\mathfrak{a}(v) = 0$. Now Proposition II.22 applies since $\mathfrak{a}$ is quasihermitean and we see that $\mathfrak{a}.v = \{0\}$. For $X \in \mathfrak{g}$ we then obtain $\mathfrak{a}.(X.v) \subseteq [X, \mathfrak{a}].v + X.(\mathfrak{a}.v) = \{0\}$, so that $\mathfrak{g}.v = \{0\}$. Thus $v = 0$ by the simplicity of V. We conclude that $\varphi(Z)$ is positive on $V \setminus \{0\}$, so that Proposition II.23 entails that V is of convex type, i.e., W_V is generating.

(2) $\Rightarrow$ (1): Since $\mathfrak{g} \neq \{0\}$, the assumption that W_V is generating implies that $W_V \neq \{0\}$. ∎

Remark III.7. The preceding result applies equally well to the complexification $(V_{\mathbb{C}}, \widetilde{\Omega})$ of a simple symplectic module (V, Ω) because the cones corresponding to both modules are the same. Hence the assertion remains true for simple complex symplectic $\mathfrak{g}$-modules. ■

Remark III.8. In view of Remark III.7, Theorem III.5 shows that the classification of simple complex symplectic modules V of convex type of the reductive Lie algebra $\mathfrak{g}$ boils down to the classification of those simple complex symplectic $\mathfrak{g}_1$-modules V_1 of convex type of a reductive Lie algebra $\mathfrak{g}_1$ whose commutator algebra is simple hermitean and whose center is at most one-dimensional (Schur's Lemma). To find the corresponding real simple module of convex type one has to apply Lemma III.2 to see whether V is of real type or not. ■

To do this, we need some information on the type of a complex irreducible representation of a compact semisimple Lie algebra.

Definition III.9. A simple complex $\mathfrak{g}$-module V is called *orthogonal* resp. *symplectic* if it admits a non-degenerate $\mathfrak{g}$-invariant complex bilinear symmetric resp. skew-symmetric form. ■

Since the $\mathfrak{g}$-invariance of a complex bilinear form is equivalent to the $\mathfrak{g}_{\mathbb{C}}$-invariance, we will be able to draw the information we need out of the following result which is essentially due to Malcev ([11]):

Proposition III.10. *Let $\mathfrak{g}$ be a complex Lie algebra with Cartan algebra $\mathfrak{h}$ and root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^\alpha$. Let V_λ denote the irreducible $\mathfrak{g}$-module with highest weight λ, write $w_0 \in \mathcal{W}$ for the unique element of the Weyl group with $w_0(\Delta^+) = -\Delta^+$ and put $m := \sum_{\alpha\in\Delta^+} \lambda(\check\alpha)$. Then one of the following three mutually exclusive cases occurs:*

(1) *$w_0(\lambda) \neq -\lambda$ and $V_\lambda \not\cong V_\lambda^*$.*

(2) *$w_0(\lambda) = -\lambda$ and m is even. Then V_λ is orthogonal.*

(3) *$w_0(\lambda) = -\lambda$ and m is odd. Then V_λ is symplectic.*

Proof. This is contained in [1, Ch. VIII, §7, no. 5, Prop. 12]. We only note that since $\lambda(\check\alpha) \in \mathbb{N}_0$ for all $\alpha \in \Delta^+$ the number m is a non-negative integer. ■

To apply this result to determine the types of representations of compact semisimple Lie algebras, one needs the following:

Proposition III.11. *Let $\mathfrak{g}$ be a compact semisimple Lie algebra and V a simple complex $\mathfrak{g}$-module. Then the following assertions hold:*

(i) *V is of real type if and only if V is orthogonal.*

(ii) *V is of complex type if and only if $V \not\cong V^*$.*

(iii) *V is of quaternionic type if and only if V is symplectic.*

Proof. Since representations of semisimple compact Lie algebras are in one-to-one correspondence with representations of the associated simply connected group which is compact, the assertions follow from [1, Ch. 9, App. 2, no. 2, Prop. 3]. ■

Combining Propositions II.10 and II.11 one can determine the type of each representation of a compact semisimple Lie algebra, so that we have access to the type of the V_2-factor in Theorem III.5. Later we will also derive a similar result for the V_1-factor (Proposition III.16).

Remark III.12. Let us return to the situation of Theorem III.5. Here $\dim \mathfrak{z}(\mathfrak{g}_1) \leq 1$ is a consequence of Schur's Lemma and the injectivity of the representation on V. Then the module V_1 is a simple module for the commutator algebra because the center acts by scalar multiples of the identity. If $\mathfrak{g}_1 = \mathfrak{z}(\mathfrak{g}_1) \cong \mathbb{R}$, then every one-dimensional complex symplectic module is of convex type and if $\mathfrak{g}_1$ is not abelian, then the condition $W_{V_1} \neq \{0\}$ implies that $W_{V_1} \cap [\mathfrak{g}_1, \mathfrak{g}_1] \neq \{0\}$ (cf. Lemma III.6, [6, III]). On the other hand this condition trivially implies $W_{V_1} \neq \{0\}$ so that it only remains to consider the case where $\mathfrak{g}_1$ is simple hermitean. ∎

For the case where $\mathfrak{g}$ is a simple hermitean Lie algebra, the classification has been carried out in [14] where it was based on the following result ([14, Th. V.12]).

Theorem III.13. *Let $\mathfrak{g}$ be a simple hermitean Lie algebra, $\mathfrak{t} \subseteq \mathfrak{k}$ a compactly embedded Cartan algebra, $Z \in i\mathfrak{z}(\mathfrak{k})$ with $\mathrm{Spec}(\mathrm{ad}\, Z) = \{0, 1, -1\}$, and V a simple complex $\mathfrak{g}_{\mathbb{C}}$-module. Let λ denote the highest weight of V with respect to a $\mathfrak{k}$-adapted positive system of roots. Then $W_V \neq \{0\}$ if and only if the following two conditions are satisfied:*

(1) $\lambda(Z) \in]0, 1[$.

(2) *If $\mathcal{W}$ denotes the Weyl group of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, then $(\mathcal{W}.\lambda)(Z)$ consists only of two elements.* ∎

To obtain further general information on the simple modules, we need the following proposition. Note that in view of [14, Th. V.3], every simple complex module of a reductive quasihermitean Lie algebra on which $\mathfrak{z}(\mathfrak{g})$ acts by purely imaginary multiples of the identity admits an invariant pseudohermitean form.

Proposition III.14. *Let V be a simple complex $\mathfrak{g}$-module, where $\mathfrak{g}$ is a quasihermitean non-compact reductive Lie algebra acting effectively on V, and B an invariant non-degenerate pseudohermitean form on V. Then the following are equivalent:*

(1) *$(V, \mathrm{Im}\, B)$ is of convex type.*

(2) *V contains exactly two K-types, i.e., V decomposes under $\mathfrak{k}$ into two simple $\mathfrak{k}$-modules.*

Proof. We first use Theorem III.5 to write V as $V_1 \otimes_{\mathbb{C}} V_2$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_2$ is compact semisimple and $\mathfrak{g}_1$ is reductive with exactly one simple hermitean ideal because $\mathfrak{g}$ is non-compact. Similarly $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{g}_2$ and the simple $\mathfrak{k}$-modules can be obtained as $V_1^j \otimes V_2$, where V_1^j is a simple $\mathfrak{k}_1$-module in V_1. Therefore the numbers of K-types in V and V_1 coincide. In view of Remark III.12, this reduces the problem to the case where $\mathfrak{g}$ is simple hermitean since the center acts by purely imaginary multiples of the identity. From now on we assume that $\mathfrak{g}$ is simple hermitean.

(1) $\Rightarrow$ (2): This follows from Proposition VII.1 in [14].
(2) $\Rightarrow$ (1): We choose $Z \in i\mathfrak{z}(\mathfrak{k})$ such that $\mathrm{Spec}(\mathrm{ad}\, Z) = \{1, 0, -1\}$. Let $V = V^+ \oplus V^-$ be the decomposition of V into simple complex $\mathfrak{k}$-modules and write λ resp. μ for the highest weights of V^+ resp. V^- with respect to a positive system Δ_k^+. We assume that $\Delta_p^+ = \{\alpha \in \Delta: \alpha(Z) > 0\}$, that λ ist the highest weight of the $\mathfrak{g}_{\mathbb{C}}$-module V, and decompose the set $\mathcal{P}_V$ as $\mathcal{P}_{V^+} \cup \mathcal{P}_{V^-}$. Since the $\mathfrak{k}$-modules V^+ and V^- are simple, the center of $\mathfrak{k}$ acts on each space by a scalar multiple of the identity. Hence $\lambda(Z) = \alpha(Z)$ for all $\alpha \in \mathcal{P}_{V^+}$ and $\mu(Z) = \alpha(Z)$ for all $\alpha \in \mathcal{P}_{V^-}$.

Let $\mathcal{W}$ denote the big Weyl group of $\mathfrak{g}$ acting on $\mathfrak{t}$. Since $\mathfrak{g}$ is simple, $\mathfrak{t}$ is a simple $\mathcal{W}$-module and $\mathrm{conv}\,\mathcal{P}_V = \mathrm{conv}(\mathcal{W}.\lambda) \subseteq i\mathfrak{t}^*$. The center of mass of $\mathcal{W}.\lambda$ is a $\mathcal{W}$-fixed point, hence equal to 0 and the affine subspace generated by $\mathcal{W}.\lambda$ is $i\mathfrak{t}^*$. Hence 0 lies in the relative interior of $\mathrm{conv}\,\mathcal{P}_V$ in $i\mathfrak{t}^*$ and we see that this set contains a weight which is positive on Z and one which is negative on Z. So we find in particular that $\lambda(Z) > 0 > \mu(Z)$. Moreover $\mu(Z) = \lambda(Z) - 1$ because $\alpha(Z) = 1$ for all positive non-compact roots since $\mathfrak{g}_{\mathbb{C}}^{\beta}.V^+ = \{0\}$ for $\beta \in \Delta_p^+$ and $\mathfrak{g}_{\mathbb{C}}^{\beta}.V^+ \subseteq V^-$ for $\beta \in -\Delta_p^+$. Therefore $\lambda(Z) \in]0,1[$ and $\langle \mathcal{W}.\lambda, Z\rangle = \{\lambda(Z), \mu(Z)\}$ contains only two elements. Now Theorem III.13 applies. ■

We recall the explicit classification from [14]. We do this by writing down the highest weights which correspond to those simple complex modules which are of convex type. These are always fundamental weights which are denoted by $\omega_1, \ldots, \omega_n$, where n is the rank of $\mathfrak{g}_{\mathbb{C}}$ (cf. [1, Ch. VI, VIII] for more details on roots and weights).

Theorem III.15. *For the simple hermitean Lie algebra $\mathfrak{g}$ the following highest weights correspond to modules of convex type:*

($\mathrm{A_n}$) $\mathfrak{g} = \mathfrak{su}(p,q)$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(p+q,\mathbb{C})$, $n = p+q-1$: ω_1 *and* ω_n.
($\mathrm{A_n}$) $\mathfrak{g} = \mathfrak{su}(n,1)$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n+1,\mathbb{C})$: $\omega_1, \ldots, \omega_n$.
($\mathrm{B_n}$) $\mathfrak{g} = \mathfrak{so}(2n-1,2)$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2n+1,\mathbb{C})$: ω_n.
($\mathrm{C_n}$) $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R})$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(n,\mathbb{C})$: ω_1.
($\mathrm{D_n}$) $\mathfrak{g} = \mathfrak{so}(2n-2,2)$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2n,\mathbb{C})$: ω_{n-1} *and* ω_n.
($\mathrm{D_n}$) $\mathfrak{g} = \mathfrak{so}^*(2n)$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2n,\mathbb{C})$: ω_1.
($\mathrm{E_6}$) $\mathfrak{g} = \mathfrak{e}_{(6,-14)}$: *no weight.*
($\mathrm{E_7}$) $\mathfrak{g} = \mathfrak{e}_{(7,-25)}$: *no weight.* ■

One remarkable consequence of this classification is that the exceptional hermitean simple algebras have no symplectic modules of convex type. Next we study the types of these representations and the possible antilinear isomorphisms because antiisomorphic symplectic $\mathfrak{g}$-modules correspond to the same real module (cf. Theorem II.26).

Proposition III.16. *Let W be a non-zero simple complex symplectic $\mathfrak{g}$-module of convex type. Then*

(i) *W is of real type if and only if W is symplectic, i.e., W carries an invariant complex symplectic form.*

(ii) *W is of quaternionic type if and only if W is orthogonal, i.e., carries an invariant non-degenerate symmetric complex bilinear form.*

Proof. (i) If W is of real type, then $W = V_{\mathbb{C}}$ holds for a real symplectic $\mathfrak{g}$-module V (Lemma II.4) and Proposition II.5 and Lemma II.11 provide an invariant non-degenerate symplectic complex bilinear form $(\Omega_V)_{\mathbb{C}}$ on W. Hence W is symplectic.

If, conversely, W is symplectic and $\Omega_{\mathbb{C}}$ is an invariant complex symplectic form, then we find an antilinear operator A on W such that

$$\Omega_{\mathbb{C}}(v, w) = B(v, Aw)$$

holds for all $v, w \in W$, where B is the pseudo-hermitean form on W with $\Omega = \operatorname{Im} B$. Then A commutes with $\mathfrak{g}$ and the fact that $\operatorname{Im} \Omega_{\mathbb{C}}$ is skew-symmetric yields that $A^\sharp = A$ with respect to the real symplectic form Ω.

Suppose that W is not of real type, i.e., W is simple over $\mathbb{R}$. Then W is of type $\mathbb{C}_I$ or $\mathbb{H}_I$ as a simple real $\mathfrak{g}$-module of convex type (cf. Lemma II.7). Thus $A^\sharp = A$ implies that $A \in \mathbb{R}\mathbf{1}$, contradicting the fact that A is antilinear.
(ii) A real bilinear symmetric form $B' \colon V \times V \to \mathbb{C}$ can be written as

$$B'(v, w) = \Omega(a.v, w) + i\Omega(b.v, w),$$

where $a, b \in \operatorname{Hom}_{\mathfrak{g}}(V)$ satisfy $a^\sharp = -a$ and $b^\sharp = -b$. Moreover, the complex bilinearity of such a form is equivalen to the condition

$$\Omega(aI.v, w) + i\Omega(b.Iv, w) = i\Omega(a.v, w) - \Omega(b.v, w),$$

which means that $b = -aI$. Now $b^\sharp = -b$ entails $-aI = (aI)^\sharp = -Ia^\sharp = Ia$.

This shows that V is orthogonal if and only if there exists $a \in \operatorname{End}_{\mathfrak{g}}(V)$ with $a^\sharp = -a$ and $Ia = -aI$. Since, according to Lemma II.7, V is of real type or of type $\mathbb{C}_I$ or $\mathbb{H}_I$, we see that V is orthogonal if and only if it is of type $\mathbb{H}_I$.

If V is of type $\mathbb{H}_I$, then the form $B'(v, w) = \Omega(J.v, w) + i\Omega(K.v, w)$ satisfies all the requirements. ■

Lemma III.17. *Let V be a simple complex $\mathfrak{g}$-module. Then $\overline{V} \cong V$ if and only if V is of real or quaternionic type. If V is of complex type and a complex symplectic $\mathfrak{g}$-module, then $\overline{V} \cong V^*$.*

Proof. (cf. [1, Ch. IX, App. 2, no. 1]). If V is of real type, then $V = W_{\mathbb{C}}$ for a simple real $\mathfrak{g}$-module W. Hence $V \cong W \oplus W$ as a real $\mathfrak{g}$-module and therefore $\overline{V} \cong W \oplus W \cong V$.

If V is of complex or quaternionic type, then $V \cong \overline{V}$ if and only if $\operatorname{End}_{\mathfrak{g}}(V)$ contains two anticommuting complex structures which is the case if and only if V is of quaternionic type.

Suppose that V is a simple complex symplectic $\mathfrak{g}$-module. Then we use Lemma II.23 to obtain a $\mathfrak{g}$-invariant pseudohermitean form on V. Hence $V^* \cong \overline{V}$. ■

The preceding results provide the information we need to use the tables on simple complex modules to determine their type.

Remark III.18. Here is some additional information on the representations which occur in Theorem III.15.

($\mathrm{A_n}$) $V_{\omega_1} = \mathbb{C}^{n+1}$ and $V_{\omega_n} = \bigwedge^n(\mathbb{C}^{n+1})$ are representations which are also simple over $\mathbb{R}$. For $\mathfrak{g} = \mathfrak{su}(p,q)$, $n = p+q-1$ with $p \geq q \geq 2$, these representations are of complex type. Here $\overline{V_{\omega_1}} \cong V_{\omega_1}^* \cong V_{\omega_n}$. For $\mathfrak{g} = \mathfrak{su}(p,1)$ we have $V_{\omega_k} = \bigwedge^k(\mathbb{C}^{p+1})$. This module is of complex type for $k \neq \frac{p+1}{2}$, and for $k = \frac{p+1}{2}$ it is of real type if k is odd and of quaternionic type otherwise. Therefore $\overline{V_{\omega_{\frac{p+1}{2}}}} \cong V_{\frac{p+1}{2}}$ and for $k \neq \frac{p+1}{2}$ we obtain $\overline{V_{\omega_k}} \cong V_{\omega_k}^* \cong V_{\omega_{n+1-k}}$ ([1, Ch. VIII, §13, no. 1]).

($\mathrm{B_n}$) $V_{\omega_n} = \bigwedge^*(\mathbb{C}^n) \cong \overline{V_{\omega_n}}$ is the spin representation. According to [1, Ch. VIII, §13, no. 2], the module V_{ω_n} is orthogonal for $n = 0,3 \bmod 4$ and symplectic for $n = 1,2 \bmod 4$. Hence it is of real type for $n = 1,2 \bmod 4$ and for $n = 0,3 \bmod 4$ it is quaternionic.

($\mathrm{C_n}$) $V_{\omega_1} = \mathbb{C}^{2n}$ is of real type because $\mathbb{R}^{2n}$ carries no $\mathrm{Sp}(n,\mathbb{R})$-invariant complex structure ([1, Ch. VIII, §13, no. 3]).

($\mathrm{D_n}$) $V_{\omega_{n-1}}$ and V_{ω_n} are the simple constituents of the spin representation on $\bigwedge^*(\mathbb{C}^n)$ which are given by the odd and the even exterior powers of $\mathbb{C}^n$. Here the situation is more complicated than for $\mathrm{B_n}$ ([1, Ch. VIII, §13, no. 4]). For $n = 0,3 \bmod 4$ there exists an invariant symmetric complex bilinear form on the spin representation and a symplectic form for $n = 1,2 \bmod 4$. If n is even, then the restriction of these forms to the simple modules $V_{\omega_{n-1}}$ and V_{ω_n} is non-degenerate and if n is odd, then both are isotropic and in duality. In the latter case both are neither orthogonal nor symplectic. It follows from Proposition III.16 that these modules are of real type for $n = 2 \bmod 4$, of quaternionic type for $n = 0 \pmod 4$, and of complex type for n odd. For n even we therefore have $\overline{V_{\omega_n}} \cong V_{\omega_n}$, $\overline{V_{\omega_{n-1}}} \cong V_{\omega_{n-1}}$, and for n odd we obtain $\overline{V_{\omega_{n-1}}} \cong V_{\omega_{n-1}}^* \cong V_{\omega_n}$.

($\mathrm{D_n}$) V_{ω_1} is the standard representation of $\mathfrak{so}^*(2n)$ on $\mathbb{C}^{2n} \cong \mathbb{H}^n$. It is of quaternionic type ([1, Ch. VIII, §13, no. 4]). ■

IV. (H_1)-homomorphisms of quasihermitean Lie algebras

In this section we explain how modules of convex type are related to equivariant embedding of bounded symmetric domains into the Siegel space, i.e., the hermitean symmetric space of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$. We will always write $\mathfrak{g}$ for a quasihermitean reductive Lie algebra and G for a simply connected group with $\mathbf{L}(G) = \mathfrak{g}$.

Definition IV.1. Let $\mathfrak{g}$ be a reductive quasihermitean Lie algebra. An element $H_0 \in \mathfrak{g}$ is called an H-element if $\mathfrak{z}_{\mathfrak{g}}(H_0) = \ker \operatorname{ad} H_0$ is a maximal compactly embedded subalgebra of $\mathfrak{g}$ and $\operatorname{Spec}(\operatorname{ad} H_0) = \{0, i, -i\}$. The pair $(\mathfrak{g}, H_0)$ is called a reductive Lie algebra of *hermitean type*.

Let $(\mathfrak{g}, H_0)$ be a reductive Lie algebra of hermitean type and set $\mathfrak{k} :=$

$\ker \operatorname{ad} H_0$. Then $\mathfrak{p} := [H_0, \mathfrak{g}]$ is a uniquely determined module complement for $\mathfrak{k}$ in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition. Note that $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{k}$. In the complexification $\mathfrak{g}_{\mathbb{C}}$, the endomorphism $\operatorname{ad} H_0$ is diagonalizable and we obtain

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^-,$$

where $\mathfrak{p}^\pm$ is the $\pm i$-eigenspace of $\operatorname{ad} H_0$. ■

Remark IV.2. Let $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$, where $\mathfrak{z}(\mathfrak{g})$ is the center, $\mathfrak{g}_0$ is the maximal compact semisimple ideal, and $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ are the non-compact simple ideals. Then an element $H_z + H_0 + \sum_{j=1}^k H_j$ is an H-element if and only if $H_0 = 0$ and H_j is an H-element in $\mathfrak{g}_j$. Since every simple hermitean Lie algebra contains exactly 2 H-elements associated to a fixed Cartan decomposition, the number of H-elements associated to a fixed Cartan decomposition in the commutator algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is 2^k. ■

Let $(\mathfrak{g}, H_0)$ be a reductive Lie algebra of hermitean type, $\mathfrak{g}_c$ the maximal compact ideal, and $\mathfrak{g}_n$ its complementary ideal. There exists a maximal closed convex invariant cone $W_{\max} = W_{\max}(H_0) \subseteq \mathfrak{g}$ with $H(W_{\max}) = \mathfrak{g}_c$, $H_0 \in W_{\max}$, and $W_{\max} \cap \mathfrak{t} = C_{\max}$ for every compactly embedded Cartan algebra $\mathfrak{t}$ containing H_0, where the positive system of roots is chosen in such a way that $\Delta_p^+ = \{\alpha \in \Delta : i\alpha(H_0) > 0\}$ (cf. [12, III]). Note that a reductive quasihermitean Lie algebra containing k simple hermitean ideals contains exactly 2^k maximal cones corresponding to the conjugacy classes of H-elements in $\mathfrak{g}'$ and that the cone $W_{\max}(H_0)$ determines the projection of H_0 onto the commutator algebra.

There also exist minimal cones $W_{\min} = W_{\min}(H_0)$ which are contained in the sum of the hermitean simple ideals of $\mathfrak{g}$ and which are uniquely determined by the condition that they contain the $\mathfrak{g}'$-component of the H-element H_0. Therefore we have exactly 2^k such cones and $W_{\min} \cap \mathfrak{t} = C_{\min}$ ([12, III]).

Definition IV.3. Let $(\mathfrak{g}, H_0)$ and $(\widetilde{\mathfrak{g}}, \widetilde{H}_0)$ be two reductive Lie algebras of hermitean type.

(a) A homomorphism $\kappa: \mathfrak{g} \to \widetilde{\mathfrak{g}}$ is called an (H_1)-homomorphism if

$$\kappa \circ \operatorname{ad} H_0 = \operatorname{ad} \widetilde{H}_0 \circ \kappa.$$

Note that this condition is equivalent to the condition that the complex linear extension $\kappa: \mathfrak{g}_{\mathbb{C}} \to \widetilde{\mathfrak{g}}_{\mathbb{C}}$ satisfies $\kappa(\mathfrak{k}_{\mathbb{C}}) \subseteq \widetilde{\mathfrak{k}}_{\mathbb{C}}$ and $\kappa(\mathfrak{p}^\pm) \subseteq \widetilde{\mathfrak{p}}^\pm$.

It is also equivalent to the condition that κ is a *Cayley homomorphism*, i.e., it respects the Cartan involutions, with the additional property that $\kappa \mid_{\mathfrak{p}}: \mathfrak{p} \to \widetilde{\mathfrak{p}}$ is complex linear with respect to the complex structures $J = \operatorname{ad} H_0 \mid_{\mathfrak{p}}$ and $\widetilde{J} = \operatorname{ad} \widetilde{H}_0 \mid_{\widetilde{\mathfrak{p}}}$.

(b) A homomorphism $\kappa: \mathfrak{g} \to \widetilde{\mathfrak{g}}$ is called an (H_2)-homomorphism if $\kappa(H_0) = \widetilde{H}_0$. It is clear that this implies in particular that κ is an (H_1)-homomorphism. ■

Note that composition of (H_1)-homomorphisms yields (H_1)-homomorphisms. We recall from [21] that (H_1)-homomorphisms of Lie algebras of hermitean type (without compact factors) are in one-to-one correspondence with strongly equivariant holomorphic maps of the corresponding bounded symmetric domains. This correspondence is set up by assigning to the (H_1)-homomorphism $\kappa: \mathfrak{g} \to \widetilde{\mathfrak{g}}$ the corresponding induced map $G/K \to \widetilde{G}/\widetilde{K}$.

Lemma IV.4. *Let $(\mathfrak{g}, H_0)$ and $(\widetilde{\mathfrak{g}}, \widetilde{H}_0)$ be two reductive Lie algebras of hermitean type and $\kappa: \mathfrak{g} \to \widetilde{\mathfrak{g}}$ a Lie algebra homomorphism. Write $p_{\mathfrak{z}}$, $p_{\widetilde{\mathfrak{g}}_n}$ resp. $p_{\widetilde{\mathfrak{g}}'}$ for the projection onto the center, $\widetilde{\mathfrak{g}}_n$ resp. the commutator algebra $\widetilde{\mathfrak{g}}'$ of $\widetilde{\mathfrak{g}}$. Then the following are equivalent:*

(1) *κ is an (H_1)-homomorphism.*

(2) *$p_{\widetilde{\mathfrak{g}}'} \circ \kappa$ is an (H_1)-homomorphism with respect to the H-element $p_{\widetilde{\mathfrak{g}}'}(\widetilde{H}_0) \in \widetilde{\mathfrak{g}}'$.*

(3) *$p_{\widetilde{\mathfrak{g}}_n} \circ \kappa$ is an (H_1)-homomorphism with respect to the H-element $p_{\widetilde{\mathfrak{g}}_n}(\widetilde{H}_0)$ of $\widetilde{\mathfrak{g}}_n$.*

For (H_2)-homomorphisms the conclusions $(1) \Rightarrow (2) \Rightarrow (3)$ hold.

Proof. [8, Lemma II.8] ∎

Proposition IV.5. *Let $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ be two reductive quasihermitean Lie algebras and $\kappa: \mathfrak{g} \to \widetilde{\mathfrak{g}}$ a Lie algebra homomorphism. Consider the following conditions:*

(1) *There exists a maximal invariant cone $\widetilde{W}_{\max} \subseteq \widetilde{\mathfrak{g}}$ such that $\kappa^{-1}(\widetilde{W}_{\max})$ is generating in $\mathfrak{g}$ and $\kappa(\mathfrak{z}(\mathfrak{g}))$ is compactly embedded.*

(2) *There exist H-elements $H_0 \in \mathfrak{g}$ and $\widetilde{H}_0 \in \widetilde{\mathfrak{g}}$ such that κ is an (H_1)-homomorphism and $\widetilde{W}_{\max} = W_{\max}(\widetilde{H}_0)$.*

Then the implication $(1) \Rightarrow (2)$ holds and $(2) \Rightarrow (1)$ holds if $\mathfrak{g}$ is semisimple without compact factors or if κ is an (H_2)-homomorphism.

Proof. [8, Proposition II.9] ∎

The preceding proposition can be used as a bridge between (H_1)-homomorphisms and symplectic modules of convex type.

Theorem IV.6. *Let $\mathfrak{g}$ be a reductive quasihermitean Lie algebra and (V, Ω) be a symplectic $\mathfrak{g}$-module of convex type. Then we can choose H-elements in such a way that $\kappa: \mathfrak{g} \to \mathfrak{sp}(V, \Omega)$ is an (H_1)-homomorphism. The converse is true if $\mathfrak{g}$ is semisimple without compact factors or if κ is an (H_2)-homomorphism.*

Proof. Let (V, Ω) be a symplectic vector space and $\mathrm{Sp}(V, \Omega)$ the corresponding symplectic group. Let further $I \in \mathrm{Sp}(V)$ denote a complex structure on V such that $I\Omega: (v, w) \mapsto \Omega(v, I.w)$ is positive definite. Then $\widetilde{H}_0 := \frac{1}{2}I$ is an H-element in the hermitean Lie algebra $\mathfrak{sp}(V, \Omega)$. The associated invariant cones are

$$\widetilde{W}_{\min}(\widetilde{H}_0) = \widetilde{W}_{\max}(\widetilde{H}_0) = \{X \in \mathfrak{sp}(V, \Omega): \varphi(X) \geq 0\},$$

where $\varphi(X) = \frac{1}{2}\Omega(X.v, v)$ is the Hamiltonian function associated to X (cf. Section II). Now let $\mathfrak{g}$ be a reductive quasihermitean Lie algebra and $\kappa: \mathfrak{g} \to \mathfrak{sp}(V, \Omega)$ a homomorphism. Then $W_V = \kappa^{-1}(\widetilde{W}_{\max})$, so that V is a symplectic $\mathfrak{g}$-module of convex type if and only if $\kappa^{-1}(\widetilde{W}_{\max})$ is generating.

Now the assertion is immediate from Proposition IV.5. ∎

Note that Theorem IV.6 explains why the classification in Theorem III.15 and the classification in [20] of the (H_1)-homomorphisms of simple hermitean Lie algebras into $\mathfrak{sp}(V, \Omega)$ yield the same representations.

Remark IV.7. The converse of the conclusion in Theorem IV.6 is false in general. One simply can take a compactly embedded one-dimensional subalgebra $\mathfrak{g}$ of $\mathfrak{sp}(V,\Omega)$ such that $\mathfrak{g} \cap W_{\max} = \{0\}$. Then the inclusion of $\mathfrak{g}$ is trivially an (H_1)-homomorphism but V is not a symplectic $\mathfrak{g}$-module of convex type. ■

V. The classification of Lie algebras with invariant cones

First we build a Lie algebra with invariant cones from basic building blocks. We need:

- a reductive quasihermitean Lie algebra $\mathfrak{l}$,
- a real vector space $\mathfrak{z}_1$,
- a symplectic $\mathfrak{l}$-module of convex type (V,Ω) with $V_{\text{fix}} = \{0\}$ such that $\mathfrak{z}(\mathfrak{l})$ acts effectively on V, and
- an $\mathfrak{l}$-invariant skew-symmetric form $q: V \times V \to \mathfrak{z}_1$.

We define the vector space

$$\mathfrak{g} := \mathfrak{g}(\mathfrak{l}, V, \mathfrak{z}_1, q) := V \times \mathbb{R} \times \mathfrak{z}_1 \times \mathfrak{l}$$

and endow it with the Lie bracket

$$[(v,t,z,X),(v',t',z',X')] = \big(X.v' - X'.v, \Omega(v,v'), q(v,v'), [X,X']\big). \tag{5.1}$$

Theorem V.1. *The space $(\mathfrak{g},[\cdot,\cdot])$ is a Lie algebra containing invariant cones. It is reductive if and only if $V = \{0\}$, and every non-reductive Lie algebra containing invariant cones arises in that way.*

Proof. Let $\mathfrak{t}_{\mathfrak{l}} \subseteq \mathfrak{l}$ be a compactly embedded Cartan algebra. According to Proposition II.22, the only element of V fixed by $\mathfrak{t}_{\mathfrak{l}}$ is 0. Therefore Proposition II.21 in [12] shows that $\mathfrak{g}$ is a Lie algebra and that

$$\begin{aligned}
\mathfrak{z}(\mathfrak{g}) &= \{0\} \times \mathbb{R} \times \mathfrak{z}_1 \times \{0\} \\
\mathfrak{n} &= V \times \mathbb{R} \times \mathfrak{z}_1 \times \{0\} \quad \text{is the nilradical} \\
\mathfrak{r} &= V \times \mathbb{R} \times \mathfrak{z}_1 \times \mathfrak{z}(\mathfrak{l}) \quad \text{is the radical, and} \\
\mathfrak{t} &= \{0\} \times \mathbb{R} \times \mathfrak{z}_1 \times \mathfrak{t}_{\mathfrak{l}}
\end{aligned}$$

is a compactly embedded Cartan algebra.

We choose a regular element $X \in \mathfrak{t}$ such that $\alpha(X) \neq 0$ holds for all $\alpha \in \mathcal{P}_V$ and B is positive definite on $V_{\mathbb{C}}^{\alpha}$ if and only if $i\alpha(X) < 0$ (Proposition II.23). Let V^+ denote the sum of all those weight spaces for which $i\alpha(X) < 0$. Then the mapping $v \mapsto v + \overline{v}, V^+ \to V$ is a bijection and we can use it to define a complex structure I on V by $I.(v+\overline{v}) := i(v - \overline{v})$. Let $v \in V_{\mathbb{C}}^{\alpha} \subseteq V^+$. Then

$$\Omega\big(I.(v+\overline{v}), v+\overline{v}\big) = \Omega\big(i(v-\overline{v}), v+\overline{v}\big) = 2i\Omega_{\mathbb{C}}(v,\overline{v}) = 2B(v,v) > 0$$

for $v \neq 0$. In view of Proposition III.13 in [12], this proves that $\mathfrak{g}$ has strong cone potential, so that Theorem III.39 in [12] shows that $\mathfrak{g}$ contains invariant cones.

If, conversely, $\mathfrak{g}$ is a non-reductive Lie algebra containing invariant cones, let $\mathfrak{t} \subseteq \mathfrak{g}$ be a compactly embedded Cartan algebra, $\mathfrak{n}$ the nilradical of $\mathfrak{g}$, $\mathfrak{s}$ a $\mathfrak{t}$-invariant Levi algebra and $\mathfrak{t}_1 \subseteq \mathfrak{t}$ a complement for the center of $\mathfrak{g}$ (cf. [12, Prop. II.11]). We set $\mathfrak{l} := \mathfrak{t}_1 + \mathfrak{s}$ and $V := [\mathfrak{t}, \mathfrak{n}]$. Note that V is invariant under $\mathfrak{l}$ ([12, Prop. II.11]). So far we have only used that $\mathfrak{g}$ contains a compactly embedded Cartan algebra.

Since $\mathfrak{g}$ contains invariant cones, it has strong cone potential and therefore there exists a $\mathfrak{k}$-adapted positive system Δ^+ and a linear functional $\omega \in \mathfrak{z}(\mathfrak{g})^*$ such that

$$\omega(i[\overline{X}_\alpha, X_\alpha]) > 0 \tag{5.2}$$

whenever $0 \neq X_\alpha \in \mathfrak{n}_{\mathbb{C}} \cap \mathfrak{g}_{\mathbb{C}}^\alpha$ (cf. Prop. III.15 and Th. III.39 in [12]). It follows in particular that $\mathfrak{g}$ and $\mathfrak{l}$ are quasihermitean (cf. Definition II.18). Now we define $\mathfrak{z}_1 := \ker \omega \cap \mathfrak{z}$, identify $\mathfrak{z}$ with $\mathbb{R} \times \mathfrak{z}_1$ via ω, and define Ω and q by

$$[v, w] = \big(\Omega(v, w), q(v, w)\big),$$

where $\Omega(v, w) = \omega([v, w])$. Note that this works since $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}(\mathfrak{g})$ by [12, Prop. II.10]. Now Ω is an $\mathfrak{l}$-invariant symplectic form on V and, in view of (5.2), Proposition II.23 shows that V is of convex type. Finally

$$\mathfrak{g} = V \times \mathbb{R} \times \mathfrak{z}_1 \times \mathfrak{l}$$

and all brackets are as in (5.1). This proves the theorem. ∎

Remark V.2. A reductive Lie algebra $\mathfrak{g}$ containing invariant cones is simply a quasihermitean reductive Lie algebra which is not compact semisimple. Therefore Theorem V.1 provides an explicit description of all Lie algebras with invariant cones. ∎

Corollary V.3. *Let $\mathfrak{g}$ be a Lie algebra with invariant cones and $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ a Levi decomposition. Then all exceptional ideals of $\mathfrak{s}$ are ideals of $\mathfrak{g}$ and therefore split as ideal direct summands.*

Proof. It follows from the classification of modules of convex type that an exceptional ideal $\mathfrak{a}$ of $\mathfrak{s}$ acts trivially on $[\mathfrak{s}, \mathfrak{r}] \subseteq [\mathfrak{t}, \mathfrak{n}]$. Therefore they act trivially on $\mathfrak{r}$ and this implies the assertion. ∎

Remark V.4. Note that the representation of a Lie algebra with invariant cones as in Theorem V.1 depends on

- the choice of $\mathfrak{l}$, and
- the choice of a suitable decomposition, i.e., a non-zero linear functional on the center.

To see that the choice of $\mathfrak{l}$ is inessential, we recall that all compactly embedded Cartan subalgebras of $\mathfrak{g}$ are conjugate and that each compactly embedded Cartan algebra $\mathfrak{t}$ uniquely determines a $\mathfrak{t}$-invariant Levi complement $\mathfrak{s}$. Let $\mathfrak{l}, \widetilde{\mathfrak{l}} \subseteq \mathfrak{t} + \mathfrak{s}$ be two different choices of an $\mathfrak{l}$-subalgebra. Let $\psi: \mathfrak{s} + \mathfrak{t} \to (\mathfrak{s} + \mathfrak{t})/\mathfrak{z}$

denote the canonical quotient homomorphism. Then $\psi\,|_{\mathfrak{l}}$ and $\psi\,|_{\widetilde{\mathfrak{l}}}$ are isomorphisms and we obtain an isomorphism

$$\widetilde{\psi} := (\psi\,|_{\widetilde{\mathfrak{l}}})^{-1} \circ \psi\,|_{\mathfrak{l}} : \mathfrak{l} \to \widetilde{\mathfrak{l}}.$$

The corresponding modules are given by $V = [\mathfrak{l}, \mathfrak{n}] = [\mathfrak{t}, \mathfrak{n}] = [\widetilde{\mathfrak{l}}, \mathfrak{n}] = \widetilde{V}$ and one easily checks that if ρ resp. $\widetilde{\rho}$ are the representations of $\mathfrak{l}$ resp. $\widetilde{\mathfrak{l}}$ on V, then $\widetilde{\rho} \circ \widetilde{\psi} = \rho$. Therefore the choice of $\mathfrak{l}$ is inessential and it determines the $\mathfrak{l}$-module V. ∎

The following observations shed some light on the choice of the functional $\beta \in \mathfrak{z}^*$. Here $\mathfrak{g}$ denotes a Lie algebra containing invariant cones and $\mathfrak{t}$ a fixed compactly embedded Cartan subalgebra.

Lemma V.5. *If Ω is an $\mathfrak{l}$-invariant symplectic structure on V such that (V, Ω) is a symplectic $\mathfrak{l}$-module of convex type, then there exists a $\mathfrak{k}$-adapted positive system Δ^+ such that B is negative definite on $V_{\mathbb{C}}^{\alpha}$ if and only if $\alpha \in \Delta_r^+$* (Definition II.18). *Here the set Δ_r^+ is uniquely determined by the isomorphy class of (V, Ω) as a symplectic $\mathfrak{l}$-module.*

Proof. In view of Remark II.10, it is clear that the requirements on the set Δ_r^+ determine the isomorphy class of (V, Ω) as a symplectic $\mathfrak{l}$-module uniquely.

Now assume that (V, Ω) is of convex type. Using Proposition II.23, we find $X \in \mathfrak{z}(\mathfrak{k})$ such that the pseudohermitean form B is negative definite on V^{α} if and only if $i\alpha(X) > 0$. Next we choose $X' \in \mathfrak{z}(\mathfrak{k})$ in such a way that $\alpha(X) \neq 0$ holds for all $\alpha \in \Delta_p$ and that the signs of $i\alpha(X)$ and $i\alpha(X')$ coincide for $\alpha \in \Delta_r$. Then $\Delta_p^+ := \{\alpha \in \Delta_p : i\alpha(X') > 0\}$ is the system of positive non-compact roots for a $\mathfrak{k}$-adapted positive system Δ^+ satisfying our requirements. ∎

Proposition V.6. *Let $\mathfrak{g}$ be a Lie algebra containing invariant cones and $\mathfrak{t} \subseteq \mathfrak{g}$ a compactly embedded Cartan subalgebra. For a $\mathfrak{k}$-adapted positive system Δ^+ we put*

$$C_{\min,\mathfrak{z}} := \operatorname{cone}\{i[\overline{X}_{\alpha}, X_{\alpha}] : \alpha \in \Delta_r^+, X_{\alpha} \in \mathfrak{g}_{\mathbb{C}}^{\alpha} \cap \mathfrak{n}_{\mathbb{C}}\}$$

and for $\beta \in \mathfrak{z}^*$ we write $\Omega_{\beta}(v, w) := \beta([v, w])$ for the associated $\mathfrak{l}$-invariant skew-symmetric form on $V := [\mathfrak{t}, \mathfrak{n}]$. Then the following assertions hold:

(i) *(V, Ω_{β}) is an $\mathfrak{l}$-module of convex type if and only if $\beta \in \operatorname{int} C_{\min,\mathfrak{z}}^{\star}$ for a $\mathfrak{k}$-adapted positive system.*

(ii) *Two symplectic $\mathfrak{l}$-modules of convex type (V, Ω_{β}) and (V, Ω_{γ}) are isomorphic if and only if the associated systems Δ_r^+ coincide.*

(iii) *If (V, Ω) is a symplectic $\mathfrak{l}$-module of convex type and Δ^+ a $\mathfrak{k}$-adapted positive system such that B is negative definite on $V_{\mathbb{C}}^{\alpha}$ if and only if $\alpha \in \Delta_r^+$, then there exists a $\beta \in \mathfrak{z}^*$ such that $(V, \Omega) \cong (V, \Omega_{\beta})$ if and only if $C_{\min,\mathfrak{z}}$ is a pointed cone.*

Proof. (i) For $X_{\alpha} \in V_{\mathbb{C}}^{\alpha} = \mathfrak{g}_{\mathbb{C}}^{\alpha} \cap \mathfrak{n}_{\mathbb{C}}$ we have

$$i\Omega_{\beta,\mathbb{C}}(X_{\alpha}, \overline{X}_{\alpha}) = i\beta([X_{\alpha}, \overline{X}_{\alpha}]) = -\beta(i[\overline{X}_{\alpha}, X_{\alpha}]).$$

We choose the $\mathfrak{k}$-adapted positive system Δ^+ as in Lemma V.5. Then the pseudohermitean form B_β is negative definite on V^α if and only if $\alpha \in \Delta_r^+$. For $\alpha \in \Delta_r^+$ we then have $\beta(i[\overline{X}_\alpha, X_\alpha]) > 0$ for $X_\alpha \neq 0$. Hence $\beta \in \operatorname{int} C^\star_{\min,\mathfrak{z}}$.

This proves the first half of (i). The other part has already been shown in the proof of Theorem V.1.

(ii) If the symplectic $\mathfrak{g}$-modules (V, Ω_β) and (V, Ω_γ) are isomorphic, then the forms B_β and B_γ are positive definite on the same weight spaces V^α. As we have seen in (i), this property entails that $\beta, \gamma \in C^\star_{\min,\mathfrak{z}}$ for the same system Δ_r^+.

To see the converse, we note that the positive system Δ_r^+ fixes the signs of B on the weight spaces $V_{\mathbb{C}}^\alpha$, hence, in view of Remark II.10, it also determines the isomorphy class of the symplectic $\mathfrak{g}$-module (V, Ω_β). It follows in particular that $(V, \Omega_\beta) \cong (V, \Omega_{\beta'})$.

(iii) It is clear that $(V, \Omega) \cong (V, \Omega_\beta)$ if and only if $\beta \in \operatorname{int} C^\star_{\min,\mathfrak{z}}$. Such a functional β exists if and only if the cone $C_{\min,\mathfrak{z}}$ is pointed. ∎

Remark V.7. In general there can be more symplectic structures on V turning it into a symplectic module of convex type as one can obtain by forms of the type Ω_β, $\beta \in \operatorname{int} C^\star_{\min,\mathfrak{z}}$. A typical example can be obtained as follows. Let $(V, \Omega) = (\mathbb{R}^2, \Omega_0)^2$ as module of $\mathfrak{sl}(2,\mathbb{R})^2$, where Ω_0 is any symplectic form on $\mathbb{R}^2$. We form the associated Lie algebra $\mathfrak{g} = V \times \mathbb{R} \times \mathfrak{l}$ as in Theorem V.1 with $\mathfrak{z}_1 = \{0\}$. Then, as we have seen in Remark II.10, the $\mathfrak{l}$-module V permits 4 different classes of symplectic structures turning it into an $\mathfrak{l}$-module of convex type. Accordingly there exist 4 different systems Δ_r^+ for $\mathfrak{g}$, but for only two of them the cone $C_{\min,\mathfrak{z}}$ is pointed. ∎

Remark V.8. If we consider only those non-reductive Lie algebras with invariant cones where $\dim \mathfrak{z} = 1$ and $\mathfrak{l}$ is given, then their isomorphy classes correspond to isomorphy classes of symplectic $\mathfrak{l}$-modules of convex type. Therefore the open problem mentioned in Remark II.10 is directly related to the classification of this type of Lie algebras. In view of Lemma V.5 we know at least that we can parametrize the isomorphy classes by a certain set of positive systems Δ_r^+. ∎

Relations to Siegel domains

Definition V.9. Let (V, A, I) be a symplectic vector space endowed with a complex structure I such that $IA(v,w) := A(v, Iw)$ is a positive definite symmetric bilinear form on V.

Let further U be a real vector space, $C \subseteq U$ an open convex cone which contains no non-trivial affine subspaces, $e \in \operatorname{int} C^\star$, and $\widehat{A}: V \times V \to U$ a skew-symmetric bilinear map such that the forms $A_u(v,w) := \langle u, \widehat{A}(v,w)\rangle$ satisfy the condition $A_e = A$ and IA_u is positive definite for $u \in \operatorname{int} C^\star$. Note that this condition is equivalent to $\widehat{A}(v, Iv) \in \overline{C} \setminus \{0\}$ for $v \in V \setminus \{0\}$.

With the data $(V, \widehat{A}, U, C)$ given, we define the *Siegel domain*

$$\mathcal{S} := \mathcal{S}(V, \widehat{A}, U, C) := \{(u,v) \in U_{\mathbb{C}} \times V : \operatorname{Im} u - \widehat{A}(v, Iv) \in C\}.$$

Note that this is a convex domain in the complex vector space $U_{\mathbb{C}} \times V$ (cf. [21, p.128]). Accordingly we say that $(V, \widehat{A}, U, C)$ is a *Siegel data.* We write

$$\mathrm{Sp}(V, \widehat{A}) := \{g \in \mathrm{End}_{\mathbb{R}}(V) \colon (\forall v, w \in V)\widehat{A}(g.v, g.w) = \widehat{A}(v, w)\}.$$

Then $\mathrm{Sp}(V, \widehat{A})$ is a reductive quasihermitean real algebraic group and the corresponding symmetric domain is the *generalized Siegel space* $\mathfrak{S}(V, \widehat{A}, C)$ consisting of all complex structures J on V such that JA_u is positive definite for all $u \in \operatorname{int} C^\star$. ∎

We recall that $\mathfrak{S}(V, \widehat{A}, C)$ does not depend on the cone C (cf. [21]). One only needs the assumption that there *exists* a cone C satisfying the requirements. Written as a condition on the U-valued map $\widehat{A}$, this means that

(C1) $\widehat{A}(v, Iv) \neq 0$ for $v \in V \setminus \{0\}$,

(C2) $D_{\widehat{A}} := \mathrm{cone}\{\widehat{A}(v, Iv) \colon v \in V\} \subseteq U$ is pointed, and

(C3) $A_e = A$ for an element $e \in \mathrm{int}(D_{\widehat{A}})^\star$.

It is clear that the requirements from above imply (C1)-(C3). Suppose, conversely, that (C1)-(C3) are satisfied. Then IA_u is positive definite for all $u \in \mathrm{int}(D_{\widehat{A}})^\star$, hence in particular for some pointed generating cone $C^\star \subseteq (D_{\widehat{A}})^\star$. Then $C := \mathrm{int}(C^\star)^\star$ satisfies the requirements.

Theorem V.10. *The tuple $(V, \widehat{A}, U)$ is a Siegel data if and only if the associated generalized Heisenberg algebra*

$$\mathfrak{n} = V \times U \quad \textit{with} \quad [(v, u), (v', u')] = \big(0, \widehat{A}(v, v')\big)$$

occurs as the nilradical of a Lie algebra containing invariant cones.

Proof. Suppose that $(V, \widehat{A}, U)$ is a Siegel data and that $C \subseteq U$ is an appropriate open pointed convex cone. We define

$$\mathfrak{g} := \mathfrak{n} \rtimes \mathfrak{sp}(V, \widehat{A}),$$

where $\mathfrak{sp}(V, \widehat{A})$ acts canonically on V and trivially on U. Then it is easy to see that $\mathfrak{g}$ is a Lie algebra whose nilradical is $\mathfrak{n}$. We claim that $\mathfrak{g}$ contains invariant cones. In view of Theorem V.1, it suffices to show that the symplectic $\mathfrak{sp}(V, \widehat{A})$-module (V, A_e) for an element $e \in C$ is of convex type. Since the inclusion $\mathfrak{sp}(V, \widehat{A}) \to \mathfrak{sp}(V, A)$ is an (H_2)-homomorphism, this follows from Theorem IV.6.

Suppose, conversely, that $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{l}$ is a Lie algebra containing invariant cones (cf. Theorem V.1). We choose a compactly embedded Cartan algebra $\mathfrak{t} \subseteq \mathfrak{g}$ and a $\mathfrak{k}$-adapted positive system of roots such that the cone $C_{\min}$ is pointed (cf. [12, III]). Let $V^+ := \sum_{\alpha \in \Delta_r^+} \mathfrak{g}_{\mathbb{C}}^\alpha$. Then the $\mathbb{R}$-linear isomorphism $V \to V^+$ can be used to obtain a complex structure I on V. For $X, Y \in V^+$ we then have

$$\widehat{A}\big(X + \overline{X}, I(X + \overline{X})\big) = [X + \overline{X}, iX - i\overline{X}] = 2[i\overline{X}, X] \in C_{\min,\mathfrak{z}},$$

where $C_{\min,\mathfrak{z}} := C_{\min} \cap \mathfrak{z}$. Since $\mathfrak{g}$ has cone potential, $\widehat{A}(v, Iv) \neq 0$ for $0 \neq v \in V$, i.e., (C1) is satisfied. Since $D_{\widehat{A}} = C_{\min,\mathfrak{z}}$ is pointed, the condition (C2) is also satisfied. Fixing $e \in \operatorname{int} C^\star$, we therefore obtain a symplectic structure A_e on V such that IA_e is positive definite and therefore (C3) holds. This completes the proof. ∎

In view of Theorem V.10, the groups $(V \oplus U) \rtimes \mathfrak{sp}(V, \widehat{A})$ are the prototypes of non-reductive Lie algebras with invariant cones since for a given generalized Heisenberg algebra $\mathfrak{n} = V \oplus U$ all algebras $\mathfrak{n} \rtimes \mathfrak{l}$ which contain invariant cones arise by certain (H_1)-homomorphisms $\kappa: \mathfrak{l} \to \mathfrak{sp}(V, \widehat{A})$ satisfying the condition that $\kappa^{-1}(W_{\max})$ is generating (cf. Theorem V.1).

Parabolic subgroups and Lie algebras with invariant cones

We have already seen that the nilpotent radicals which occur in Lie algebras with invariant cone are precisely those generalized Heisenberg algebras which arise as Siegel data. The connection between non-reductive Lie algebras with invariant cones and symmetric domains is much deeper as we will explain in this subsection.

Let $(\mathfrak{g}, H_0)$ be a simple Lie algebra of hermitean type and

$$\kappa: \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}$$

an (H_1)-homomorphism.

We put

$$H_\kappa := \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad X_\kappa := \kappa \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We write $\mathfrak{b}_\kappa = (U + V) \rtimes \mathfrak{z}_\mathfrak{g}(H_\kappa)$, where

$$U = \mathfrak{g}(\operatorname{ad} H_\kappa; 2) \quad \text{and} \quad V = \mathfrak{g}(\operatorname{ad} H_\kappa; 1).$$

Then $\mathfrak{b}_\kappa$ is a maximal parabolic subalgebra of $\mathfrak{g}$ (cf. [21]) and $\mathfrak{n} := U + V$ is a generalized Heisenberg algebra since U is central in $\mathfrak{n}$ and $[V, V] \subseteq U$.

The following facts can be found in [21, p.113]. We have

$$\mathfrak{z}_\mathfrak{g}(H_\kappa) = \mathfrak{g}_\kappa^{(1)} \oplus \mathfrak{g}_\kappa^{(2)},$$

where $\mathfrak{g}_\kappa^{(1)} = \mathfrak{l}_2 \oplus \mathfrak{g}_\kappa$ and $\mathfrak{g}_\kappa^{(2)} = \mathbb{R} H_\kappa \oplus \mathfrak{g}_\kappa^{(2)\prime}$, where $\mathfrak{l}_2$ is a compact ideal in $\mathfrak{z}_\mathfrak{g}(H_\kappa)$ which is maximal with respect to the property of being compactly embedded in $\mathfrak{g}$, $\mathfrak{g}_\kappa$ is simple hermitean, and the commutator algebra of $\mathfrak{g}_\kappa^{(2)}$ is either $\{0\}$ or simple and non-compact.

Let A denote the symplectic structure on V given by the element $X_\kappa \in U$ by

$$A(v, v') = -\tfrac{1}{4} \langle X_\kappa, [v, v'] \rangle.$$

Then we obtain an (H_2)-homomorphism $\rho_V: \mathfrak{g}_\kappa^{(1)} \to \mathfrak{sp}(V, A)$ with respect to $\frac{1}{2} I \in \mathfrak{sp}(V, A)$, and $[\mathfrak{g}_\kappa^{(1)}, U] = \{0\}$.

We identify U with U^* via the scalar product given by

$$\langle X, Y \rangle := -B(X, \theta Y),$$

where θ is a Cartan involution of $\mathfrak{g}$ and B is the Cartan-Killing form. Then the orbit of X_κ under the group $G_\kappa^{(2)} := \langle \exp \mathfrak{g}_\kappa^{(2)} \rangle$ in U is an irreducible self-dual homogeneous cone, denoted C, and we have a skew-symmetric mapping

$$\widehat{A}: V \times V \to U, \quad (X, Y) \mapsto -\tfrac{1}{4}[X, Y]$$

satisfying $I A_u = I(u \circ \widehat{A}) \gg 0$ for every $u \in C$.

Proposition V.11. *The Lie algebra* $\mathfrak{p}_\kappa := \mathfrak{n} \rtimes \mathfrak{g}_\kappa^{(1)}$ *contains invariant cones,* $\mathfrak{n}$ *is its nilradical, and* $\mathfrak{g}_\kappa^{(1)}$ *contains one hermitean factor.*

Proof. If $V = \{0\}$, then $\mathfrak{p}_\kappa$ is reductive quasihermitean and there is nothing to prove. Assume that $V \neq \{0\}$.

We know already that $\mathfrak{g}_\kappa^{(1)}$ is reductive and quasihermitean with one hermitean factor, and since $[\mathfrak{g}_\kappa^{(1)}, U] = \{0\}$, we have $\rho_V(\mathfrak{g}_\kappa^{(1)}) \subseteq \mathfrak{sp}(V, \widehat{A})$.

From the positive definiteness of the form IA_u for all $u \in C$, it follows that $(V, \widehat{A}, U, C)$ is a Siegel data. Moreover $\mathfrak{g}_\kappa^{(1)}$ acts effectively with $V_{\rm fix} = \{0\}$ (cf. [Sa80, p.112]). In view of Theorem V.1, it remains to show that V is a symplectic $\mathfrak{g}_\kappa^{(1)}$-module of convex type which follows from the observation that ρ_V is an (H_2)-homomorphism (Theorem V.10). ∎

The preceding result shows how Lie algebras containing invariant cones arise as subalgebras of maximal parabolic subalgebras of hermitean simple Lie algebras. The size of $\mathfrak{g}_\kappa^{(1)}$ in $\mathfrak{z}_\mathfrak{h}(H_\kappa)$ depends on the rank of the homomorphism κ (cf. [21]). If the rank is minimal, then $\mathfrak{g}_\kappa^{(2)} = \mathbb{R}H_\kappa$ ([21, p.113]), so that $\mathfrak{z}_\mathfrak{g}(H_\kappa) = \mathbb{R}H_\kappa \oplus \mathfrak{g}_\kappa^{(1)}$ and therefore $\mathfrak{b}_\kappa = \mathfrak{p}_\kappa \rtimes \mathbb{R}H_\kappa$. Moreover, the inspection of the restricted root system of $\mathfrak{g}$ shows that in this case $U = \mathbb{R}X_\kappa$, hence $\mathfrak{n}$ is a Heisenberg algebra. Moreover, the representation ρ_V of $\mathfrak{g}_\kappa^{(1)}$ is irreducible over $\mathbb{R}$ ([21, p.112]). Therefore the $\mathfrak{g}_\kappa^{(1)}$-module V is contained in the list of representations in Theorem III.15.

It is instructive to have a closer look at this situation. The situation is relatively simple for the class of those hermitean algebras arising via skewhermitean forms.

Example V.12. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and V a $\mathbb{K}$-left vector space endowed with a non-degenerate skewhermitean form $\langle \cdot, \cdot \rangle$, i.e.,

$$\overline{\langle v, w\rangle} = -\langle w, v\rangle, \quad \lambda\langle v, w\rangle = \langle \lambda v, w\rangle \quad \text{and} \quad \langle v, \lambda w\rangle = \langle v, w\rangle\overline{\lambda}.$$

The basic bulding block for such vector spaces is $\mathbb{K}^2$ endowed with the form $\langle (x,y),(x',y')\rangle := x\overline{y'} - y\overline{x'}$. We write $\mathfrak{su}(V)$ for the Lie algebra of the group $\mathrm{SU}(V) := \mathrm{U}(V) \cap \mathrm{Sl}(V)$, where $\mathrm{U}(V)$ denotes the group of all $\mathbb{K}$-linear isometries.

Up to equivalence, we have the following cases:

(1) $\mathbb{K} = \mathbb{R}$, $V = \mathbb{R}^{2n}$, $\mathfrak{su}(V) = \mathfrak{sp}(n, \mathbb{R})$.

(2) $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^{p+q}$, $\mathfrak{su}(V) = \mathfrak{su}(p,q)$.

(3) $\mathbb{K} = \mathbb{H}$, $V = \mathbb{H}^n$, $\mathfrak{su}(V) = \mathfrak{so}^*(2n)$.

Note that this covers all simple hermitean Lie algebras up to $\mathfrak{so}(n,2)$ and the two exceptional ones.

Let $v_0 \in V$ be a non-zero isotropic vector. We choose $v_1 \in V$ with $\langle v_1, v_0\rangle = 1$. Set $V' := v_0^\perp \cap v_1^\perp$. Then $V \cong \mathbb{K}v_0 \oplus \mathbb{K}v_1 \oplus V' \cong \mathbb{K}^2 \oplus V'$. For $\mathbb{K}^2$, the real rank of $\mathfrak{su}(\mathbb{K}^2)$ is 1 and the corresponding (H_1)-homomorphism $\kappa\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{su}(\mathbb{K}^2)$ is the natural inclusion. We have in particular

$$H_\kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using the embedding $\mathfrak{su}(\mathbb{K}^2) \to \mathfrak{su}(V)$ obtained by splitting the $\mathbb{K}^2$-factor, we see that

$$\mathfrak{z}_{\mathfrak{su}(V)}(H_\kappa) = \mathbb{R}H_\kappa \oplus \mathfrak{su}(V'), \quad \mathfrak{p}_\kappa \cong (V' + \mathbb{R}X_\kappa) \rtimes \mathfrak{su}(V')$$

and

$$\mathfrak{b}_\kappa = \{X \in \mathfrak{su}(V) \colon X.v_0 \in \mathbb{K}v_0\}. \qquad \blacksquare$$

Example V.13. For the exceptional hermitean algebra $\mathfrak{g} = \mathfrak{e}_{(6,-14)}$ one finds for κ of rank 1 that $\mathfrak{g}_\kappa^{(1)} \cong \mathfrak{su}(5,1)$ and that V is a real module of dimension 20, hence a real form of $\bigwedge^3(\mathbb{C}^6)$. $\blacksquare$

Example V.14. For the exceptional hermitean algebra $\mathfrak{g} = \mathfrak{e}_{(7,-25)}$ one finds for κ of rank 1 that $\mathfrak{g}_\kappa^{(1)} \cong \mathfrak{so}(2,10)$ and that V is a real module of dimension 32, hence a factor of the corresponding spin representation. $\blacksquare$

Example V.15. For $\mathfrak{g} = \mathfrak{so}(2,n)$ we obtain $\mathfrak{g}_\kappa^{(1)} \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(n-2)$ and that V is a real module of dimension $2(n-2)$. More precisely, it is the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^{n-2}$, where the factors are endowed with the natural representations. $\blacksquare$

VI. More on the structure of Lie algebras with invariant cones

The following theorem was the original motivation for us to consider the structure of a Lie algebra with invariant cones from the point of view presented in Section II.

Theorem VI.1. *Let $\mathfrak{g}$ be a Lie algebra with invariant cones, $\mathfrak{t} \subseteq \mathfrak{g}$ a compactly embedded Cartan algebra, and $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ a $\mathfrak{t}$-invariant Levi decomposition. Let $\alpha, \beta \in \Delta$ with $\mathfrak{g}_{\mathbb{C}}^\alpha \subseteq \mathfrak{r}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}^\beta \subseteq \mathfrak{s}_{\mathbb{C}}$, and $X_\beta \in \mathfrak{g}_{\mathbb{C}}^\beta$ with*

$$\beta([X_\beta, \overline{X}_\beta]) = 2.$$

Then

$$\alpha([X_\beta, \overline{X}_\beta]) \in \{-1, 0, 1\}.$$

Proof. Let $\mathfrak{l}$ be the intersection of $\mathfrak{g}$ with the smallest conjugation invariant subalgebra containing $\mathfrak{g}_{\mathbb{C}}^\beta$. Then $\mathfrak{l} \cong \mathfrak{sl}(2,\mathbb{R})$ ([12, Prop. II.8]). Therefore Proposition II.28 applies to the module $\mathfrak{m} = [\mathfrak{t}, \mathfrak{r}]$ of convex type (Theorem V.1) and it follows that every non-trivial simple $\mathfrak{l}$-submodule is of dimension at most 2. Now α is a $\mathfrak{t}_{\mathbb{C}}$-weight of $\mathfrak{m}_{\mathbb{C}}$ and therefore the restriction of these weights to $\mathfrak{l}_{\mathbb{C}} \cap \mathfrak{t}_{\mathbb{C}}$ are the corresponding weights for $\mathfrak{l}$. But $\mathfrak{m}$ contains only two types of simple $\mathfrak{l}$-submodules: one-dimensional trivial modules and two-dimensional modules. Therefore $\alpha([X_\beta, \overline{X}_\beta]) \in \{-1, 0, 1\}$. $\blacksquare$

Theorem VI.1 has some important consequences for the structure of Lie algebras with invariant cones.

Corollary VI.2. *Let $\mathfrak{g}$ be a Lie algebra with invariant cones, Δ^+ a $\mathfrak{k}$-adapted positive system of roots, $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ a $\mathfrak{t}$-invariant Levi decomposition, and $\alpha, \beta \in \Delta_p^+$ such that $\mathfrak{g}_{\mathbb{C}}^\alpha \subseteq \mathfrak{r}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}^\beta \subseteq \mathfrak{s}_{\mathbb{C}}$. Then the following assertions hold:*

(i) $[\mathfrak{g}_{\mathbb{C}}^\alpha, \mathfrak{g}_{\mathbb{C}}^\beta] = \{0\}$.

(ii) $[\mathfrak{g}_{\mathbb{C}}^{-\alpha}, \mathfrak{g}_{\mathbb{C}}^{-\beta}] = \{0\}$.

(iii) $[\mathfrak{g}_{\mathbb{C}}^\alpha, \mathfrak{g}_{\mathbb{C}}^{-\beta}] = \{0\}$ *or* $\alpha - \beta \in -\Delta_p^+$.

(iv) *The subalgebra* $\mathfrak{p}_{\mathbb{C}}^+ := \sum_{\alpha \in \Delta_p^+} \mathfrak{g}_{\mathbb{C}}^\alpha$ *is abelian.*

Proof. (i) Let $X_\beta \in \mathfrak{g}_{\mathbb{C}}^\beta$ with $\beta([X_\beta, \overline{X}_\beta]) = 2$. Then

$$\alpha([X_\beta, \overline{X}_\beta]) \in \{0, 1\} \tag{6.1}$$

since α is a positive non-compact root and Δ^+ is $\mathfrak{k}$-adapted which implies that $C_{\min} \subseteq C_{\max}$ (cf. [12, Th. III.20]).

Suppose that $\{0\} \neq [\mathfrak{g}_{\mathbb{C}}^\alpha, \mathfrak{g}_{\mathbb{C}}^\beta] \subseteq \mathfrak{g}_{\mathbb{C}}^{\alpha+\beta}$. Then $\alpha + \beta \in \Delta_p^+$ and $(\alpha + \beta)([X_\beta, \overline{X}_\beta]) \geq 2$. This contradicts Theorem VI.1.

(ii) This follows from (i) by interchanging Δ^+ and $-\Delta^+$.

(iii) If

$$\{0\} \neq [\mathfrak{g}_{\mathbb{C}}^\alpha, \mathfrak{g}_{\mathbb{C}}^{-\beta}] \subseteq \mathfrak{g}_{\mathbb{C}}^{\alpha-\beta} \subseteq \mathfrak{r}_{\mathbb{C}},$$

then $\alpha - \beta \in \Delta_p$ and $(\alpha - \beta)([X_\beta, \overline{X}_\beta]) \leq -1$, so that $\alpha - \beta \in -\Delta_p^+$ follows from (6.1).

(iv) It follows from (i) that root spaces in $\mathfrak{r}_{\mathbb{C}}$ and those in $\mathfrak{s}_{\mathbb{C}}$ commute. On the other hand those in $\mathfrak{r}_{\mathbb{C}}$ commute by [12, Prop. II.10] and those in $\mathfrak{s}_{\mathbb{C}}$ by [7, Lemma 7.7]. ∎

Corollary VI.3. *We keep the assumptions of* Corollary VI.2 *and set* $\Delta_r := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^\alpha \subseteq \mathfrak{r}_{\mathbb{C}}\}$ *(cf. Definition II.18) and* $\mathfrak{m}_{\mathbb{C}}^+ := \bigoplus_{\alpha \in \Delta_r^+} \mathfrak{g}_{\mathbb{C}}^\alpha$. *Then the mapping*

$$\mathfrak{m}_{\mathbb{C}}^+ \to \mathfrak{m} := [\mathfrak{t}, \mathfrak{r}], \quad X \mapsto X + \overline{X}$$

is a bijection which induces a complex structure on $\mathfrak{m}$ *given by*

$$I.(X + \overline{X}) = i(X - \overline{X}).$$

Let $\mathfrak{s} = (\mathfrak{k} \cap \mathfrak{s}) + \mathfrak{p}$ *be a Cartan decomposition of* $\mathfrak{s}$. *Then the complex structure on* $\mathfrak{m}$ *is invariant under* $\mathfrak{k}$ *and the restrictions of the operators* $\operatorname{ad} X$, $X \in \mathfrak{p}$ *are antilinear, i.e.,*

$$\operatorname{ad} X|_{\mathfrak{m}} \circ I = -I \circ \operatorname{ad} X|_{\mathfrak{m}}$$

for all $X \in \mathfrak{p}$.

Proof. The invariance under $\mathfrak{k}$ follows from that fact that the subalgebra $\mathfrak{m}_{\mathbb{C}}^+ = \sum_{\alpha \in \Delta_r^+} \mathfrak{g}_{\mathbb{C}}^\alpha$ is invariant under $\mathfrak{k}$ and this is the i-eigenspace for I on $\mathfrak{m}_{\mathbb{C}}$. The $(-i)$-eigenspace is $\mathfrak{m}_{\mathbb{C}}^- := \overline{\mathfrak{m}_{\mathbb{C}}^+}$ which is also $\mathfrak{k}$-invariant.

For $X \in \mathfrak{g}_{\mathbb{C}}^\alpha \subseteq \mathfrak{s}_{\mathbb{C}}$, $\alpha \in \Delta_p^+$, it follows from Corollary VI.2(iii) that $\operatorname{ad} X$ maps $\mathfrak{m}_{\mathbb{C}}^+$ into $\mathfrak{m}_{\mathbb{C}}^-$ and vice versa. This means that $\operatorname{ad} X$ anticommutes with I on $\mathfrak{m}_{\mathbb{C}}$. The same holds for $\overline{X} \in \mathfrak{g}_{\mathbb{C}}^{-\alpha}$. Then $\operatorname{ad}(X + \overline{X})$ also anticommutes with I on $\mathfrak{m}$ and this proves the assertion. ∎

Corollary VI.4. *We set* $\Delta_{p,s} := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^\alpha \subseteq \mathfrak{s}_{\mathbb{C}}\}$ *and* $\mathfrak{p}_s^\pm := \sum_{\alpha \in \Delta_{p,s}^+} \mathfrak{g}_{\mathbb{C}}^\alpha$. *Then*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_s^- \oplus \mathfrak{m}_{\mathbb{C}}^- \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}^+ \oplus \mathfrak{p}_s^+$$

is a 5*-grading of* $\mathfrak{g}_{\mathbb{C}}$.

Proof. In view of Corollary VI.2, this is an elementary verification. The crucial point is that $[\mathfrak{p}_s^+, \mathfrak{m}_{\mathbb{C}}^-] \subseteq \mathfrak{m}_{\mathbb{C}}^+$ and $[\mathfrak{p}_s^-, \mathfrak{m}_{\mathbb{C}}^+] \subseteq \mathfrak{m}_{\mathbb{C}}^-$. ∎

Remark VI.5. Let us say that a Lie algebra $\mathfrak{g}$ is *admissible* if $\mathfrak{g} \oplus \mathbb{R}$ contains invariant cones. If $\mathfrak{g}$ has this property but does not itself contain invariant cones, then it must be compact semisimple ([12, Th. III.39]). Hence Theorem VI.1 and all its consequences also remains true for this slightly extended class of Lie algebras since it is trivially true for compact semisimple Lie algebras. ∎

Example VI.6. An instructive example which illustrates the above results is the Lie algebra $\mathfrak{g} = \mathfrak{h}_n \rtimes \mathfrak{sp}(n, \mathbb{R})$, where $\mathfrak{h}_n$ is the $(2n+1)$-dimensional Heisenberg algebra. Here the compact roots Δ_k are the roots of the compact Lie algebra $\mathfrak{u}(n)$, hence

$$\Delta_k^+ = \{\varepsilon_k - \varepsilon_j : 1 \leq k < j \leq n-1\}$$

and the non-compact positive roots of $\mathfrak{sp}(n, \mathbb{R})$ are given by

$$\Delta_{p,s}^+ = \{\varepsilon_k + \varepsilon_j : 1 \leq k, j \leq n\}.$$

In $\mathfrak{g}$ we also have positive roots corresponding to the root spaces contained in $(\mathfrak{h}_n)_{\mathbb{C}}$ which are given by

$$\Delta_r^+ = \{\varepsilon_i : 1 \leq j \leq n\}.$$ ∎

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