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On Algebraic and Semialgebraic Groups and Semigroups

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0 . This is a "semi"lecture[1] whose background is as follows: The LIE-theory of *subsemigroups* of a LIEgroup, as it is considered in the compact presentation by K. H. HOFMANN in volume 1, no. 1 of this seminar [4], largely reduces to a LIE-theory of *subgroups* in the algebraic case. Specifically, if the group G in question and the subsemigroup S are both affine algebraic varieties the S is in fact a subgroup. This fact is well known in the theory of algebraic semigroups and a consequence of a theorem on subsemigroups of algebraic LIE groups published by C. CHEVALLEY in the second volume of his book on Lie groups in 1951 [2]. We shall rewrite the theorem using HOPF algebras and consider several consequences. The second part of the lecture deals with the main consequence. It answers the question which class of subsemigroups of an algebraic group one should consider in order to develop a "satisfactory" LIE-theory of semigroups in algebraic LIE groups by methods close to algebraic geometry. In the case of a real base field we propose the consideration of semialgebraic semigroups and illustrate this proposal by some basic facts and interesting examples as well.

1. The CHEVALLEY theorem and its consequences

Theorem 1.1. *Let $\mathbb{K}$ denote an infinite base field, and $\mathcal{P}$ be the* HOPF *algebra of polynomial functions on* $\mathrm{GL}(n;\mathbb{K})$. *Assume that $S \subseteq \mathrm{GL}(n;\mathbb{K})$ is a subsemigroup, and $\mathfrak{a} = \mathrm{ann}_{\mathcal{P}} S$ its annihilator ideal in the algebra $\mathcal{P}$. Then we have the following conclusions*

i) $\mathfrak{a}$ *is a coideal in $\mathcal{P}$, too, i.e.,* $\Delta\mathfrak{a} \subseteq \mathfrak{a} \otimes \mathcal{P} + \mathcal{P} \otimes \mathfrak{a}$;

ii) $\mathcal{G}(\mathfrak{a}) = \{g \in \mathcal{G}(\mathcal{P}) : (\mathrm{id}_{\mathcal{P}} \otimes g)\Delta\mathfrak{a} \subseteq \mathfrak{a}\}$ *is a group;*

iii) $G = \mathrm{ann}_{\mathrm{GL}(n;\mathbb{K})}\,\mathfrak{a}$ *is an affine algebraic group canonically isomorphic to* $\mathcal{G}(\mathfrak{a})$, *and* $\mathfrak{a} = \mathrm{ann}_{\mathcal{P}} G$.

By Δ we denote the coproduct of the HOPF algebra $\mathcal{P}$, while $\mathcal{G}(\mathcal{P})$ denotes the structure group of $\mathcal{P}$. In other words, $\mathcal{G}(\mathcal{P})$ is the set of homomorphisms of the commutative algebra $\mathcal{P}$ onto $\mathbb{K}$ with convolution as multiplication. The canonical isomorphism between G and $\mathcal{G}(\mathfrak{a})$ is the restriction of the canonical isomorphism which maps every $g \in \mathrm{GL}(n;\mathbb{K})$ onto the homomorphism $\mathcal{P} \to \mathbb{K}$ defined by

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evaluation at g.
A sketchy proof of ii) and iii) runs as follows: The mapping $g \mapsto r(g)$ with $r(g) = (\mathrm{id}_{\mathcal{P}} \otimes g)\Delta \in \operatorname{Aut} \mathcal{P}$ is just the so-called *right regular representation* of $\mathcal{G}(\mathcal{P})$ on the linear space $\mathcal{P}$. Now ii) follows from Lemma 1 of CHEVALLEY stating: *if* $\mathfrak{a}$ *is a linear subspace of* $\mathcal{P}$ *and* $r(g)\mathfrak{a} \subseteq \mathfrak{a}$ *for every* $g \in \mathcal{G}(\mathcal{P})$ *,then* $r(g)\mathfrak{a} = \mathfrak{a}$. If $\mathcal{G}'(\mathfrak{a}) = \operatorname{ann}_{\mathcal{G}(\mathcal{P})} \mathfrak{a}$, then evidently we have $\mathcal{G}'(\mathfrak{a}) \subseteq \mathcal{G}(\mathfrak{a})$, for $r(g)\mathfrak{a} = (\mathrm{id}_{\mathcal{P}} \otimes g)\Delta\mathfrak{a} \subseteq \mathfrak{a}g(\mathcal{P}) = \mathfrak{a}$. Assume $g_0 \in \mathcal{G}'(\mathfrak{a})$. Then once more by Lemma 1 we get $r(g_0)\mathfrak{a} = \mathfrak{a}$, and for every $p \in \mathfrak{a}$ there exists a $q \in \mathfrak{a}$ such that $p = r(g_0)q$. Let $\varepsilon \in \mathcal{G}(\mathcal{P})$ denote the identity, i.e., ε is the counit of $\mathcal{P}$, then it holds that $\varepsilon(p) = \varepsilon(r(g_0)q) = (\varepsilon \otimes g_0)\Delta q = g_0(q) = 0$, and for every $g \in \mathcal{G}(\mathcal{P})$ we have $g(p) = (\varepsilon \otimes g)\Delta p = \varepsilon(r(g)p) = 0$. This implies $\mathcal{G}(\mathfrak{a}) \subseteq \mathcal{G}'(\mathfrak{a})$, and thus, finally, iii).

1.2. Consequences.

A. A ZARISKI-closed subsemigroup of an affine algebraic group over an infinite base field is an affine algebraic group.

B. The group of units of an affine algebraic semigroup is an affine algebraic group [6].

In the following we assume the base field to be algebraically closed.
C. Let $\mathcal{H}$ denote an affine HOPF algebra (i.e., $\mathcal{H}$ is commutative and finitely generated). Every linear subspace $\mathfrak{a}$, which is an ideal and a coideal of $\mathcal{H}$, is annihilated by the augmentation or counit ε of $\mathcal{H}$, that is, $\varepsilon(\mathfrak{a}) = 0$, and is invariant under the antipode or symmetry σ of $\mathcal{H}$, that is $\sigma(\mathfrak{a}) \subseteq \mathfrak{a}$. In short: Every biideal of an affine HOPF algebra is a HOPF ideal [5].

D. Every quotient bialgebra of an affine HOPF algebra is a quotient HOPF algebra.

E. If S is an affine algebraic semigroup, then for every $s \in S$ there exists a power s^k which belongs to a subgroup of S [6].
There is an "elementary" proof of this statement, which is sketched as follows: Assume $S \subseteq \mathrm{Mat}(n; \mathbb{K})$ to be an affine algebraic semigroup. Denote by G'_m the group of $n \times n$ matrices of the form

$$\begin{pmatrix} A & O \\ O & O \end{pmatrix}; \quad A \in \mathrm{GL}(m; \mathbb{K}), \quad 1 \leq m \leq n,$$

$G'_m \cong GL(m; \mathbb{K})$. If $s \in S$ and $\operatorname{rank} s = m$, then there exists a matrix $t \in \mathrm{GL}(n; \mathbb{K})$ such that $s' = tst^{-1}$ has Jordan normal form. There is a power of s' such that $s'^k \in G'_m$. The group $G_m = t^{-1}G'_m t$ is isomorphic to $\mathrm{GL}(m; \mathbb{K})$ and $H = S \cap G_m$ is not void since $s^k \in H$. Also, H is a group by the theorem.

F. If S is an affine algebraic semigroup, then for every $s \in S$ there exists an idempotent e_0 such that for suitable powers of s we have $e_0 s^k = s^k = s^k e_0$ [6].

G. If there is only one idempotent in an affine algebraic monoid S, then S is a group [6].

2. Semialgebraic Semigroups

2.1. Foundations

From now on we take the base field $\mathbb{K}$ to be the field of real numbers $\mathbb{R}$. We denote the HOPF algebra of polynomial functions on $\mathrm{GL}(n) = \mathrm{GL}(n;\mathbb{R})$ by

$$\mathcal{P}(n) = \mathbb{R}[X_{11}, \dots, X_{nn}, x]/(x \det X - 1).$$

Let G denote a real affine algebraic group

$$G = \mathrm{ann}_{\mathrm{GL}(n)}\, \mathfrak{a}, \quad \text{and} \quad \mathfrak{a} = \mathrm{ann}_{\mathcal{P}(n)} G,$$

where $\mathfrak{a}$ denotes an ideal and a coideal of $\mathcal{P}(n)$, contained in the kernel of the augmentation ε, and invariant under the antipode σ of $\mathcal{P}(n)$

$$\Delta\mathfrak{a} \subseteq \mathfrak{a} \otimes \mathcal{P}(n) + \mathcal{P}(n) \otimes \mathfrak{a}, \qquad \varepsilon(\mathfrak{a}) = 0, \sigma(\mathfrak{a}) \subseteq \mathfrak{a}.$$

Accordingly,

$$\mathcal{P} = \mathcal{P}(G) = \mathcal{P}(n)/\mathfrak{a}$$

denotes the HOPF algebra of polynomial functions on G.

Let $S \subseteq G$ denote a subsemigroup of G. We call S a *semialgebraic semigroup* if

$$S = \{s \in G; \quad r_j(s) \geq 0, r_j \in \mathcal{P}, \quad j = 1, \dots, l\}.$$

Then S is a *"closed" affine semialgebraic* set. For simplicity we restrict the considerations to "closed" semialgebraic semigroups.
If $\mathfrak{b} = \mathrm{ann}_{\mathcal{P}} S$ denotes the annihilator of S in $\mathcal{P}$, then $\mathfrak{b}$ is an ideal and a coideal of $\mathcal{P}$ and by 1.2C it is contained in the kernel of ε and is invariant under σ, hence $\mathcal{P}^* = \mathcal{P}/\mathfrak{b}$ is a real HOPF algebra defining a real affine algebraic subgroup G^* of G, which contains S as a subsemigroup. G^* is the "smallest" real affine algebraic group which contains S as a subsemigroup.

In the sequel we may assume $G = G^*$. Let $S \subseteq G$ denote a closed affine semialgebraic set

$$S = \{s \in G; \quad r_j(s) \geq 0, r_j \in \mathcal{P}, \quad j = 1, \dots, l\}.$$

The cone

$$\mathcal{P}_+(S) = \{p \in \mathcal{P} : (\forall s \in S) \quad p(s) \geq 0\}$$

defines a semiorder on $\mathcal{P}$:

$$p \leq_S q \text{ iff } q - p \in \mathcal{P}_+(S).$$

We observe that $p \leq_S q$ and $q \leq_S p$ implies $q - p \in \mathfrak{b} = \mathrm{ann}_{\mathcal{P}} S$. Let us identify the tensor product $\mathcal{P} \otimes \mathcal{P}$ with the algebra of polynomial functions on $G \times G$.

The semialgebraic set S is a semialgebraic semigroup iff

i) $$\Delta\mathcal{P}_+(S) \subseteq (\mathcal{P} \otimes \mathcal{P})_+(S \times S),$$

it is a semialgebraic monoid iff, moreover, it is true that

ii) $$\varepsilon(p) \geq 0 \text{ for every } p \in \mathcal{P}_+(S),$$

and it is a semialgebraic group iff, in addition, the following condition is satisfied:

iii) $$\sigma(\mathcal{P}_+(S)) \subseteq \mathcal{P}_+(S).$$

If S is a semialgebraic semigroup, then, by our assumption $G = G^*$, the relations $p \leq_S q$ and $q \leq_S p$ imply $p = q$ for every $p, q \in \mathcal{P}$.
Let $\mathcal{G}(\mathcal{P})$ denote the structure group of the HOPF algebra $\mathcal{P}$. Then $\mathcal{G}(\mathcal{P}) \cong G$ canonically, and we have the isomorphism

$$\begin{aligned} S &\cong \{\varsigma \in \mathcal{G}(\mathcal{P}) : \varsigma(\mathcal{P}_+(S)) \subseteq \mathbb{R}_+\} \\ &= \{\varsigma \in \mathcal{G}(\mathcal{P}) : (\forall p, q \in \mathcal{P}) \quad p \leq_S q \text{ implies } \varsigma(p) \leq \varsigma(q)\}. \end{aligned}$$

Assume G to be an *irreducible* real affine algebraic group, and denote by $\mathcal{R} = \mathcal{R}(G)$ the field of fractions of the polynomial functions $\mathcal{P} = \mathcal{P}(G)$. I.e., $\mathcal{R}$ is the field of *rational functions* on G. Set

$$S = \{s \in G : r_j(s) \geq 0, r_j \in \mathcal{P}, (j = 1, \ldots, l)\}.$$

Then S is a semialgebraic subset of G. We denote by $\sum \mathcal{R}^2$ the cone consisting of the sums of squares of elements from $\mathcal{R}$, and by $\sum \mathcal{R}^2(r_j;\ (j = 1, \ldots, l))$ the cone in $\mathcal{R}$ generated by $r_1, \ldots, r_l$. The elements of the cone $\sum \mathcal{R}^2(r_j;\ (j = 1, \ldots, l))$ are of the form $f = f_0 + \sum_{i=1}^k f_i t_i$, where $f_i \in \sum \mathcal{R}^2$, $i = 1, \ldots, k$, and where the $t_i, (i = 1, \ldots, k)$ are products of the $r_1, \ldots, r_l$. The *Positivstellensatz* ([1], pp. 84, 93, 95) implies the following equation:

$$\mathcal{P}_+(S) = \sum \mathcal{R}^2(r_j;\ (j = 1, \ldots, l)) \cap \mathcal{P}.$$

Keeping in mind that the direct product $G \times G$ is irreducible, too, and using once more the Positivstellensatz we get the following criterion:

Let G denote an irreducible real affine algebraic group. A semialgebraic subset

$$S = \{s \in G : r_j(s) \geq 0, r_j \in \mathcal{P}, \quad j = 1, \ldots, l\}$$

of G is a semialgebraic semigroup iff

i) $$\Delta r_j = f_{0j} + \sum_{i=1}^{k} f_{ij}(t_{ij} \otimes t'_{ij}), \quad j = 1, \ldots, l,$$

where the f_{ij}, $i = 0, 1, \ldots, k$, $j = 1, \ldots, l$ are sums of squares of elements from the field of fractions of $\mathcal{P} \otimes \mathcal{P}$, and where the t_{ij}, t'_{ij}, $i = 1, \ldots, k$, $j = 1, \ldots, l$ are products of the $r_1, \ldots, r_l$.

Let us consider the case $G = \mathrm{GL}(n)$. Write $\mathcal{R}(n)$ for the field of fractions of $\mathcal{P}(n)$, then $\mathcal{R}(n)$ is isomorphic to $R(X) = R(X_{11}, \ldots, X_{nn})$. Define a semialgebraic set by the equation

$$S = \{s \in \mathrm{GL}(n);\ r_j(s) \geq 0,\ r_j \in \mathcal{P}(n), \quad j = 1, \ldots, l\},$$

then we may choose $r_j \in R[X] = R[X_{11}, \ldots, X_{nn}]$, $j = 1, \ldots, l$. Now assume $G = \mathrm{GL}(n) \times \mathrm{GL}(n)$. Then $\mathcal{P}(G) \cong \mathcal{P}(n) \otimes \mathcal{P}(n)$. Denote the field of fractions by $\mathcal{R}(n,n)$. Now $\mathcal{R}(n,n)$ is isomorphic to $R(X,Y) = R(X_{11}, \ldots, X_{nn}, Y_{11}, \ldots, Y_{nn})$. Using the Positivstellensatz the criterion looks as follows:

The semialgebraic set

$$S = \{s \in \mathrm{GL}(n) : r_j(s) \geq 0,\, r_j \in R[X], \quad j = 1, \ldots, l\}$$

is a semialgebraic semigroup iff

i) $$\Delta r_j = f_{0j} + \sum_{i=1}^{k} f_{ij}(t_{ij} \otimes t'_{ij}) \quad j = 1, \ldots, l,$$

where the f_{ij}, $i = 0, 1, \ldots, k$, $j = 1, \ldots, l$ *are sums of squares of elements from* $\mathcal{R}(n,n)$, $f_{ij} \in \sum \mathcal{R}^2(n,n)$, *and where the* t_{ij}, t'_{ij}, $i = 1, \ldots, k$, $j = 1, \ldots, l$ *are products of the* $r_1, \ldots, r_l \in R[X]$.

We may rewrite the necessary and sufficient condition i) using the isomorphisms indicated above and having in mind that the coproduct Δ is defined by matrix multiplication as follows:

i′) $$r_j(XY) = f_{0j}(X,Y) + \sum_{i=1}^{k} f_{ij}(X,Y) t_{ij}(X) t'_{ij}(Y) \quad j = 1, \ldots, l.$$

Here $r_j(XY)$ is a short notation for $r_j\left(\sum_{i=1}^n X_{1i}Y_{i1}, \ldots, \sum_{i=1}^n X_{ni}Y_{in}\right)$; moreover, the functions $f_{ij}(X,Y) = f_{ij}(X_{11}, \ldots, X_{nn}, Y_{11}, \ldots, Y_{nn})$ belong to $\sum R(X,Y)^2$, i.e., are sums of squares of rational functions in $2n^2$ variables, and the functions $t_{ij}(X) = t_{ij}(X_{11}, \ldots, X_{nn})$ are products of the $r_1(X), \ldots, r_l(X)$, while the functions $t'_{ij}(Y) = t'_{ij}(Y_{11}, \ldots, Y_{nn})$ are products of the $r_1(Y), \ldots, r_l(Y)$.

Let S denote a semialgebraic monoid. Assume $S \subseteq G$ with a real affine algebraic group G, and denote by $L(G)$ the LIE algebra of G. We define the LIE *wedge* of S by the following equation

$$\begin{aligned} W(S) &= \{u \in L(G); \quad (\forall p \in \mathcal{P}_+(S),\, t \in \mathbb{R}_+) \quad p(\exp tu) \geq 0\} \\ &= \{u \in L(G); \quad (\forall t \in \mathbb{R}_+) \quad r_j(\exp tu) \geq 0, \quad j = 1, \ldots, l\}. \end{aligned}$$

Evidently only those elements $p \in \mathcal{P}_+(S)$ are important, which satisfy the equation $p(e) = \varepsilon(p) = 0$. Regarding the $r_1, \ldots, r_l \in \mathcal{P}$ defining the set S, we may assume $r_1(e) = \cdots = r_k(e) = 0$, and $r_{k+1}(e) > 0, \ldots, r_l(e) > 0$. Then

$$\begin{aligned} W(S) &= \{u \in L(G); \quad (\forall p \in \mathcal{P}_+(S) \text{ with } p(e) = 0, \quad t \in \mathbb{R}_+) \quad p(\exp tu) \geq 0\} \\ &= \{u \in L(G); \quad (\forall t \in \mathbb{R}_+) \quad r_i(\exp tu) \geq 0, \quad i = 1, \ldots, k\}. \end{aligned}$$

2.2 Examples

1) Let $G = D(n)$ denote the group of regular real $n \times n$ diagonal matrices,

$$S = D_+^m(n) = \{s \in D(n); \quad s_{ii} \geq 1,\, i = 1, \ldots, m, \quad s_{jj} \geq 0,\, j = m+1, \ldots, n\}.$$

The set $D_+^m(n)$ is a semialgebraic monoid with the following LIE wedge:

$$W(D_+^m(n)) = \{u \in R^{n^2}; \quad u_{ij} = 0,\, i \neq j, \quad u_{ii} \geq 0,\, i = 1, \ldots, m\}.$$

The group of units consists of the regular diagonal matrices having 1 at the first m places in the diagonal; it is isomorphic to $D(n-m)$. Note that $D^0(1)$ is the multiplicative group of positive real numbers, and that $D^1(1)$ is the multiplicative monoid of real numbers ≥ 1.

2) Assume $G = \mathrm{GL}(n)$, and put

$$S = \mathrm{SL}_+(n) = \{s \in \mathrm{GL}(n) :\ \det s \geq 1\}.$$

Then $\mathrm{SL}_+(n)$ is a semialgebraic monoid with LIE wedge

$$W(\mathrm{SL}_+(n)) = \{u \in R^{n^2}; \quad \mathrm{Tr}\, u \geq 0\}.$$

The group of units of $\mathrm{SL}_+(n)$ is $\mathrm{SL}(n)$.

3) Assume $G = \mathrm{GL}(n)$, put

$$\begin{aligned} S = \mathrm{GL}_+^m(n) \quad = \quad & \{s \in \mathrm{GL}(n); \quad s_{ii} \geq 1,\, i = 1, \ldots, m, \\ & s_{ij} \geq 0,\, i \neq j \text{ or } i = j = m+1, \ldots, n\}. \end{aligned}$$

The set $\mathrm{GL}_+^m(n)$ is a semialgebraic monoid with LIE wedge

$$W(\mathrm{GL}_+^m(n)) = \{u \in R^{n^2}; \quad u_{ii} \geq 0,\, i = 1, \ldots, m, \quad u_{ij} \geq 0,\, i \neq j\}.$$

Now $\mathrm{GL}_+^0(n)$ is the monoid of nonnegative regular $n \times n$ matrices.

4) Define $\mathrm{SL}_+^m(n) = \mathrm{GL}_+^m(n) \cap \mathrm{SL}_+(n)$. Then $\mathrm{SL}_+^m(n)$ is a semialgebraic monoid and

$$W(\mathrm{SL}_+^m(n)) = W(\mathrm{GL}_+^m(n)) \cap W(\mathrm{SL}_+(n)).$$

In the case $m = n$ we have

$$W(\mathrm{SL}_+^n(n)) = W(\mathrm{GL}_+^n(n)).$$

5) By $s\begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix}$ we denote the k-minor of the matrix s given by the rows with indices $i_1 \cdots i_k$ and the columns with the indices $j_1 \cdots j_k$. Assume $G = \mathrm{GL}(n)$, and put

$$S = \mathrm{GL}_{++}^m(n) = \left\{ s \in \mathrm{GL}(n); \quad s\begin{pmatrix} i_1 \cdots i_k \\ i_1 \cdots i_k \end{pmatrix} \begin{cases} \geq 1 & \text{if } 1 \leq i_1 \leq \cdots \leq i_k \leq m, \\ \geq 0 & \text{otherwise} \end{cases} \right\}.$$

The set $\mathrm{GL}_{++}^m(n)$ is a semialgebraic monoid. The LIE wedge of $\mathrm{GL}_{++}^m(n)$ equals the LIE wedge of $\mathrm{GL}_+^m(n)$. Here $\mathrm{GL}_{++}^0(n)$ is the monoid of completely nonnegative regular $n \times n$ matrices. If $n \geq 3$ then

$$\mathrm{GL}_{++}^n(n) \subset \mathrm{SL}_+^n(n) \subset \mathrm{GL}_+^n(n),$$

and

$$W(\mathrm{GL}^n_{++}(n)) = W(\mathrm{SL}^n_+(n)) = W(\mathrm{GL}^n_+(n)).$$

6) Let G denote the group of "similarities" in euclidean n-space:

$$G = \{g \in \mathrm{GL}(n); \quad g = \alpha h, \alpha \in R,\ \alpha \neq 0, \text{ and } h \in SO(n)\}.$$

We define

$$S = \{s \in G; \quad s = \alpha h,\ \alpha \geq 1, \text{ and } h \in SO(n)\}.$$

The set S is a semialgebraic monoid with the LIE wedge

$$W(S) = \{\alpha u \in R^{n^2}; \quad \alpha \geq 0 \text{ and } u \in so(n)\}.$$

S is the monoid of "expanding" similarities, isomorphic to the direct product of the monoid $D^1(1)$ and the orthogonal group $SO(n) : S \cong D^1(1) \times SO(n)$; its LIE wedge is isomorphic to $\mathbb{R}_+ \oplus so(n)$; its unit group is $SO(n)$.

7) The last example is a little bit beyond the context of this lecture. Consider the set

$$S = Osc(n) = \{s \in \mathrm{GL}^0_{++}(n); s_{ij} > 0, \text{ if } |i - j| \leq 1\}.$$

The set $Osc(n)$ is the semialgebraic semigroup of *oscillation matrices* ([3], p.447); it is not a monoid, and it is not "closed".

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