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Endomorphisms of Stable Planes

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ABSTRACT. Endomorphisms of stable planes are introduced, and it is shown that these are injective, locally constant or collapsed. Examples are studied, and it is shown that there are stable planes admitting “substantially more” endomorphisms than automorphisms.

It seems to be a rather popular feeling that geometry is the domain of groups and symmetry, whereas asymmetry and semigroups have been banished to other fields in mathematics. The purpose of this note is to show that this feeling is justified in the area of locally compact *connected* affine or projective planes, but that the situation is completely different if one studies hyperbolic planes and their generalizations.

Definition 1.

a) A *linear space* $(P, \mathcal{L})$ consists of a non-empty set P (“points”) and a system $\mathcal{L}$ of subsets of P (“lines”) such that
- for each pair (p, q) of distinct points there is exactly one set $L \in \mathcal{L}$ (denoted by pq) such that $\{p, q\} \subseteq L$,
- each member of $\mathcal{L}$ contains at least two elements, and $P \notin \mathcal{L}$.

b) A *stable plane*[1] $(P, \mathcal{L})$ is a linear space whose point space P and line space $\mathcal{L}$ are equipped with non-discrete[2] Hausdorff[3] topologies such that
- the join map $\vee: P^2 \setminus \{(p,p) | p \in P\} \to \mathcal{L}: (p, q) \mapsto pq$ is continuous,
- the set $\mathcal{D} := \left\{(L, M) \in \mathcal{L}^2 \;\middle|\; L \neq M, L \cap M \neq \emptyset\right\}$ is open in $\mathcal{L} \times \mathcal{L}$, and the intersection map $\wedge: \mathcal{D} \to P: (L, M) \mapsto L \wedge M$ is continuous (where $L \wedge M$ is defined by $\{L \wedge M\} = L \cap M$).
- there exists a quadrangle in P (i.e., four points such that no three of them are contained in a line).

The fact that $\mathcal{D}$ is open in $\mathcal{L}^2$ shall be referred to as *stability.*

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[1] In a number of papers by R. Löwen or the author, stable planes are studied under the additional assumption that P and $\mathcal{L}$ are locally compact and of positive covering dimension, see [13] for an overview. We do not need these assumptions in the present note, thus we resume the terminology of [7].

[2] Note that this, together with the geometrical assumptions that define a stable plane, already implies that there are no isolated points in any line, cf. [7: 1.2].

[3] In fact, the Hausdorff property is an easy consequence of the remaining axioms, cf. [3: 4.2].

Definition 2. Let $(P,\mathcal{L})$ and $(P',\mathcal{L}')$ be linear spaces. A mapping $\pi\colon P\to P'$ is called a *lineation* from $(P,\mathcal{L})$ to $(P',\mathcal{L}')$ if for each line $L\in\mathcal{L}$ there exists a line $L'\in\mathcal{L}'$ (not necessarily unique) such that $L^\pi\subseteq L'$. A lineation π is called *collapsed* if the image P^π is contained in some line $H\in\mathcal{L}'$.

Lemma 3. *Let $\pi\colon(P,\mathcal{L})\to(P',\mathcal{L}')$ be an injective*[4] *lineation of stable planes.*

a) There exists a unique mapping $\lambda\colon\mathcal{L}\to\mathcal{L}'$ such that $L^\pi\subseteq L^\lambda$ for each $L\in\mathcal{L}$.

b) If π is non-collapsed then λ is injective.

c) If π is continuous then λ is continuous.

Proof. Assume that L and M are two lines such that $L^\lambda=M^\lambda$. If $p\in L$ and $q\in M\setminus\{p\}$ then $(pq)^\lambda=L^\lambda$. Hence we may assume that L meets M. Let z be a point that is not contained in $L\cup M$. By stability, there exists a line H through z that meets both L and M in different points. This implies that $z^\pi\in H^\lambda=L^\lambda$, and π is collapsed.
Finally assume that π is continuous. Let $\mathcal{U}$ be a neighborhood of L^λ for some line $L\in\mathcal{L}$. Choosing two points $p,q\in L$ we infer that there are neighborhoods U' of p^π and V' of q^π, respectively, such that $U'V'\subseteq\mathcal{U}$. Now there are neighborhoods U of p and V of q, respectively, such that $U^\pi\subseteq U'$ and $V^\pi\subseteq V'$. Consequently, UV is a neighborhood of L such that $(UV)^\lambda=U^\pi V^\pi\subseteq U'V'\subseteq\mathcal{U}$. ■

In the sequel, let $(P,\mathcal{L})$ and $(P',\mathcal{L}')$ be stable planes, and assume that $\pi\colon P\to P'$ is a continuous lineation from $(P,\mathcal{L})$ to $(P',\mathcal{L}')$.

Lemma 4. *Assume that $p,q\in P$ are points such that $p^\pi\neq q^\pi$. If there is a point $r\in P\setminus pq$ such that $r^\pi=q^\pi$ then there is a neighborhood U of p such that $U^\pi\subseteq p^\pi q^\pi$, i.e., the restriction $\pi|_U$ is collapsed.*

Proof. By stability, there is a neighborhood $U\subseteq V$ of p such that each line in $qU:=\{qu\mid u\in U\}$ meets the line pr in a point of V. Consider the mapping

$$\alpha\colon U\to pr\cap V\colon u\mapsto qu\wedge pr.$$

Since $u^\alpha\in qu$, we know that $u^{\alpha\pi}\in q^\pi u^\pi$ (recall that $q^\pi\neq u^\pi$). On the other hand, we have that $u^\alpha\in pr$, hence $u^{\alpha\pi}\in p^\pi r^\pi=p^\pi q^\pi$. Now $u^{\alpha\pi}\neq q^\pi$ by the choice of V, and we infer that $p^\pi r^\pi=u^{\alpha\pi}q^\pi=u^\pi q^\pi$. Thus $U^\pi\subseteq p^\pi q^\pi$. ■

Lemma 5. *Assume that there are a neighborhood U in P, a point $z\in P$ and a line $L\in\mathcal{L}'$ such that $U^\pi\subseteq L$ but $z^\pi\notin L$. Then the restriction $\pi|_U$ is locally constant.*

Proof. Let u,v be points in U such that u,v,z are collinear. Then $v^\pi=z^\pi v^\pi\wedge L=z^\pi u^\pi\wedge L=u^\pi$. Hence the mapping $\zeta\colon zU\to L\colon zu\mapsto u^\pi$ is well-defined. For each $v\in U$ and each $y\in W\setminus zv$ there exists a neighborhood $\mathcal{U}$ of zv in $\mathcal{L}_z$ such that the line yv meets each member of $\mathcal{U}$ in a point of U. Using the mapping $\eta\colon yU\to L\colon yu\mapsto u^\pi$ we infer that $\zeta|_{\mathcal{U}}$ is constant. Consequently, the restriction $\pi|_V$ is constant, where $V:=\{u\in U\mid zu\in\mathcal{U}\}$ is a neighborhood of v. ■

[4] I.e., the mapping $\pi\colon P\to P'$ is injective.

Theorem 6. *If a continuous lineation $\pi: (P, \mathcal{L}) \to (P', \mathcal{L}')$ of stable planes is not injective, then π is collapsed or π is locally constant.*

Proof. Let $q, r \in P$ be two points such that $q^\pi = r^\pi$. Since π is not collapsed, there are points p and s such that p^π, q^π, s^π are not collinear. In particular, we may assume that $p \notin qr$. By (4) and (5), there is a neighborhood V of p such that $\pi|_V$ is constant. If x is a point such that $x^\pi \neq p^\pi$ then (4) and (5) apply again, yielding a neighborhood of x such that the restriction of π to this neighborhood is constant. Since $P^\pi \neq \{p^\pi\}$ there exists at least one such point, and interchanging the roles of p and x gives such a neighborhood for each point y with $y^\pi = p^\pi$. ∎

Theorem (6) is a generalization of S. Breitsprecher's result [1: 3.5] for the case of topological projective planes.

Examples 7.

a) A typical example for a non-injective, non-collapsed lineation is the epimorphism from the projective plane over the field of rational numbers, or over the field of p-adic numbers (for the sake of compactness) onto the projective plane over a field of characteristic p, cf. [6: p.182].

b) Similar constructions work for each field $F(x)$ of rational functions [6: p.182]; one obtains lineations from the projective plane over $F(x)$ onto the projective plane over F. Note that the latter can be embedded in the plane over $F(x)$, i.e., we may consider these examples as "endomorphisms".

c) Epimorphisms onto finite projective planes play a prominent role in the theory of compact disconnected projective planes. See T. Grundhöfer's paper [4] for recent results and further references.

d) For any stable plane $(P, \mathcal{L})$ and any triangle x, y, z in P there are neighborhoods X, Y, Z of x, y, z, respectively, such that XY, XZ, YZ are mutually disjoint neighborhoods[5] of the lines xy, xz, yz in L, respectively. Hence the mapping

$$\pi: X \cup Y \cup Z \to X \cup Y \cup Z : p \mapsto \begin{cases} x & \text{if } p \in X \\ y & \text{if } p \in Y \\ z & \text{if } p \in Z \end{cases}$$

is a lineation from the stable plane $\mathcal{P}$ that is induced on the point set $X \cup Y \cup Z$ onto $\mathcal{P}$. The same procedure applies to any finite subset S of P with the property that no line meets S in more than two points.

Theorem (6) and the examples motivate the introduction of a category **StP** of stable planes, whose morphisms are continuous injective non-collapsed lineations. This implies that images under morphisms are stable planes. In contrast, for a category of projective planes one would take as morphisms those lineations whose image contains a quadrangle (and hence is a projective plane). In **StP**, we exclude morphisms whose image is discrete.

Additional topological assumptions make our morphisms especially well-behaved:

[5] For disjoint sets $A, B \subseteq P$, we write $AB := \{ab \mid a \in A, b \in B\} \subseteq \mathcal{L}$.

Theorem 8. *Let $\mathcal{P} = (P, \mathcal{L})$ be a stable plane, and assume that P is locally compact and of positive, finite covering dimension. Then each endomorphism π of $\mathcal{P}$ is an open mapping. Moreover: If λ is the mapping defined in* (3) *then (π, λ) is an isomorphism from $(P, \mathcal{L})$ onto $(P^\pi, \mathcal{L}_{P^\pi})$, where $\mathcal{L}_{P^\pi} := \{L \cap P^\pi \mid L \in \mathcal{L}\} \setminus \{\emptyset\}$.*

Proof. Since π is continuous and injective, it induces a homeomorphism from $\bar{U}$ onto $\bar{U}^\pi$. Consequently, the sets U and U^π are homeomorphic. From the domain invariance property (cf. [8: Th. 11,b)]) we infer that U^π is open in P. By assumption, the mapping $\pi: P \to P^\pi$ is a bijection. According to (3), we have that $\lambda: \mathcal{L} \to \mathcal{L}_{P^\pi}$ is also a bijection. This implies that π^{-1} is a lineation (with line mapping λ^{-1}): If p, q, r are three points in P then $r^\pi \in p^\pi q^\pi = (pq)^\lambda$ if, and only if, $(rp)^\lambda = (pq)^\lambda$. Hence $rp = pq$, and $r \in pq$. Since π is an open mapping, the restriction $\pi: P \to P^\pi$ is a homeomorphism. ∎

Note that, in the situation of (8), both λ and λ^{-1} are continuous by (3,c); this reflects the fact that in each stable plane the topology of the line space is determined uniquely by the topology of the point space [7: 1.4,1.5].

If P is compact, then the domain invariance property is not needed in the proof of (8). On the other hand, compactness of P is equivalent to the assumption that any two lines in $(P, \mathcal{L})$ meet [7: 1.27]. Since compact projective planes of positive dimension are connected [12: Cor. 7.7], we obtain from (6) and (8):

Corollary 9. *Let $\mathcal{P} = (P, \mathcal{L})$ be a compact projective plane of positive finite covering dimension. Then each continuous lineation from $\mathcal{P}$ to $\mathcal{P}$ is either collapsed or an automorphism.* ∎

This generalizes H. Salzmann's result [11] on *flat* projective planes (i.e., compact projective planes of dimension 2), which was proved by the use of coordinate methods. See also [10: V, §4] for similar results in that direction.

As a consequence of (8), we have that each semigroup of endomorphisms of a stable plane whose point space is locally compact and of positive finite covering dimension is "almost a group":

Theorem 10. *Assume that $\mathcal{P} = (P, \mathcal{L})$ is a stable plane, and let Σ be a set of endomorphisms of $\mathcal{P}$, endowed with the compact-open topology derived from the action on P. If Σ is closed under composition then Σ is a topological semigroup. If for each element $\pi \in \Sigma$ the image P^π has non-empty interior in P then Σ is cancellative*[6].

Proof. This follows easily from the facts that P^π is a neighborhood in P and that a lineation $\psi: (P, \mathcal{L}) \to (P, \mathcal{L})$ is determined by the restriction $\psi|_U$ for each neighborhood U in P. The latter is a consequence of the property that for each point $p \in P$ there exist two lines L, M through p that meet U in more than one point. ∎

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[6] I.e., for $\pi, \alpha, \beta \in \Sigma$, the implications $\pi\alpha = \pi\beta \Rightarrow \alpha = \beta$ and $\alpha\pi = \beta\pi \Rightarrow \alpha = \beta$ hold.

Actually, there are many examples of semigroups of endomorphisms that are embedded in a group of automorphisms of a (larger) stable plane:

11. An important class of examples of stable planes is obtained in the following way: In a compact projective plane $\mathcal{P} = (P, \mathcal{L})$, let U be an open non-empty subset of P, and let $\mathcal{L}_U := \{L \cap U \mid L \in \mathcal{L}\} \setminus \{\emptyset\}$. Then $\mathcal{U} := (U, \mathcal{L}_U)$ is a stable plane. In such situations, one obtains in an obvious way a semigroup of endomorphisms of $\mathcal{U}$:

$$\mathrm{End}_{\mathcal{P}}(\mathcal{U}) := \{\pi \mid \pi \text{ is a continuous collineation of } \mathcal{P} \text{ such that } U^\pi \subseteq U\}$$

Moreover:

Theorem 12. *Let $\mathcal{P} = (P, \mathcal{L})$ be a compact projective plane, and let U be an open non-empty subset of P. Let $\mathrm{Aut}(\mathcal{P})$ be the group of all continuous collineations of $\mathcal{P}$, endowed with the compact-open topology derived from the action on P. Then the following hold:*

a) $\mathrm{End}_{\mathcal{P}}(\mathcal{U})$ is a subsemigroup of $\mathrm{Aut}(\mathcal{P})$. The topology that is induced from $\mathrm{Aut}(\mathcal{P})$ on $\mathrm{End}_{\mathcal{P}}(\mathcal{U})$ coincides with the compact-open topology derived from the action on U.

b) If there exists some $\pi \in \mathrm{Aut}(\mathcal{P})$ such that $\bar{U}^\pi \subseteq U$ then $\mathrm{End}_{\mathcal{P}}(\mathcal{U})$ has non-empty interior.

Proof. By definition of compact-open topology, the set $\{\pi \in \mathrm{Aut}(\mathcal{P}) \mid \bar{U}^\pi \subseteq U\}$ is open in $\mathrm{Aut}(\mathcal{P})$. ∎

13. Let $\mathcal{P}$ be the projective plane over the real numbers $\mathbb{R}$, and let

$$H := \{\mathbb{R}(x, y, 1) \mid x^2 + y^2 < 1\}.$$

Then the stable plane $\mathcal{H} := (H, \mathcal{L}_H)$ is the *real hyperbolic plane* (Klein's model). Obviously, the diagonal matrix $\mathrm{diag}(\frac{1}{2}, \frac{1}{2}, 1) \in \mathrm{GL}_3\mathbb{R}$ induces a collineation π of $\mathcal{P}$ such that $\bar{H}^\pi \subset H$. Consequently, the semigroup $\mathrm{End}_{\mathcal{P}}(\mathcal{H})$ has non-empty interior in the 8-dimensional Lie group $\mathrm{Aut}(\mathcal{P}) \cong \mathrm{PGL}_3\mathbb{R}$. Thus we have:

Theorem 14. *There exists a topological semigroup Ω of continuous injective lineations of the real hyperbolic plane such that:*

a) The topological space underlying Ω is an 8-dimensional manifold (with boundary).

b) The boundary of Ω contains the full group of collineations of $\mathcal{H}$, which is a Lie group of dimension 3. ∎

The semigroup Ω is known from the theory of Lie semigroups and invariant cones, cf. [9]. The next example, which is obtained similarly, is perhaps even more striking:

Example 15. Let U be an open set of points in the real projective plane $\mathcal{P}$ such that U is contained in the interior of some conic C, and that

$$\{\pi \in \mathrm{Aut}(\mathcal{P}) \mid U^\pi = U\}$$

consists of the identity alone (this may be achieved by deleting a suitable closed set of points). Then there exists a simply connected neighborhood $V \subset U$. Since the conic C can be mapped into V by an element of $\mathrm{Aut}(\mathcal{P})$, the semigroup $\mathrm{End}_{\mathcal{P}}(\mathcal{U})$ has non-empty interior in the 8-dimensional Lie group $\mathrm{Aut}(\mathcal{P})$.
Similar constructions work in the projective planes over the complex numbers, Hamilton's quaternions, or Cayley's octonions; as well as in many non-classical planes.

For the case of locally compact connected affine planes of finite covering dimension, we obtain that there exist no proper endomorphisms at all (in contrast to (14)):

Theorem 16. *Let $\mathcal{A} = (A, \mathcal{L}_A)$ be a locally compact connected affine plane of finite covering dimension. Then each continuous lineation from $\mathcal{A}$ to $\mathcal{A}$ is either collapsed or an automorphism of $\mathcal{A}$.*

Proof. From (6) we infer that π is an endomorphism of $\mathcal{A}$. According to (8), the mapping π is open. In particular, for each line $L \in \mathcal{L}$ the intersection $L \cap A$ is open in L. Assume that $x \in A \setminus A^\pi$. For any point $a \in A$, the line $a^\pi x$ contains another point $b^\pi \in A^\pi$. Since π is not collapsed, there exists a point $c \in A$ such that $c^\pi \notin a^\pi b^\pi$. Every line through c^π contains a point of $A^\pi \setminus \{c^\pi\}$. Therefore there is exactly one line L through c^π that does not contain a point of $(a^\pi b^\pi) \cap A^\pi$. Since intersection points are unique in $\mathcal{A}$, we obtain that $L = c^\pi x$. This contradicts the fact that there exists a parallel to $a^\pi b^\pi$ through c^π. ∎

Remark 17. V. Corbas has proved [2] that each surjective lineation from an affine plane onto an affine plane is an isomorphism. In our result (16), the assumption of surjectivity is replaced by topological assumptions.

Even in the case of disconnected affine planes, the procedure described in (11) does not give any new lineations:

Theorem 18. *Let $\mathcal{P} = (P, \mathcal{L})$ be a projective plane, and let $\mathcal{A} = (A, \mathcal{L}_A)$ be the affine plane, where $A := P \setminus L$ for some $L \in \mathcal{L}$. Then each element of $\mathrm{End}_{\mathcal{P}}(\mathcal{A})$ is a collineation of $\mathcal{A}$.*

Proof. This implies that $\mathrm{End}_{\mathcal{P}}(\mathcal{A})$ equals the stabilizer of L in $\mathrm{Aut}(\mathcal{P})$. ∎

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