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Convexity properties of Graßmannians

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In this note we study the images of various moment maps $\Phi_u: G_p(\mathbb{C}^n) \to \mathfrak{u}^*$, where $G_p(\mathbb{C}^n)$ is the Graßmann-manifold of p-planes in $\mathbb{C}^n$ and $\mathfrak{u}^*$ the (real) dual of the Lie algebra $\mathfrak{u}$ of a compact group U acting on $G_p(\mathbb{C}^n)$ in a Hamiltonian way.

More precisely, if we let $\mathfrak{g}_{\mathbb{C}} = \mathrm{gl}(n, \mathbb{C})$ act on $\mathbb{C}^n$ via the identity representation then the p-fold wedge product of this representation is a highest weight module $V = \wedge^p \mathbb{C}^n$ of $G = \mathrm{Gl}(n, \mathbb{C})$ with highest weight vector

$$v_0 = e_1 \wedge \ldots \wedge e_p$$

and highest weight

$$\lambda_0(\,\mathrm{diag}(a_1, \ldots, a_n)) = \sum_{j=1}^{p} a_j,$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis for $\mathbb{C}^n$ and the set $\mathfrak{t}_{\mathbb{C}}$ of diagonal matrices in $\mathrm{gl}(n, \mathbb{C})$ is used as Cartan subalgebra. The ordering of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is chosen such that the corresponding Borel algebra consists of the upper triangular matrices in $\mathrm{gl}(n, \mathbb{C})$. The stabilizer of the line $[v_0] \in \mathbb{P}(V)$ generated by v_0 is the maximal parabolic subgroup

$$P_{\mathbb{C}} = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{Gl}(n, \mathbb{C}) \colon A \in \mathrm{Gl}(p, \mathbb{C}), D \in \mathrm{Gl}(q, \mathbb{C})\}$$

and the action of $\mathrm{Gl}(n, \mathbb{C})$ induces an embedding

$$G_p(\mathbb{C}^n) \cong G_{\mathbb{C}}/P_{\mathbb{C}} \cong G_{\mathbb{C}} \cdot [v_0] \to \mathbb{P}(V).$$

The choice of a Hermitean metric on V gives a Fubini-Study metric on $\mathbb{P}(V)$ which in turn induces a Kähler metric on $G_p(\mathbb{C}^n)$. We choose the metric in such a way that $\mathrm{U}(n) \subseteq \mathrm{Gl}(n, \mathbb{C})$ acts unitarily on V. Then $U(n)$ also preserves the Kähler metric on $G_p(\mathbb{C}^n)$. This in particular means that the action of $\mathrm{U}(n)$ preserves the symplectic structure of $G_p(\mathbb{C}^n)$, which is given by the imaginary part of the Kähler metric. Let $\dot{\varphi}: \mathfrak{u}(n) \to \mathcal{X}(G_p(\mathbb{C}^n))$ be the derived action which associates with a Lie algebra element X the vector field on $G_p(\mathbb{C}^n)$ whose flow is given by $\exp tX$. Moreover we consider the map

$$\psi: C^\infty(G_p(\mathbb{C}^n)) \to \mathcal{X}(G_p(\mathbb{C}^n)),$$

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which maps a function to the corresponding Hamiltonian vector field. Choose a linear map

$$\kappa: \mathfrak{g} \to C^\infty\big(G_p(\mathbb{C}^n)\big)$$

such that $\psi \circ \kappa = \dot{\varphi}$ and define

$$\Phi: G_p(\mathbb{C}^n) \to \mathfrak{u}(n)^*$$

via

$$\langle \Phi(x), X\rangle = \big(\kappa(X)\big)(x) \quad X \in \mathfrak{u}(n).$$

Then Φ is called a moment map if it is $\mathrm{U}(n)$-equivariant. Note at this point that the construction of the moment map just given of course works for any group action $G \times M \to M$ preserving the symplectic structure and not only for $\mathrm{U}(n) \times G_p(\mathbb{C}^n) \to G_p(\mathbb{C}^n)$. Using the fact that moment maps are characterized by $\psi \circ \kappa = \dot{\varphi}$ and the equivariance it is not hard to show that the moment map for the action of $\mathrm{U}(n)$ on $G_p(\mathbb{C}^n)$ is just the restriction to $G_p(\mathbb{C}^n)$ of the moment map for the action of $\mathrm{U}(n)$ on $\mathbb{P}(V)$ which is given by

$$\langle \Phi([v]), X\rangle = i\frac{(X \cdot v \mid v)}{(v \mid v)}$$

or, when we introduce homogeneous coordinates with respect to an orthonormal basis, by

$$\Phi\big((z_0 : \ldots : z_N)\big) = \frac{i}{\sum_{k=0}^N |z_k|^2}(z_j\overline{z}_k)_{j,k=0,\ldots,N},$$

where $\dim V = N+1$. Moreover, when we restrict the action to a subgroup U of $\mathrm{U}(n)$, then the corresponding moment map $\Phi_\mathfrak{u}$ is Φ followed by the canonical projection $\mathfrak{u}(n)^* \to \mathfrak{u}^*$, where $\mathfrak{u} = \mathbf{L}(U) \subseteq \mathfrak{u}(n)$.

Thes moment maps and their images are closely related to subsemigroups of $G = \mathrm{Gl}(n, \mathbb{R})$ which have been studied in the context of control theory and harmonic analysis. The common feature of the examples to be described below is the following:

Let $G_p(\mathbb{R}^n)$ be the real Graßmannian of p-planes in $\mathbb{R}^n$ which may be constructed as the set of real points of $G_p(\mathbb{C}^n)$ and H be a closed subgroup of $\mathrm{Gl}(n, \mathbb{R})$. Further let U be a compact form of H, i.e., the complexifications $\mathfrak{h}_\mathbb{C}$ and $\mathfrak{u}_\mathbb{C}$ with $\mathfrak{h} = \mathbf{L}(H)$ agree. Note that $\mathrm{Gl}(n, \mathbb{R})$ acts transitively on $G_p(\mathbb{R}^n)$. We assume that H has at least two open orbits in $G_p(\mathbb{R}^n)$. Then our examples will satisfy the following conditions

(i) $\Phi_\mathfrak{u}(G/P)$ is convex, where $P = P_\mathbb{C} \cap \mathrm{Gl}(n, \mathbb{R})$.
(ii) $\Phi_\mathfrak{u}(HP/P)$ is open in the linear span of $\Phi_\mathfrak{u}(G/P)$ and dense in $\Phi_\mathfrak{u}(G/P)$.
(iii) $\Phi_\mathfrak{u}|_{HP/P}$ is a diffeomorphism onto its image.

The semigroups alluded to earlier are given by

$$S = \{g \in G: gHP/P \subseteq HP/P\}.$$

We note in passing that the complex conjugation interchanges the Kähler metric with its complex conjugate, so it is antisymplectic, i.e., it interchanges the symplectic form with its negative. Moreover the moment map Φ anti-commutes with complex conjugation, i.e., it satisfies

$$\Phi(\overline{x}) = -\overline{\Phi(x)}.$$

In particular, the image of G/P is contained in $i\mathfrak{g}^*$.

Example 1. We let H be the subgroup of diagonal matrices and $p = 1$, then $G_p(\mathbb{C}^n) = \mathbb{P}(\mathbb{C}^n)$. In this case the open H-orbits on $G_p(\mathbb{R}^n) = \mathbb{P}(\mathbb{R}^n)$ are given by the sets

$$\{(z_0 : \ldots : z_{n-1}) \in \mathbb{P}(\mathbb{R}^n) : \varepsilon_j z_j > 0 \quad j = 0, \ldots, n-1\},$$

where the ε_j are fixed and equal to 1 or -1. All together we have 2^n open H-orbits and the statements (i) and (ii) are special cases of Duistermaat's extension of the Atiyah-Guillemin-Sternberg convexity theorem which deals with fixed point sets of antisymplectic involutions. It is of course not necessary to appeal to this theorem here, since here $\mathfrak{u} = i\mathfrak{h}$ and the moment map is given by

$$\Phi\big((z_0 : \ldots : z_{n-1})\big) = \frac{i}{\sum_{k=0}^{N} |z_k|^2}(|z_0|^2, \ldots, |z_{n-1}|^2)$$

so that it is elementary to check that $\Phi_{\mathfrak{u}}\big(\mathbb{P}(\mathbb{R}^n)\big) = \Phi_{\mathfrak{u}}\big(\mathbb{P}(\mathbb{C}^n)\big)$ is an n-dimensional simplex satisfying (i) and (ii).

The semigroup S in this case consists of all elements in $\mathrm{Gl}(n, \mathbb{R})$ with non-negative entries (cf. [9] for more information).

Example 2. Let $H = \mathrm{O}(p, q)$, where $n = p + q$. We write elements of G and $\mathfrak{g} = \mathrm{gl}(n, \mathbb{R})$ as block matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A of size $p \times p$ and D of size $q \times q$. The usual Cartan involutions $g \mapsto (g^\top)^{-1}$ and $X \mapsto -X^\top$ are both denoted by θ. Let τ be given by

$$g \mapsto J\theta(g)J, \quad X \mapsto J\theta(X)J$$

on G and $\mathfrak{g}$ respectively, where

$$J = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}.$$

Then $H = G^\tau = \mathrm{O}(p, q)$ and G/H is a pseudo-Riemannian symmetric space. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q} = \mathfrak{k} + \mathfrak{p}$ be the eigenspace decompositions of $\mathfrak{g}$ with respect to τ and θ for the eigenvalues 1 and -1. Then

$$\mathfrak{q} \cap \mathfrak{p} = \{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A, D \text{ symmetric } \},$$

$$\mathfrak{h} \cap \mathfrak{k} = \{\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A, D \text{ skew} - \text{symmetric } \},$$

$$\mathfrak{h} \cap \mathfrak{p} = \{\begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix}\}$$

and

$$\mathfrak{a} = \{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A, D \text{ diagonal } \}$$

is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}, \mathfrak{p}$ and $\mathfrak{q}$.

Note that $\mathfrak{u} := (\mathfrak{h} \cap \mathfrak{k}) + i(\mathfrak{h} \cap \mathfrak{p})$ is a compact form of $\mathfrak{h}$ and the image of G/P under the corresponding moment map $\Phi_{\mathfrak{u}}$ has to be contained in the dual of $\mathfrak{u} \cap i\mathfrak{g} = i(\mathfrak{h} \cap \mathfrak{p})$. The key idea to prove the convexity of this image is to find a suitable maximal abelian subspace of $i(\mathfrak{h} \cap \mathfrak{p})$, study the image of the moment map for the corresponding torus and then use equivariance properties to describe $\Phi_{\mathfrak{u}}(G/P)$.

Consider the elements

$$X_i = \begin{pmatrix} 0 & B_i \\ B_i^\top & 0 \end{pmatrix},$$

where all entries of B_i are zero except for the entry at the i-th row and the i-th column which is 1. Then

$$\mathfrak{b} := \sum_{i=1}^{\min(p,q)} \mathbb{R} X_i$$

is maximal abelian in $\mathfrak{h} \cap \mathfrak{p}$. But $\mathfrak{h} = (\mathfrak{k} \cap \mathfrak{h}) + (\mathfrak{p} \cap \mathfrak{h})$ is the Cartan decomposition associated with $\theta|_{\mathfrak{h}}$ so there belongs a Weyl group to the data $(\mathfrak{h}, \mathfrak{b})$. This Weyl group which we denote by $W_{\mathfrak{b}}$ is isomorphic to the symmetric group on $\min(p,q)$ elements, extended by the multiplications with -1 in each coordinate with the obvious action by permutation of the entries.

If $p \leq q$ we write the elements of $\mathfrak{g}$ as block matrices with nine entries, where the diagonal entries are of size $p \times p, p \times p$ and $(q-p) \times (q-p)$ (the case $p > q$ can be treated analogously and we will not mention that case explicitly). Then conjugation by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -\mathbf{1}_p & \mathbf{1}_p & 0 \\ \mathbf{1}_p & \mathbf{1}_p & 0 \\ 0 & 0 & \mathbf{1}_{(q_p)} \end{pmatrix}$$

is an automorphism of $\mathfrak{g}_{\mathbb{C}}$ which maps $\mathfrak{a}_{\mathbb{C}}$ onto a Cartan algebra $\mathfrak{t}^\sharp_{\mathbb{C}}$ containing $\mathfrak{b}$ via

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} B+A & B-A & 0 \\ B-A & B+A & 0 \\ 0 & 0 & 2C \end{pmatrix}.$$

There is another Weyl group, denoted by $W^\sharp$, associated to the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}^\sharp_{\mathbb{C}})$. This Weyl group is of course isomorphic to the Weyl group for the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ which is simply the permutation group on n letters. Each element of $W_{\mathfrak{b}}$ can be obtained from an element of $W^\sharp$ by restriction to $\mathfrak{b}$. Let $\mathfrak{t}^\sharp = \mathfrak{u}(n) \cap \mathfrak{t}^\sharp_{\mathbb{C}}$ and

$$\mathrm{p}\colon \mathfrak{u}(n)^* \to i\mathfrak{b}^*$$

then the crucial property of our Weyl groups is

$$(*) \qquad \mathrm{p}\big(\mathrm{conv}(W^\sharp \cdot \mu)\big) = \mathrm{conv}\big(W_{\mathfrak{b}} \cdot \mathrm{p}(\mu)\big)$$

for all $\mu \in i(\mathfrak{t}^\sharp)^*$.

Let $T^\sharp$ and $T_\mathfrak{b}$ be the tori in $\mathrm{U}(n)$ with Lie algebras $\mathfrak{t}^\sharp$ and $i\mathfrak{b}$. Note the following inclusions:

$$\begin{array}{ccc} \mathfrak{t}^\sharp & \longleftarrow & i\mathfrak{b} \\ \downarrow & & \downarrow \\ \mathfrak{u}(n) & \longleftarrow & \mathfrak{u}. \end{array}$$

We collect the corresponding moment maps in a commutative diagramm

$$\begin{array}{ccccc} & & (\mathfrak{t}^\sharp)^* & & \\ & \nearrow & \uparrow \Phi^\sharp & \searrow & \\ \mathfrak{u}(n)^* & \xleftarrow{\Phi} & G_p(\mathbb{C}^n) & \xrightarrow{\Phi_\mathfrak{b}} & i\mathfrak{b}^* \, . \\ & \searrow & \downarrow \Phi_\mathfrak{u} & \nearrow & \\ & & \mathfrak{u}^* & & \end{array}$$

As in the case of Example 1

$$\Phi^\sharp(G_\mathbb{C}/P_\mathbb{C}) = \Phi^\sharp(G/P)$$

is a simplex and hence $\Phi_\mathfrak{b}(G/P)$, being a projection of $\Phi^\sharp(G/P)$ is also a convex polytop. In fact, we may again apply Duistermaat's results and this time [2], Proposition 4.2, implies

$$\Phi_\mathfrak{b}(G/P) = \Phi_\mathfrak{b}(\overline{B\cdot[v_0]}),$$

where B is the analytic subgroup of G with Lie algebra $\mathfrak{b}$. Moreover the general theory says that $\Phi^\sharp(G/P)$ is the convex hull of the Weyl group orbit $W^\sharp\cdot\lambda^\sharp$ for any extremal weight $\lambda^\sharp$ of the module V with respect to $\mathfrak{t}^\sharp_\mathbb{C}$. But such a weight is given by

$$\lambda^\sharp\colon \begin{pmatrix} A & B & 0 \\ B & A & 0 \\ 0 & 0 & C \end{pmatrix} \mapsto \sum_{i=1}^{p}(b_i - a_i) + \sum_{k=1}^{q} c_k,$$

where $A = \mathrm{diag}(a_1,\ldots,a_p), B = \mathrm{diag}(b_1,\ldots,b_p)$ and $C = \mathrm{diag}(c_1,...,c_q)$. Applying the projection p and $(*)$ we find

$$(**) \qquad \Phi_\mathfrak{b}(G/P) = \mathrm{conv}\big(W_\mathfrak{b}\cdot \mathrm{p}(\lambda^\sharp)\big).$$

But the projection $\mathrm{p}(\lambda^\sharp) = \mu_0$ corresponds to the element

$$\begin{pmatrix} 0 & B_0 \\ B_0^\top & 0 \end{pmatrix} \in \mathfrak{b}$$

with $B_0 = \sum_{i=1}^{\min(p,q)} B_i$. Thus the image of $\Phi_\mathfrak{b}$ is the cube which one obtains from $\begin{pmatrix} 0 & B_0 \\ B_0^\top & 0 \end{pmatrix}$ upon the action of the Weyl group $W_\mathfrak{b}$.

Now we are ready to use equivariance: Let $K_H = \mathrm{O}(p)\times\mathrm{O}(q)$ be the maximal compact subgroup of H contained in $\mathrm{U}(n)$ and consider the Cartan decomposition

$$H = K_H B K_H.$$

Note that $K_H \subseteq P$ so that the U-equivariance of Φ implies

$$\Phi(H\cdot[v_0]) = \Phi(K_H B\cdot[v_0]) = \mathrm{Ad}^*(K_H)\Phi(B\cdot[v_0]).$$

Analogously, we have

$$\Phi_{\mathfrak{u}}(H\cdot[v_0]) = \mathrm{Ad}^*(K_H)\Phi_{\mathfrak{b}}(B\cdot[v_0]) = \mathrm{Ad}^*(K_H)\Phi_{\mathfrak{b}}(B\cdot\cdot[v_0]).$$

We set

$$D := \mathrm{Ad}^*(K_H)\Phi_{\mathfrak{b}}(G/P) = \Phi_{\mathfrak{u}}(\overline{H\cdot[v_o]}) \subseteq i(\mathfrak{h}\cap\mathfrak{p})^*.$$

Let $\mu\in D$. Then

$$\mathrm{conv}\big(\mathrm{Ad}^*(K_H)\mu\big) \subseteq D. \tag{$\dagger$}$$

In fact, let $\mu' \in \mathrm{conv}\big(\mathrm{Ad}^*(K_H)\mu\big)$. Then μ' corresponds to an element of $i(\mathfrak{h}\cap\mathfrak{p})^*$ so that there exists an element $k\in K_H$ such that $\mathrm{Ad}^*(k)\mu'\in i\mathfrak{b}^*$. Let $[v]\in G/P$ be such that $\mu=\Phi([v])$ then

$$\begin{aligned}
\mathrm{p}(\mathrm{conv}\big(\mathrm{Ad}^*(K_H)\mu\big) &= \mathrm{conv}\big(\mathrm{p}(\Phi(K_H\cdot[v]))\big)\\
&\subseteq \mathrm{conv}\big(\Phi_{\mathfrak{b}}(K_H\cdot[v])\big)\\
&\subseteq \mathrm{conv}\big(\Phi_{\mathfrak{b}}(G/P)\big)\\
&= \Phi_{\mathfrak{b}}(G/P)\subseteq D
\end{aligned}$$

and hence

$$\mathrm{Ad}^*(k)\mu' = \mathrm{p}(\mathrm{Ad}^*(k)\mu')\in D.$$

Thus we have

$$\mu'\in\mathrm{Ad}^*(K_H)D = D$$

which proves ($\dagger$). Now let $\mu\in\Phi_{\mathfrak{b}}(G/P)\subseteq i(\mathfrak{h}\cap\mathfrak{p})^*$, and $k\in K_H$ be such that $\mathrm{Ad}^*(k)\mu\in i\mathfrak{b}^*$. But then

$$\mathrm{Ad}^*(k)\mu\in i\mathfrak{b}^*\cap\Phi_{\mathfrak{u}}(G/P)\subseteq\Phi_{\mathfrak{b}}(G/P)\subseteq D$$

and hence $\mu\in\mathrm{Ad}^*(k)D\subseteq D$. Thus we have

$$\Phi_{\mathfrak{b}}(G/P) = \Phi_{\mathfrak{b}}(\overline{H\cdot[v_0]}) = D.$$

Using ($\dagger$), ($*$) and ($**$) we calculate

$$\begin{aligned}
D &= \mathrm{Ad}^*(K_H)\Phi_{\mathfrak{b}}(G/P)\\
&= \mathrm{Ad}^*(K_H)\mathrm{conv}(W_{\mathfrak{b}}\cdot\mu_0)\\
&\subseteq \mathrm{conv}(\mathrm{Ad}^*(K_H)\cdot\mu_0)\\
&\subseteq D,
\end{aligned}$$

since the action of $W_{\mathfrak{b}}$ is induced by $N_{K_H}(\mathfrak{b})$. Thus, in particular D is convex and we have shown (i) and the density part of (ii) for this example. We can even say more:

$$\Phi_{\mathfrak{b}}(G/P) = \{\begin{pmatrix} 0 & gBh\\ h^{-1}B^\top g^{-1} & 0\end{pmatrix} : g\in \mathrm{O}(p), h\in\mathrm{O}(q), B\in\Phi_{\mathfrak{b}}(G/P)\}$$

which is linearly isomorphic to the set of linear contractive mappings from $\mathbb{C}^q$ to $\mathbb{C}^p$ with the usual norm. This can be seen by mapping the image of the linear operator gBh isometrically to $\mathbb{C}^q$ and then considering the polar decomposition.

In order to show (iii) we study $\Phi_{\mathfrak{u}}|_{B\cdot[v_0]}$ in more detail. Consider the elements

$$v_i := e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{p+i} \wedge e_{i+1} \wedge \ldots \wedge e_p$$

then

$$(\exp tX_i)\cdot v_i = (\cosh t)v_i + (\sinh t)v_0.$$

The map

$$(t_1,\ldots,t_{\min(p,q)}) \mapsto \prod_{i=1}^{\min(p,q)} \exp t_i X_i \cdot [v_0]$$

is a diffeomorphism $\mathbb{R}^{\min(p,q)} \to B\cdot[v_0]$. If we use it as coordinate chart for $B\cdot[v_0]$ and further identify $i\mathfrak{b}^*$ with $\mathbb{R}^{\min(p,q)}$ via the dual basis of $\{X_1,\ldots,X_{\min(p,q)}\}$, then $\Phi_{\mathfrak{u}}|_{B\cdot[v_0]}$ is given by

$$(t_1,\ldots,t_s) \mapsto (\tanh\frac{t_1}{2},\ldots,\tanh\frac{t_s}{2}),$$

where $s = \min(p,q)$. In order to prove (iii) it now suffices to consider the mappings

$$\varphi_1 = \big((k,b)\mapsto khP\big)\colon K_H \times B \to HP/P$$

and

$$\varphi_2 = \big((k,f)\mapsto \mathrm{Ad}^*(k)f\big)\colon K_H \times i\mathfrak{b}^* \to i(\mathfrak{h}\cap\mathfrak{p})^*.$$

In fact,

$$\varphi_2\big(k, \mathrm{p}_{\mathfrak{h}} \circ \Phi(b\cdot[v_0])\big) = \Phi_{\mathfrak{h}} \circ \varphi_1(k, b\cdot[v_0])$$

and the claim follows since the fibers of φ_1 and of φ_2 are both given by the Weyl group $W_{\mathfrak{b}}$.

The semigroup S in this case is given by

$$\overline{H \exp c_{\max} H}$$

with

$$c_{\max} = \{\mathrm{diag}(a_1,\ldots,a_n)\colon (\forall j \le p < k)\ a_j \ge a_k\}$$

(cf. [6, §5.7).

Example 3. We conclude this report with an example which indicates that the simultaneous occurence of the properties (i)-(iii) and semigroups as shown in Examples 1 and 2 is not purely accidental. Let $H = \mathrm{Gl}(n,\mathbb{R})$, and $G = \mathrm{Gl}(n,\mathbb{C})$ embedded as the diagonal in $G_{\mathbb{C}} = \mathrm{Gl}(n,\mathbb{C}) \times \mathrm{Gl}(n,\overline{\mathbb{C}})$, where $\overline{\mathbb{C}}$ indicates that on this copy of $\mathbb{C}$ we have the opposite complex structure. As maximal parabolic in $\mathfrak{g}$ we choose

$$\mathcal{P} = \{\begin{pmatrix} a & v^\top \\ 0 & D \end{pmatrix} \in \mathrm{sl}(n,\mathbb{C})\colon a\in\mathbb{C}, v\in\mathbb{C}^n\}.$$

Then $G/P = \mathbb{P}(\mathbb{C}^n)$, $G_{\mathbb{C}}/P_{\mathbb{C}} = \mathbb{P}(\mathbb{C}^n) \times \mathbb{P}(\overline{\mathbb{C}}^n)$, and a compact form of H is $\mathrm{U}(n)$. The moment map $\Phi_{\mathfrak{u}}: G_{\mathbb{C}}/P_{\mathbb{C}} \to \mathfrak{u}(n)^*$ associated with this action is

$$\Phi_{\mathfrak{h}}\big(z_0 : \ldots : z_{n-1}), (w_0 : \ldots : w_{n-1})\big) = \frac{i}{2}\Big(\frac{z_j \overline{z}_k}{\sum_{k=0}^{n-1} |z_k|^2} + \frac{\overline{w_j} w_k}{\sum_{k=0}^{n-1} |w_k|^2}\Big)_{j,k=0,\ldots,n-1}$$

and, when restricted to G/P,

$$\Phi_{\mathfrak{h}}\big((z_0 : \ldots : z_{n-1})\big) = \frac{i}{\sum_{k=0}^{n-1} |z_k|^2} (\operatorname{Re} z_j \overline{z}_k).$$

Whereas the image is a disk for $n = 2$ it is non-convex for $n = 3$. This can be seen from a somewhat tedious comutation using affine coordinates on the projective plane. Note that for $n \geq 3$ no proper subsemigroup S of $\mathrm{Gl}(n, \mathbb{C})$ is known which has non-empty interior and satisfies $\mathrm{Gl}(n, \mathbb{R}) \subseteq S$.

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