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Irreducibility of Gaussian regular representations of a group of germs of real analytic diffeomorphisms

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1. Introduction

The regular representation of a finite dimensional Lie group decomposes into a direct sum of all unitary representations (compact case) or into a direct integral of almost all unitary representations with respect to the Plancherel measure (non-compact case, see [W]). Therefore it is natural to study the situation in the case of infinite dimensional groups. Due to A. Weil there is no Haar measure on groups which are not locally compact, but it is possible to construct an analogue of the regular representation for infinite dimensional Lie groups. The left regular representation is defined on $H = L_2(\overline{G}, d\mu)$ by

$$T(\phi)f(\psi) := \sqrt{\frac{d\mu(L_{\phi^{-1}}\psi)}{d\mu(\psi)}}f(L_{\phi^{-1}}\psi), \ \phi \in G, \psi \in \overline{G},$$

where $\overline{G}$ is a topological group which contains G as a dense subset, $d\mu$ is a measure on $\overline{G}$ quasinvariant under G (see [Os]) and L_ϕ is the left action.

Contrarily to the locally compact case in the case of infinite dimensional groups it is possible that the regular representation is irreducible for certain measures $d\mu$. (This is impossible in the locally compact case because the operators of the right and the left regular representation commute.) First such results were obtained by N.I. Nessonov and A.V. Kosyak for the group of finite upper-triangular matrices of infinite order with units on the diagonal (see [K1], [K2]). This group is an inductive limit of finite dimensional Lie groups. It arises the question if such an result is possible for (weak) projective limits of finite dimensional Lie groups.

This article is devoted to the irreducibility of certain Gaussian regular representations of a diffeomorphism group of germs of the real line. The group is a subgroup of the group of lower-triangular matrices with units on the diagonal and a dense subgroup of the projective limit of the finite dimensional groups of n-Jets.

The proof is based on the fact that the operators x_i, $\partial_i + m_i x_i$, $i \in \mathbb{N}$ of an irreducible representation of the infinite Heisenberg algebra in $L_2(\mathbb{R}^\infty, d\mu_m)$ may be approximated by the generators of one-parameter subgroups of G.

This article is organized as follows: in section 2 we present some facts about the group G. In section 3 the analogue of a left regular representation is discussed. These representations depend on on a certain infinite dimensional measure $d\mu_m$

quasiinvariant under G. We give a characterization of all diagonal GAUSSIAN measures quasiinvariant under G. In section 4 the generators of a class of one-parameter subgroups of the left regular representation are calculated. In section 5 we present the idea of the proof of the irreducibility for certain measures $d\mu_m$.

2. Preliminaries

Our purpose in this section is to introduce the basic objects of the consideration.

Definition 2.1. By G we call the group of all real analytic transformations $\phi(x)$ of neighbourhoods of zero of the real line with $\phi(0) = 0$ and $\phi'(0) = 1$, that is,

$$G := \{\phi(x) = x + a_1x^2 + a_2x^3 + \cdots; a_n \in R;\ \limsup |a_n|^{1/n} < \infty\}$$

equipped with the multiplication rule $\phi \circ \psi(x) := \psi(\phi(x))$.

Endow G with a topological structure: consider the vector spaces G_R, $(R > 0)$

$$G_R = \{\phi(x) = a_0x + a_1x^2 + \cdots; a_n \in R; \limsup |a_n|^{1/n} < 1/R\}$$

and the seminorm system

$$|\phi|_r := \sup_{|x|<r} |\phi(x)|;\ r = R(1 - 1/n);\ n > 1$$

or the equivalent seminorm system (see [K])

$$|\phi|_M := \sup_{k\geq 1} |a_kM^k|;\ M < R.$$

Then G_R is a FRECHET space. Equip $\bigcup_{R>0} G_R$ with the inductive limit topology. Identify G with the hyperplane $a_0 = 1$. Then the group G is a topological space.

Proposition 2.1. *G is a topological group.*

3. The definition of the left regular representation

Consider the embedding of G into the group

$$F = \{\psi(x) = x + y_1x^2 + y_2x^3 + \cdots \mid y_i \in R\}$$

of formal power series: $G \subset F \simeq \mathbb{R}^\infty$,
and the left action of G on F

$$L_\phi\ \psi(x) := \phi \circ \psi(x) = \psi(\phi(x)), \quad \phi \in G, \psi \in F.$$

L_ϕ may be rewritten in matrix form

$$\begin{aligned}
L_\phi\psi &= L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ a_{12} & 1 & 0 & \cdots \\ a_{13} & a_{12} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} \\
&= Ay + b, \quad \text{where} \\
a_{ij} &= \sum_{\substack{i_1+i_2+\cdots+i_{j+1}=i+1 \\ i_1,\cdots,i_{j+1}>0}} a_{i_1-1}\cdots a_{i_{j+1}-1}, \quad a_0 := 1.
\end{aligned}$$

This is obtained by a comparison of the coefficients of the power series ϕ, ψ and $\phi \circ \psi$. The L_ϕ are affine maps of $F \simeq \mathbb{R}^\infty$.

For sequences of positive real numbers $m = (m_n)_{n\geq 1}$ on F
GAUSSIAN diagonal measures are defined:

$$d\mu_m = \bigotimes_{n=1}^{\infty} \sqrt{\frac{m_n}{\pi}} e^{-m_n y_n^2} dy_n \ .$$

In the following measures $d\mu_m$ are considered where the positive sequence $m = (m_n)_{n\geq 1}$ satisfies the condition $(*)$:

$$(*) \qquad \sum_{n=1}^{\infty} (n+1)^2 \frac{m_{n+1}}{m_n} < \infty.$$

Lemma 3.1. *Suppose that $m = (m_n)_{n\geq 1}$ satisfies the condition $(*)$. Then*

(i) $\exists C > 0 :\ m_n < C^n/(n!)^2$.

(ii) $\forall M > 0 :\ \sum_{k\in\mathbb{N}} m_n M^n < \infty$.

(iii) $\forall M \in \mathbb{N} : \quad \sum_{i,n\in\mathbb{N}} \binom{n+i}{n}^2 M^{2i} \frac{m_{n+i}}{m_n} < \infty$.

Proof. (i): Condition $(*)$ implies that it exists a $C' > 0$ with $\frac{m_{n+1}}{m_n}(n+1)^2 < C'$, that is, $m_{n+1} < m_n C'/(n+1)^2$ and from this by successively inserting we get $m_{n+1} < m_1 C'^n/(n+1)!^2$. The proposition follows with $C := \max\{C', m_1\}$.
(ii) turns out from (i).
(iii):

$$\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \binom{n+i}{n}^2 M^{2i} \frac{m_{n+i}}{m_n}$$

$$= \sum_{i=1}^{\infty} \frac{1}{i!^2} \sum_{n=1}^{\infty} (n+1)^2 (n+2)^2 \cdots (n+i)^2 \frac{m_{n+i}}{m_{n+i-1}} \cdots \frac{m_{n+2}}{m_{n+1}} \frac{m_{n+1}}{m_n} M^{2i}$$

$$\leq \sum_{i=1}^{\infty} \frac{1}{i!^2} \sum_{n_1, n_2, \cdots, n_i = 1}^{\infty} \left(\frac{(n_1+1)^2 \ldots (n_i+1)^2 m_{n_1+1} \ldots m_{n_i+1}}{m_{n_1} m_{n_2} \cdots m_{n_i}} \right) M^{2i}$$

$$= \sum_{i=1}^{\infty} \frac{1}{i!^2} (\sum_{n=1}^{\infty} (n+1)^2 \frac{m_{n+1}}{m_n})^i M^{2i} = f(M^2 \sum_{n=1}^{\infty} (n+1)^2 \frac{m_{n+1}}{m_n}) < \infty ,$$

where $f(x) = \sum_{i=1}^{\infty} \frac{x^i}{i!^2}$ is a power series with an infinite radius of convergence. ∎

Let the image of the measure $d\mu_m$ under the map $L_\phi, \phi \in G$ be denoted by $d\mu_m^{L_\phi}$. The next proposition is devoted to a characterization of all diagonal GAUSSIAN measures quasiinvariant under G.

Proposition 3.2. *The measures $d\mu_m$ and $d\mu_m^{L_\phi}$ are equivalent for all $\phi \in G$ if and only if they satisfy the condition $(*)$.*

Proof. According to [S-F] the GAUSSIAN measures $d\mu_m$ and $d\mu_m^{Ay+b}$ are equivalent if and only if $Mb \in \ell_2$ and $MAM^{-1} - I$ is a HILBERT-SCHMIDT operator. Here is

$$M = \begin{pmatrix} \sqrt{m_1} & 0 & 0 & \cdots \\ 0 & \sqrt{m_2} & 0 & \cdots \\ 0 & 0 & \sqrt{m_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

1.($(*)$ is necessary): When (1) is taken into account in particular $a_{n+1,n} = (n+1)a_1$ is obtained. It follows for the HILBERT-SCHMIDT norm

$$|MAM^{-1} - I|_2^2 = \sum_{j<i} a_{ij}^2 \frac{m_i}{m_j} \geq \sum_{n=1}^{\infty} a_{n+1,n}^2 \frac{m_{n+1}}{m_n}$$

$$= \sum_{n=1}^{\infty} (n+1)^2 a_1^2 \frac{m_{n+1}}{m_n} = a_1^2 \sum_{n=1}^{\infty} (n+1)^2 \frac{m_{n+1}}{m_n} .$$

Hence we get the proposition choosing $\phi \in G$ with $a_1 \neq 0$.

2.($(*)$ is sufficient): Because $\phi = x + a_1 x^2 + a_2 x^3 + \cdots \in G$ is a power series with a finite radius of convergence there exists a $M > 0$ with $|a_i| \leq M^i$. We find from (1) that

$$|a_{ij}| \leq \sum_{\substack{i_1+\cdots+i_{j+1}=i+1 \\ i_1,\cdots,i_{j+1}>0}}^{\infty} M^{i-j} = \binom{i}{j} M^{i-j} .$$

Applying Lemma 3.1(iii) we obtain

$$|MAM^{-1}-I|_2^2=\sum_{j<i}a_{ij}^2\frac{m_i}{m_j}=\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}a_{n+i,n}^2\frac{m_{n+i}}{m_n}$$

$$\leq\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\binom{n+i}{n}^2 M^{2i}\frac{m_{n+i}}{m_n}<\infty\ .$$

Moreover, we have $Mb\in\ell_2$, since $|a_i|<M^i$ and Lemma 3.1(ii). ■

Because of Proposition 3.2 the following definition makes sense:

Definition 3.1. Let $m=(m_n)_{n\geq 1}$ be a positive real sequence satisfying $(*)$. Then on $H=L_2(F,d\mu_m)$ the *left regular representation* is defined by

$$T_m(\phi)f(\psi):=\sqrt{\frac{d\mu_m(L_{\phi^{-1}}\psi)}{d\mu_m(\psi)}}f(L_{\phi^{-1}}\psi).$$

Remark. Definition 3.1 allows the following generalization to topological groups B, which are not locally compact. Instead of B consider the triple $(B',B,d\mu)$. Here B' is a topological group, which contains B as a dense subgroup and $d\mu$ is a quasiinvariant measure under the left action of B on B' (see [Os]). The left regular representation is defined by

$$T_{d\mu}^{B',B}(g_0)f(g):=\sqrt{\frac{d\mu_m(L_{g_0^{-1}}g)}{d\mu_m(g)}}f(L_{g_0^{-1}}g),\quad g\in B',g_0\in B,$$

in particular $T_m=T_{d\mu_m}^{F,G}$.

Proposition 3.3. *The representation T_m is unitary and continuous.*

4. The derived representation of the left regular representation

Our purpose in this subsection is to calculate the generators of some one-parameter subgroups of G in the left regular representation.

An easy calculation shows:

Lemma 4.1. *The subsets $G_n=\left\{\frac{x}{\sqrt[n]{1-tnx^n}}:\ t\in R\right\}\subset G$; $n\in\mathbb{N}$ form one-parameter subgroups.*

Let A_n the selfadjoint operator which generates the one-parameter subgroup $T_m(G_n)$ of the left regular representation (see [R-S]). Further consider the in H dense set $D=lin\{\bigcup_{p\geq 1}C_0^\infty(\mathbb{R}^p)\}$ of continuations of C^∞-functions with compact support in $\mathbb{R}^p$ on $\mathbb{R}^\infty$.

Proposition 4.2. *There is $D\subset D(A_n)$ and*

$$A_n|_D f=-\left(\sum_{i=0}^{\infty}(i+1)y_i\left(\frac{\partial}{\partial_{i+n}}-m_{i+n}y_{i+n}\right)\right)f,\ y_0:=1.$$

Remark. The domain D is not invariant under the action of A_n.

5. Irreducibility

Consider the left regular representation T_m. According to Proposition 3.2 T_m is well defined for sequences $m = (m_n)_{n\geq 1}$ satisfying condition $(*)$, that is, $\sum_{k=1}^{\infty}(i+1)^2\frac{m_{i+1}}{m_i} < \infty$.

Theorem 5.1. *T_m is irreducible for sufficiently fastly decreasing sequences $(m_n)_{n\geq 1}$.*

Indication of Proof. Let W be the von Neumann algebra generated by the operators $T_m(\phi), \phi \in G$. We have to show that $W' = \{\lambda I\}$.

The self-adjoint operators A_n are affiliated to the subgroups generated by themselves and therefore they are also affiliated to W.

For $n \geq 1$ consider the sequence of symmetric operators

$$C_{n,k} := \frac{-1}{(n+1)m_{n+k}}(A_{n+k}A_k + A_kA_{n+k}), \quad k \in \mathbb{N}.$$

Define C_n as the weak graph limit of the $C_{n,k}$, i.e., $C_n\phi := h$, if there is a sequence $\phi_k \to \phi, \;\; \phi_k \in D(C_{n,k})$ such that $\langle C_{n,k}\phi_k, g\rangle \longrightarrow_{k\to\infty} \langle h, g\rangle \;\; \forall g \in H$.

Lemma 5.2. *(i) $\lim_{k\to\infty}\langle C_{n,k}f, g\rangle = \langle y_nf, g\rangle$ for $f \in D, g \in H$.*
(ii) $\overline{C_n}$ is equal to the selfadjoint multiplication operator y_n.

Since the A_n are affiliated to W, the $A_{n+k}A_k + A_kA_{n+k}$ commute with W on D, because

$$CA_lA_kf = A_lCA_kf = A_lA_kCf$$

for $f \in D, C \in W'$. That is,

$$\langle (A_{n+k}A_k + A_kA_{n+k})f, C^*g\rangle = \langle (A_{k+n}A_k + A_kA_{n+k})Cf, g\rangle,$$

where $f \in D(C_n), g \in H$. Multiplying both sides with $\frac{-1}{(n+1)m_{n+k}}$ and taking the limit for $k \to \infty$ one obtains according to Lemma 5.2(i):

$$\langle C_nf, C^*g\rangle = \langle CC_nf, g\rangle = \langle C_nCf, g\rangle \;\; \forall g \in H,$$

that is,

$$CC_nf = C_nCf, \quad f \in D.$$

We take the closure and conclude because of Lemma 5.2(ii) and the essentially selfadjointness of $y_n|_D$

$$Cy_n \subseteq y_nC,$$

i.e., y_n is affiliated to W.

Next we want to show that the operators

$$D_n := \overline{\frac{\partial}{\partial y_n} - m_ny_n}$$

are also affiliated to W.

Consider the sequence

$$\begin{array}{rcccccc}
A_n|_D & = & D_n & +\,2y_1 D_{n+1} & +\,3y_2 D_{n+2} & +\,4y_3 D_{n+3} & +\cdots \\
A_{n+1}|_D & = & & D_{n+1} & +\,2y_1 D_{n+2} & +\,3y_2 D_{n+3} & +\cdots \\
A_{n+2}|_D & = & & & D_{n+2} & +\,2y_1 D_{n+3} & +\cdots \\
A_{n+3}|_D & = & & & & D_{n+3} & +\cdots \\
\vdots & = & & & & & \ddots
\end{array}$$

Choose successively polynomials $p_i(y_1, \cdots, y_i)$ with the GAUSS algorithm such that

$$E_{n,k} := A_n - \sum_{i=1}^{k} p_i A_{n+i}$$

and

$$E_{n,k}|_D = D_n + q_1^k D_{n+k+1} + q_2^k D_{n+k+2} + \cdots$$

with certain polynomials q_i^k.

Define E_n as the graph limit of the symmetric operators $E_{n,k}$ that is, $E_n\phi := h$ if there is a sequence $\phi_k \to \phi$, $\phi_k \in D(E_{n,k})$ such that $\lim_{k\to\infty} E_{n,k}\phi_k = h$.

Lemma 5.3. *The following statements hold:*
(i) $\forall f \in D: \quad \lim_{k\to\infty} E_{n,k} f = D_n f$.
(ii) D_n *is essentially selfadjoint on* D.
(iii) E_n *is equal to the selfadjoint operator* D_n.

For $f \in D$, $C \in W'$ we get:

$$CE_{n,k}f = \sum_{i=0}^{n} Cp_i A_{i+n} f = \sum_{i=0}^{n} p_i A_{n+i} Cf = E_{n,k}Cf$$

since p_i, A_{n+i} are affiliated to W. Taking the limit on both sides for $k \to \infty$ we observe:

$$CE_n f = E_n Cf, \quad f \in D.$$

We take the closure and obtain because of Lemma 5.3(ii), (iii)

$$CD_n \subseteq D_n C, \quad C \in W',$$

i.e., D_n is affiliated to W.

The affiliated operators y_n, iD_n form an irreducible representation of the infinite HEISENBERG algebra (see [S]). Therefore we have $W' = \{\lambda I\}$. This completes the proof of the theorem. ∎

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