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On the exponential function of an invariant Lie semigroup

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Introduction

It is a well known fact in the theory of Lie groups that the exponential function $\exp: \mathfrak{L}(G) \to G$ is surjective if $\mathfrak{L}(G)$ is a compact Lie algebra. In [7] the first author has presented a control theoretic proof of this result. The major advantage of the control theoretic viewpoint is that it permits generalizations which prove useful in the Lie theory of semigroups. In this paper we are aiming at the result that the exponential function of an invariant Lie semigroup in a Lie group with compact Lie algebra is surjective. Clearly, this generalizes the above result on Lie groups to Lie semigroups. We will obtain this result from a more general theorem about surjectivity properties of the exponential function of an invariant Lie semigroup. This theorem is a very special case of more general result concerning ordered manifolds with affine connections where the order is defined by a cone field invariant under parallel transport ([8]). The results in [8] apply not only to invariant Lie semigroups but also to orderd symmetric spaces as considered in [3].

In this paper we restrict ourselves to the case of invariant subsemigroups of Lie groups. In this case the results are much more easier to prove so that the technical difficulties do not obscure the main ideas. Another advantage is that *Pontrjagin's Maximum Principle* appears more naturally and is easier to describe in this case. This is mostly due to the fact that the tangent and cotangent bundle of a Lie group permit nice trivializations.

The authors are gratefully indepted to E. B. Vinberg for many fruitful discussions on the application of the characteristic function of a cone to Lie semigroups.

The characteristic function and the length functional of a cone

Definition 1. A *cone* C in a finite dimensional vector space V is a closed convex subset which is invariant under multiplication with positive scalars. We say that C is *pointed* if $C \cap C = \{0\}$ and that C is *generating* if $C - C = V$, i.e., if $\mathrm{int}(C) \neq \emptyset$. The *dual cone* is $C^* := \{\omega \in \widehat{V} : \omega(C) \subseteq \mathbb{R}^+\}$, where $\widehat{V}$ denotes the linear dual of V. The group of linear automorphisms of V leaving C invariant is denoted $\mathrm{Aut}(C)$. Note that this is a closed subgroup of $\mathrm{Gl}(V)$ and therefore a Lie group.

Let C be a pointed generating cone in the n-dimensional vector space V. We fix a Lebesgue measure $\mu_{\widehat{V}}$ on $\widehat{V}$. The associated *characteristic function* $\varphi = \varphi_C$ of C is defined by

$$\varphi(x) = \int_{C^*} e^{-\langle \omega, x\rangle} d\mu_{\widehat{V}}(\omega).$$

This function depends on the Haar measure on $\widehat{V}$. Therefore we say that a function φ' on $\mathrm{int}(C)$ is a *characteristic function of* C if it is proportional to φ. This property is independent of the choice of the Haar measure on $\widehat{V}$. For more information on the characteristic function of a cone we refer to [1].

Moreover, we define $\psi = \psi_C$ on V by

$$\psi(c) = \begin{cases} \varphi(c)^{-\frac{1}{n}} & \text{for } c \in \mathrm{int}(C) \\ 0 & \text{for } c \notin \mathrm{int}(C). \end{cases}$$

We say that a function ψ' on V is a *length functional of* C if it is proportional to ψ. This does not depend on the choice of Haar measure on $\widehat{V}$. ■

Proposition 2. *The function ψ has the following properties:*

(i) $\psi(c) > 0$ *for* $c \in \mathrm{int}(C)$.

(ii) $\psi(g.c) = |\det(g)|^{\frac{1}{n}} \psi(c)$ *for* $g \in \mathrm{Aut}(C)$.

(iii) $\psi(\lambda c) = \lambda\psi(c)$ *for* $\lambda \in]0, \infty[$.

(iv) ψ *is continuous on* V.

(v) ψ *is smooth on* $\mathrm{int}(C)$ *and the second derivative* $d^2 \log \psi(x)$ *is negative definite at each point* $x \in \mathrm{int}(C)$, *hence* $\log \psi$ *is strictly concave on* $\mathrm{int}(C)$.

(vi) *The mapping* $x \mapsto x^\sharp := d(\log \psi)(x)$ *is a diffeomorphism of* $\mathrm{int}(C)$ *onto* $\mathrm{int}(C^*)$ *with the property that* $(\lambda x)^\sharp = \frac{1}{\lambda} x^\sharp$.

(vii) *There exists a norm* $\|\cdot\|$ *on* V *such that* $\psi \leq \|\cdot\|$.

(viii) $d\psi(x)(x) = \psi(x)$, *i.e.,* $x^\sharp(x) = 1$ *for all* $x \in \mathrm{int}(C)$.

(ix) *Let* $\omega \in \mathrm{int}(C^*)$ *and* $c > 0$. *Then there exists a unique element* $x \in C$ *with*

$$\omega(x) = \min\{\omega(y) : \psi(y) = c\}.$$

This element satisfies $x^\sharp = \frac{1}{\omega(x)}\omega$.

(x) ψ *is concave on* C.

Proof. [8, I.5] ■

If

$$C = \left\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 \geq \sqrt{x_1^2 + \cdots + x_n^2}\right\},$$

then

$$\psi(x_0,\ldots,x_n) = \sqrt{x_0^2 - x_1^2 - \cdots - x_n^2}\psi(1,0,\ldots,0)$$

is the well known length function from Lorentz geometry on Minkowski space.

From now on we suppose that C is a pointed generating cone in the Lie algebra $\mathfrak{g}$ of the connected Lie group G.

Definition 3. We recall that an absolutely continuous curve $\gamma\colon [a,b] \to G$ is said to be a *conal curve* if the bounded measurable function

$$u_\gamma : [a,b] \to \mathfrak{g}, \quad t \mapsto d\lambda_{\gamma(t)^{-1}}\big(\gamma(t)\big)\gamma'(t)$$

takes almost everywhere values in the cone C.

As in Lorentzian geometry the length functional permits to measure the "length" of a, not necessarily conal, curve as

$$L(\gamma) := \int_a^b \psi\big(u_\gamma(t)\big)\ dt.$$

Note that a non-constant curve may have length zero if all of its tangents, i.e., the values of u_γ, lie on the boundary of the cone C.

The main advantage in the group setting is that the properties of this length functional are very well reflected by the functional

$$L_\psi\colon L_c^\infty(\mathbb{R},\mathfrak{g}) \to \mathbb{R}, \quad u \mapsto \int_{-\infty}^{\infty} \psi\big(u(t)\big)\ dt,$$

where $L_c^\infty(\mathbb{R},\mathfrak{g})$ denotes the space of all essentially bounded functions with compact support with values in the Lie algebra $\mathfrak{g}$. Note that this space is a union of the Banach spaces $L_c^\infty([-n,n],\mathfrak{g})$. ■

Applications to invariant Lie semigroups

Definition 4. For a closed subsemigroup S of a Lie group G we define the *tangent wedge*

$$\mathfrak{L}(S) := \{X \in \mathfrak{g} : \exp \mathbb{R}^+ X \subseteq S\}.$$

We say that S is a *Lie semigroup* if S is reconstructable from its tangent wedge in the sense that

$$S = \overline{\langle \exp \mathfrak{L}(S)\rangle},$$

i.e., S is the smallest closed subsemigroup of G containing $\exp \mathfrak{L}(S)$. We say that a Lie semigroup S is

(i) *generating* if its tangent wedge generates the Lie algebra $\mathfrak{g}$,

(ii) *invariant* if it is invariant under all inner automorphisms of G. Note that this is equivalent to the invariance of $\mathfrak{L}(S)$ under the adjoint action of G on $\mathfrak{g}$, and

(iii) *pointed* if its group of units $H(S) = S \cap S^{-1}$ is reduced to $\{\mathbf{1}\}$.

For a Lie semigroup $S \subseteq G$ we define

$$\operatorname{comp}(S) := \{s \in S : S \cap sS^{-1} \text{ is compact}\}.$$

If we consider the quasi order

$$g \leq_S g' \quad :\Longleftrightarrow \quad g' \in gS$$

on G, then $\operatorname{comp}(S)$ consists of all points s such that the order interval $[\mathbf{1}, s] = \{g \in G : \mathbf{1} \leq_S g \leq_S s\} = S \cap sS^{-1}$ is compact. ■

The following theorem is our main result. As we will see later, it is possible to drop the assumptions that S is pointed and generating.

Theorem 5. *Let $S \subseteq G$ be an invariant Lie semigroup. Then*

$$\operatorname{comp}(S) \subseteq \exp \mathfrak{L}(S).$$

■

First we consider the special case where $C := \mathfrak{L}(S)$ is a pointed generating invariant cone in the Lie algebra $\mathfrak{g}$. We use the length functional of C to get a distance function on the group G from the length functional on conal curves.

Definition 6. Let $g, h \in G$ and $\Omega_{g,h}$ denote the set of all conal curves from g to h in G. We define

$$g \prec_S h \quad :\Leftrightarrow \quad \Omega_{g,h} \neq \emptyset.$$

Note that $g \prec_S h$ implies that $g \preceq_S h$ but that the converse does not hold in general (cf. [9], [10]). We set

$$d(g,h) := \begin{cases} 0 & \text{for } g \not\prec_S h \\ \sup\{L(\gamma) : \gamma \in \Omega_{g,h}\} & \text{for } g \prec_S h. \end{cases}$$

Furthermore we set $d(g) := d(\mathbf{1}, g)$. ■

Lemma 7. *The distance function d has the following properties:*

(i) If $p \prec_S q \prec_S r$, then $d(p,r) \geq d(p,q) + d(q,r)$.

(ii) $d(gx, gy) = d(x,y)$ for all $g, x, y \in G$.

(iii) $d(s) > 0$ iff $s \in \operatorname{int}(S)$.

(iv) $d(s) < \infty$ and $\mathbf{1} \prec_S s$ for all $s \in \operatorname{comp}(S)$.

Proof. (i) This follows by concatenation of conal curves.

(ii) This is a direct consequence of the fact that $L(\lambda_g \circ \gamma) = L(\gamma)$, where λ_g is the left multiplication with g.
(iii) Suppose that $d(s) > 0$. Then there exists a conal curve $\gamma: [0,T] \to G$ with $\gamma(0) = \mathbf{1}$, $\gamma(T) = s$ and $L(\gamma) > 0$. Whence there exists $t_0 \in]0,T[$ such that $\psi\big(u_\gamma(t_0)\big) > 0$. Let $\alpha(t) := \gamma(t_0)^{-1}\gamma(t+t_0)$. Then $\alpha: [0, T-t_0] \to G$ is a conal curve starting in $\mathbf{1}$. We choose a neighborhood U of $\mathbf{1}$ in G and $V \subseteq \mathfrak{g}$ such that $\exp: V \to U$ is a diffeomorphism. Now

$$(\exp|_V^{-1} \circ \alpha)'(0) = d\exp(0)^{-1}\alpha'(0) = u_\gamma(t_0) \in \operatorname{int} C.$$

Therefore $(\exp|_V^{-1} \circ \alpha)(0) = 0$ entails that there exists $t > 0$ with $(\exp|_V^{-1} \circ \alpha)(t) \in \operatorname{int}(C)$. We conclude that

$$\alpha(t) \in \exp\big(\operatorname{int}(C)\big) \subseteq \operatorname{int}(S).$$

Putting all these things together yields

$$\gamma(T) = \gamma(t_0)\alpha(t)\big(\gamma(t_0+t)^{-1}\gamma(T)\big) \in S \operatorname{int}(S) S \subseteq \operatorname{int}(S).$$

For the converse, assume that $s \in \operatorname{int}(S)$. Pick $X \in \operatorname{int}(C)$ and choose $t > 0$ such that $\exp(tX) \in \operatorname{int}(sS^{-1}) = s\operatorname{int}(S)^{-1}$. Then $\mathbf{1} \prec_S \exp(tX) \prec_S s$ and (i) shows that

$$d(s) = d(s,\mathbf{1}) \geq d\big(s, \exp(tX)\big) + d\big(\exp(tX), \mathbf{1}\big) \geq d\big(\exp(tX)\big) \geq t\psi(X) > 0.$$

(iv) Let $s \in \operatorname{comp}(S)$ and $\gamma \in \Omega_{\mathbf{1},s}$. In view of Proposition 2(vii) there exists a euclidean norm $\|\cdot\|$ on $\mathfrak{g}$ such that $\Psi(c) \leq \|c\|$ for all $c \in \mathfrak{g}$. The norm $\|\cdot\|$ defines a left-invariant Riemannian metric on G. We also note that such a metric is complete. Using [10, 1.24] we see that there exists an upper bound $L > 0$ for the Riemannian arc length of all conal curves $\gamma \in \Omega_{\mathbf{1},s}$. Whence $L(\gamma) \leq L$ holds also for the ψ-arc length. Thus $d(s,\mathbf{1}) \leq L < \infty$. The last assertion follows from [10, 1.23, 1.24]. ∎

The preceding lemma proposes the following strategy for the proof of Theorem 5. Pick $s \in \operatorname{comp}(S)$, and assume for a moment that $d(s) > 0$. Then $d(s) < \infty$. Find a conal curve γ from $\mathbf{1}$ to s which realizes the distance $d(s)$ as ψ-arclength. Show that γ, after reparametrization, is a one-parameter group.

Proposition 8. *Let $s \in \operatorname{comp}(S)$ and suppose that $d(s) > 0$. Then there exists a conal curve $\gamma: [0,T] \to S$ such that*

$$L(\gamma) = d(s) = \max\{L(\alpha) : \alpha \in \Omega_{\mathbf{1},s}\}.$$

Proof. We know from Lemma 7(iv) that $d(s) < \infty$. Therefore we find a sequence $\gamma_n \in \Omega_{\mathbf{1},s}$, $\gamma_n: [0,T_n] \to G$, such that $L(\gamma_n) \to d(s)$. Let $\omega \in \operatorname{int}(C^*)$. Since L is invariant under reparametrization we may assume that the functions u_{γ_n} take values in the compact convex set $C_\omega := \{c \in C : \omega(c) = 1\}$ and that $\gamma_n(0) = \mathbf{1}$. Note that this set is a base of the cone C.

In view of Proposition 1.24 in [10] we know that the sequence

$$T_n = L_\omega(\gamma_n) := \int_0^{T_n} \langle \omega(\tau), u_{\gamma_n}(\tau) \rangle \, d\tau$$

is bounded by a certain number $T > 0$. Let $X \subseteq L^\infty([0,T], \mathfrak{g})$ denote the set of all functions taking values in the cone C. This is a norm closed convex cone in this Banach space. Moreover, the functional L_ψ is concave (Proposition 2(x)) and norm continuous, and the sequence u_{γ_n} is contained in a weak-$*$-compact subset. Let u be a cluster point of this sequence.

Choose $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ with $L(\gamma_n) = L_\psi(u_{\gamma_n}) > d(s) - \varepsilon$. The set

$$X_\varepsilon := \{ v \in X : L_\psi(v) \geq d(s) - \varepsilon \}$$

is norm-closed and convex, therefore weak-$*$-closed. It follows that

$$L_\psi(u) \geq d(s) - \varepsilon.$$

For the solution γ of the initial value problem

$$\gamma(0) = \mathbf{1}, \quad \gamma'(t) = d\lambda_{\gamma(t)}(\mathbf{1})u(t)$$

on the interval $[0,T]$, this entails that $u_\gamma = u$ and the sequence γ_n has a subsequence converging uniformly to γ. Thus $\gamma(T) = \lim_{k\to\infty} \gamma_k(T) = s$ and $\gamma \in \Omega_{\mathbf{1},s}$ satisfies

$$L(\gamma) = L_\psi(u) \geq d(s) - \varepsilon.$$

Since ε was arbitrary, it follows that $d(s) = L(\gamma)$. ∎

To prove that a distance maximizing curve is essentially a one-parameter group, we apply Pontrjagins Maximum principle ([13]). To see how this works, we consider the following control system on the group $G \times \mathbb{R}$. The set of *admissible controls* is the cone

$$\mathcal{U} := \{ u \in L_c^\infty(\mathbb{R}^+, \mathfrak{g}) : u(t) \in C \ \text{ a.e.} \}$$

and the system is given by

$$(\dagger) \qquad (\gamma, l)'(t) = \Big(d\lambda_{\gamma(t)}(\mathbf{1})u(t), \psi\big(u(t)\big) \Big), \qquad (\gamma, l)(0) = (\mathbf{1}, 0).$$

Suppose that the curve $\gamma \colon [0,T] \to G$ is a distance maximizing curve from $\mathbf{1}$ to s. Define $l(t) := \int_0^t \psi\big(u_\gamma(\tau)\big) \, d\tau$. Then (γ, l) is a solution of the system $(\dagger)$ and since every other solution (γ_1, l_1) of this system with $\gamma_1(t) = s$ satisfies $l_1(t) \leq l(t)$, the point $\big(s, \gamma(T)\big)$ lies on the boundary of the set of reachable points. Therefore the Maximum Principle, as stated in [7, 4.1] provides a non-zero solution (ω, τ) of the adjoint system

$$(\omega, \tau)'(t) = \big(\omega(t) \circ \operatorname{ad} u(t), 0 \big)$$

(cf. [7, 5.1]) satisfying

$$(*) \qquad \tau(t)\psi\big(u(t)\big) + \langle \omega(t), u(t)\rangle = 0 = \min_{c\in C}\big(\tau(t)\psi(c) + \langle \omega(t), c\rangle\big).$$

In view of Theorem 5.1 in [7] we even have an explicit form of ω given by

$$\omega(t) = \omega(0) \circ \mathrm{Ad}\,\big(\gamma(t)\big).$$

Proposition 9. *If γ is a distance maximizing curve from $\mathbf{1}$ to s and $d(s) > 0$, then $s \in \exp \mathfrak{L}(S)$.*

Proof. We consider the solution (ω, τ) of the adjoint system. Set $\tau_0 := \tau(t)$. Suppose that $\omega(0) = 0$. Then $\tau_0 \neq 0$ and therefore

$$\tau_0\psi\big(u(t)\big) = 0$$

entails that $\psi\big(u(t)\big) = 0$, i.e., $u(t) \in \partial C$ for almost all $t \in [0,T]$. This contradicts the assumption that $d(s) = L_\psi(u) > 0$. Whence $\omega(0) \neq 0$.

For $v \in \partial C$ we find that

$$0 \leq \tau_0\psi(v) + \langle \omega(0), v\rangle = \langle \omega(0), v\rangle.$$

This shows that $\omega(0) \in C^*$, and, since C and therefore C^* is invariant under $\mathrm{Ad}(G)$, we find that $\omega(t) \in C^*$ for all $t \in [0,T]$.

Suppose that $\tau_0 \geq 0$. Then $\tau_0\psi\big(u(t)\big) + \langle \omega(t), u(t)\rangle = 0$ entails that both summands are zero. Pick t with $\psi\big(u(t)\big) > 0$. Then $\langle \omega(t), u(t)\rangle > 0$ and we have a contradiction which shows that $\tau_0 < 0$. After rescaling of ω we assume that $\tau_0 = -1$.

Pick $t \in [0,T]$ with $u(t) \in \mathrm{int}(C)$. Then

$$\psi\big(u(t)\big) = \langle \omega(t), u(t)\rangle > 0$$

and for $v \in C$ with $\psi(v) = \psi\big(u(t)\big)$ the assertion $(*)$ implies that

$$\langle \omega(t), v\rangle \geq \psi(v) = \psi\big(u(t)\big) = \langle \omega(t), u(t)\rangle.$$

In view of Proposition 2(ix) this means that

$$d(\log\psi)\big(u(t)\big) = \frac{1}{\psi\big(u(t)\big)} d\psi\big(u(t)\big) = \frac{1}{\langle \omega(t), u(t)\rangle}\omega(t).$$

We conclude that

$$d\psi\big(u(t)\big) = \omega(t) \in \mathrm{int}(C^*).$$

It follows in particular that $\omega(s) \in \mathrm{int}(C^*)$ holds for all $s \in [0,T]$ because C^* is invariant under the coadjoint action.

Let $X \in \mathfrak{g}$. Since $\mathfrak{g}$ contains a pointed generating invariant cone, the group G is unimodular, i.e., $\det\big(\mathrm{Ad}(g)\big) = 1$ for all $g \in G$. Now the invariance of C and Proposition 2(ii) show that

$$\psi \circ e^{\mathrm{ad}\, X} = \psi \qquad \forall X \in \mathfrak{g}.$$

Whence

$$\begin{aligned} 0 &= \frac{d}{ds}\Big|_{s=0} \psi\big(u(t)\big) = \frac{d}{ds}\Big|_{s=0} \psi\big(e^{s\,\mathrm{ad}\,X} u(t)\big) \\ &= d\psi\big(u(t)\big) \circ \mathrm{ad}\,X\big(u(t)\big) = \omega(t) \circ \mathrm{ad}\,X\big(u(t)\big) \\ &= -\omega(t) \circ \mathrm{ad}\,u(t)(X) = -\omega'(t)(X). \end{aligned}$$

Since X was arbitrary, ω is constant. Next Proposition 2 and $d\psi\big(u(t)\big) = \omega(t) = \omega(0)$ yield that the values of u lie on a line $\mathbb{R}^+X$ in the interior of C. Finally this proves that $s \in \exp \mathbb{R}^+X$. ∎

Proposition 10. *If S is a pointed generating invariant Lie semigroup, then*

$$\mathrm{comp}(S) \subseteq \exp \mathfrak{L}(S).$$

Proof. First we pick $s \in \mathrm{int}(S) \cap \mathrm{comp}(S)$. Then $d(s) > 0$ (Lemma 7) and with Propositions 8 and 9 we see that $s \in \exp \mathfrak{L}(S)$.

Now let $s \in \mathrm{comp}(S)$ be arbitrary. Since $\mathrm{comp}(S)$ is open in S ([9, V.9]) and $\mathrm{int}(S)$ is dense, it follows from the first part of the proof that $s \in \overline{\exp \mathfrak{L}(S)}$. Let $s = \lim_{n\to\infty} \exp X_n$ with $X_n \in \mathfrak{L}(S)$. If the sequence X_n is bounded, there exists a convergent subsequence and $s \in \exp \mathfrak{L}(S)$ follows. If not, we may assume that $\|X_n\| \to \infty$ for a norm $\|\cdot\|$ on $\mathfrak{g}$. Passing to a subsequence we also assume that $\frac{1}{\|X_n\|}X_n \to X \in \mathfrak{L}(S) \setminus \{0\}$. For $t \in \mathbb{R}^+$ this entails that

$$\mathbf{1} \leq_S \exp\Big(t\frac{1}{\|X_n\|}X_n\Big) \leq_S \exp(X_n)$$

and therefore

$$\mathbf{1} \leq_S \exp(tX) \leq_S s.$$

Since $s \in \mathrm{comp}(S)$, it is impossible that the non-compact set $\exp(\mathbb{R}^+X)$ is contained in the order interval $[\mathbf{1}, s]$ ([11, III.11]), a contradiction. ∎

Proof. (of Theorem 5) Let $H(S) := S \cap S^{-1}$. We may assume that $\mathrm{comp}(S) \neq \emptyset$. Then it follows in particular that $H(S)$ is a compact normal subgroup of G. Let $G_1 := G/H(S)$ and $\pi\colon G \to G_1$ denote the quotient homomorphism. Then $S_1 := \pi(S)$ is an invariant Lie semigroup with $H(S_1) = \{\mathbf{1}\}$ and $\mathrm{comp}(S_1) = \pi\big(\mathrm{comp}(S)\big)$ ([9, V.8]).

Let $s \in \mathrm{comp}(S)$. Then $\pi(s) \in \mathrm{comp}(S_1)$. Let $W_1 := \mathfrak{L}(S_1)$, $\mathfrak{g}_2 := W_1 - W_1$, G_2 the group $\langle \exp \mathfrak{g}_2 \rangle$ endowed with its Lie group topology, and $j\colon G_2 \to \overline{\langle \exp \mathfrak{g}_2 \rangle}$ the corresponding injective morphism of Lie groups. Then $S_2 := \overline{\langle \exp_{G_2} W_1 \rangle}$ is an invariant Lie semigroup in G such that $\mathfrak{L}(S_2) = W_1$ is generating in $\mathfrak{g}_2$. Now it follows from [12, IV.4] that there exists an element $s_2 \in S_2$ such that $j(s_2) = \pi(s)$. We claim that $s_2 \in \mathrm{comp}(S_2)$. If this is false, then there exists $X \in W_1$ such that $\exp_{G_2}(\mathbb{R}^+w) \subseteq [\mathbf{1}, s_2]$ ([9, VI.3]). Then $\exp_{G_1}(\mathbb{R}^+X) \subseteq [\mathbf{1}, \pi(s)]$ contradicts the fact that $\pi(s) \in \mathrm{comp}(S_1)$.

Now Proposition 10 provides $X \in W_1$ with $\exp_{G_2} X = s_2$. Then $\pi(s) = \exp_{G_1} X$. Let $\mathfrak{q} \subseteq \mathfrak{g}$ be an $H(S)$-invariant vector space complement for

$\mathfrak{h} := \mathfrak{L}\big(H(S)\big)$. Then $[\mathfrak{h}, \mathfrak{q}] = \{0\}$ and $\mathfrak{L}(S) = \mathfrak{h} + \mathfrak{L}(S) \cap \mathfrak{q}$. Let $X' \in \mathfrak{L}(S) \cap \mathfrak{q}$ such that $d\pi(\mathbf{1})X' = X$. Then

$$\pi(\exp_G X') = \exp_{G_1} X = \pi(s).$$

Therefore there exists $h \in H(S)$ such that $s = (\exp_G X')h$. The compactness of the group $H(S)$ now implies the existence of $Y \in \mathfrak{h}$ such that $\exp Y = h$. Since $[Y, X'] = 0$, this leads to

$$s = \exp X' \exp Y = \exp(X' + Y)$$

with $X' + Y \in \mathfrak{L}(S)$. ∎

Corollary 11. *If $\mathfrak{g} = \mathfrak{L}(G)$ is a compact Lie algebra and $S \subseteq G$ an invariant Lie semigroup, then $S = \exp \mathfrak{L}(S)$.*

Proof. (Sketch, cf. [8, IV.9]) This is a standard reduction argument. The new complication, in comparison with the proof of Theorem 5, is that $H(S)$ may be non-compact. This does not really matter because the non-compact factor of $H(S)$ is a central vector group in G. ∎

We note that the last result is a contribution to the theory of divisible subsemigroups of Lie groups (cf. [4, 5]). Corollary 11, together with [2, II.7.3, V.6.11] characterizes the divisible pointed subsemigroups of Lie groups with compact Lie algebra as those that are invariant.

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