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The Divisibility Problem for subsemigroups of Lie groups

K. H. Hofmann and W. A. F. Ruppert

An element d of a semigroup S is called *divisible* if it has roots of arbitrary order; that is, for every $n \in \mathbb{N}$ there is an element d_n in S such that $d_n^n = d$. If the elements d_n can be taken in a prescribed subset D of S then d is said to be *divisible in* D.

In the algebraic as well as in the topological theory of groups and semigroups divisibility is the major basic concept which allows the introduction of a linear structure by defining an exponential function: If an element is divisible then there is usually a good chance to find a (rational or, by continuous extension, real) one-parameter semigroup passing through it; combining 'sufficiently many' such one-parameter semigroups gives rise to an exponential function. The solution of Hilbert's fifth problem and the theory of divisible compact topological semigroups are familiar examples for the successfull application of these ideas.

In the Lie theory of semigroups new aspects of divisibility show up: For closed subsemigroups of Lie groups the exponential image of a zero neighborhood in the tangent wedge is in general *not* a neighborhood of the identity, hence divisibility of the whole semigroup— though known to imply surjectivity of the exponential function— does not *a priori* imply local divisibility. Also, if the tangent wedge of a closed subsemigroup is a Lie semialgebra, or, equivalently, if it generates a locally divisible *local semigroup in some Campbell-Hausdorff neighborhood in the Lie algebra*, then it does not folllow *a priori* that the semigroup itself is locally divisible.

The "Divisibility Problem," hithereto still open, asks whether the tangent wedge of a closed divisible subsemigroup in a connected Lie group is a Lie semialgebra.

The aim of the subsequent notes is to give a short introduction to the so far existing divisibility theory for subsemigroups of Lie groups, and to present, in condensed and abridged form, a few results recently obtained by the authors.

1. Local and global divisibility for subsemigroups of Lie groups

We first recapitulate the essential concepts and facts. Unless specified otherwise we use the notation and terminology of [3]; all unexplained terms can be looked up there.

As in [3], we let the local Lie theory of semigroups take place in the Lie algebra rather than in the Lie group: we work inside a given *Campbell-Hausdorff neighborhood* B, that is, a star-shaped zero neighborhood in the Lie algebra, so that triple products with respect to the Campbell-Hausdorff multiplication $*$ are defined.

We say that a subset S of B is a *local semigroup with respect to* B if $0 \in S$ and $(S * S) \cap B \subseteq S$. The *tangent wedge* $\mathfrak{L}(S)$ of S is defined to be the set W of all subtangent vectors of S at 0; this set is a *Lie wedge*, i.e. it is a closed convex cone and invariant under the action of $e^{\operatorname{ad} x}$ for any $x \in H(W) := W \cap -W$. (Note that any two local semigroups in the same germ have the same tangent wedge.)

One of the fundamental results of Lie theory says that every Lie wedge W is the tangent wedge of a local semigroup. Furthermore, every local semigroup S in $\mathfrak{g}$ with $\mathfrak{L}(S) = W$ has interior points in the smallest Lie algebra $\mathfrak{g}(W)$ containing W, regardless whether W has interior points in $\mathfrak{g}(W)$ or not. Thus for many purposes (in particular, in dealing with the Divisibility Problem) it will cause no loss of generality if we assume that $\mathfrak{g} = \mathfrak{g}(W)$.

If, on the other hand, W is a wedge in $\mathfrak{g}$ such that for some Campbell-Hausdorff neighborhood B the intersection $W \cap B$ is a local semigroup with respect to B then W is called a *semialgebra*. The wedge W is called an *invariant wedge* if it is invariant under the maps $e^{\operatorname{ad} x}$, for all $x \in \mathfrak{g}$ (not only for $x \in H(W)$). It can be shown that invariant wedges are semialgebras.

Suppose that S is a *preanalytic* subsemigroup of a Lie group G; that is, the subgroup $G(S)$ generated by S in G is analytic. Then for any Campbell-Hausdorff neighborhood B on which the exponential map is injective the set $\exp^{-1}(S \cap \exp B)$ is a local semigroup with respect to B. By a slight abuse of notation we also write $\mathfrak{L}(S)$ for the tangent wedge of this local semigroup and call it the *tangent wedge of* S. If S is closed in $G(S)$ (in particular, if S is closed in G) then $\mathfrak{L}(S) = \{x \in \mathfrak{g} \mid \exp x \in S\}$.

A preanalytic subsemigroup S of G is called *strictly infinitesimally generated* if it is generated (as a subsemigroup of G) by $\exp \mathfrak{L}(S)$. If S is strictly infinitesimally generated then it is preanalytic and the Lie algebra of $G(S)$ is generated (as a Lie algebra) by $\mathfrak{L}(S)$.

A local semigroup S with respect to a Campbell-Hausdorff neighborhood B is called *strictly infinitesimally generated* if it is locally generated by $\mathfrak{L}(S) \cap B$. Note that if S is a strictly infinitesimally generated subsemigroup in the Lie group G and B is a Campbell-Hausdorff neighborhood in $\mathfrak{g}$ then the induced local semigroup $\exp^{-1}(S \cap \exp B)$ need not be strictly infinitesimally generated.

A closed [local] semigroup S is *infinitesimally generated* if it contains a dense strictly infinitesimally generated [local] subsemigroup.

Let us now turn to divisibility.

1.1. Definition. (i) A subset D of a local or global semigroup is called *divisible in itself*, or *divisible* for short, if each of its elements is divisible in D. A divisible semigroup is sometimes also called *a globally divisible semigroup*, for emphasis.

(ii) A subset D of a Lie group with $\mathbf{1} \in D$ is called *locally divisible* if $\mathbf{1}$ has a neighborhood basis in D consisting of divisible subsets. Similarly, a subset C of a Campbell-Hausdorff neighborhood B in the Lie algebra $\mathfrak{g}$ with $0 \in C$ is called *locally divisible* if 0 has a neighborhood basis in C which consists of divisible subsets.

Note that local divisibility does *not* mean simply that there exists a divisible neighborhood of the identity. Also, it is not clear from the onset that the semigroup S must be locally divisible if the tangent wedge of S generates a locally divisible local semigroup with respect to some Campbell-Hausdorff neighborhood.

The reason for this kind of subtleties lies in the fact that we actually use the concept 'local' in two different settings: in the semigroup setting ('local' with respect to a subset of a Lie group), and in the setting of local semigroups generated by the tangent wedge ('local' with respect to a subset of a Campbell-Hausdorff neighborhood B in a Lie algebra, 'growing' by successive multiplication as far as B reaches).

Recall that a Lie group is called *exponential* if the exponential function is surjective, that is, if every point lies on a one-paramter subgroup. In [7] a Lie group G is called *weakly exponential* if the image under the exponential function is dense in G. We adapt these definitions to subsets of semigroups, only replacing the term 'weakly exponential' by 'densely exponential').

1.2. Definition. Let A be a subset of G.

(i) A point $a \in A$ is said to be *exponential in* A if there is an element $x \in \mathfrak{g}$ such that $\exp x = a$ and $\exp[0,1]{\cdot}x \subseteq A$.

(ii) If every element of A is exponential in A then A is said to be *exponential.* If the exponential elements of A are dense in A then we say that A is *densely exponential.*

(iii) Suppose that A is closed and contains the identity. Then A is said to be *locally exponential* if there exists a basis of exponential $\mathbf{1}$-neighborhoods in A.

On the level of Lie algebras, a set $M \subseteq \mathfrak{g}$ is *locally exponential* if it contains 0 and has a neighborhood base at 0 consisting of star-shaped sets. Thus a local semigroup S is locally exponential if and only if the intersections of its Lie wedge W with the Campbell-Hausdorff neighborhoods of 0 form a neighborhood basis of 0 in S.

The following result provides the basis for the structure theory of semialgebras (HOFMANN and LAWSON [6] (1988), cf. also [3], Theorem IV.1.31, on p. 295):

1.3. Theorem. *For a local semigroup S with Lie wedge W the following assertions are equivalent:*

(a) *S is locally divisible;*

(b) *there exists **at least one** Campbell-Hausdorff neighborhood B such that $S \cap B$ is locally divisible;*

(c) *S is locally exponential;*

(d) *W is a semialgebra, i. e., there exists a Campbell Hausdorff neighborhood B such that $(B \cap W) * (B \cap W) \subseteq W$;*

(e) *for **all** Campbell-Hausdorff neighborhoods B we have $(B \cap W) * (B \cap W) \subseteq W$;*

(f) *for every point $x \in W$ we have $[x, T_x] \subseteq T_x$, where T_x denotes the tangent space of W at x.*

Note that the above assertion (f) implies that the vector subspace $V = W - W$ spanned by W is a Lie subalgebra of $\mathfrak{g}$ (since $T_x = W - W = V$ for all points x in the algebraic interior of W). Thus if $G = G(S)$ then W is *generating* in $\mathfrak{g}$, that is, $\mathfrak{g} = W - W$. In this case assertion (f) is also equivalent to

(f′) *for every C^1-point in the boundary of W we have $[x, T_x] \subseteq T_x$.*

An immediate consequence of the above result is that the tangent wedge of a closed submonoid S of a connected Lie group is a semialgebra if S is locally divisible. KARL HERMANN NEEB [9] has shown that the converse holds if, in addition, S is infinitesimally generated.

Similar to the local case globally divisible closed semigroups are exponential (HOFMANN and LAWSON [5] (1983), cf. [3], Theorem V.6.5, on p. 460):

1.4. Theorem. *A closed subsemigroup of a connected Lie group is globally divisible if and only if it is exponential.*

In view of this result the following problem seems to be natural— in fact at first glance one would suspect that Theorem 1.4 already furnishes all essential facts needed for a solution:

Problem 1:

(i) *Is the Lie wedge of a closed divisible subsemigroup a semialgebra?*

(ii) *Are closed divisible subsemigroups of connected Lie groups locally divisible?*

Obviously, if the answer to question (ii) is 'yes' then also (i) is settled in the affirmative.

In 1983 HOFMANN and LAWSON indeed have given an affirmative answer to question (i) for a special class of closed divisible subsemigroups. This class comprises the closed divisible submonoids with trivial group of units, and, in addition, those which are subsemigroups of connected Lie groups with trivial maximal compact subgroups ([6]). Let us recall the formulation of the results by HOFMANN and LAWSON as recorded in [3] (Theorem V.6.10, on p. 462):

1.5. Definition. Let G be Lie group with Lie algebra $\mathfrak{g}$. A wedge W in $\mathfrak{g}$ *disperses in* G if there exists an open neighborhood B of 0 in $\mathfrak{g}$ such that $B \cap W = B \cap \exp^{-1}(\exp W)$.

$$\begin{array}{ccccc}
\boxed{\begin{array}{c} S \text{ is} \\ \text{divisible} \end{array}} & \begin{array}{c} \not\Leftarrow \\ \overset{?}{\Rightarrow} \end{array} & \boxed{\begin{array}{c} S \text{ is} \\ \text{locally divisible} \end{array}} & \Longleftrightarrow & \boxed{\begin{array}{c} \mathfrak{L}(S) \text{ generates a loc.} \\ \text{div.loc. semigroup} \end{array}} \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
\boxed{\begin{array}{c} S \text{ is} \\ \text{exponential} \end{array}} & \begin{array}{c} \not\Leftarrow \\ \overset{?}{\Rightarrow} \end{array} & \boxed{\begin{array}{c} S \text{ is} \\ \text{locally exponential} \end{array}} & \Longleftrightarrow & \boxed{\begin{array}{c} \mathfrak{L}(S) \text{ is a} \\ \text{semialgebra} \end{array}} \\
\Downarrow \not\Uparrow & & \Downarrow \not\Uparrow & & \\
\boxed{\begin{array}{c} S \text{ is densely} \\ \text{exponential} \end{array}} & \Longrightarrow & \boxed{\begin{array}{c} S \text{ is locally} \\ \text{densely exponential} \end{array}} & &
\end{array}$$

Fig.1: Implications between divisibility properties

1.6. Theorem. *Let S be a closed divisible subsemigroup of a Lie group G with group of units $H(S)$ and suppose that there exists an open neighborhood U of the identity such that any compact connected subgroup of G is contained in $H(S)$ whenever it is contained in the tube $UH(S)$. Then the tangent wedge $\mathfrak{L}(S)$ of S is a Lie semialgebra which disperses in G.*

But in fact, under the given assumptions, Theorem 1.6 also yields an affirmative answer to question (ii). We only have to apply the following observation (note that for divisible closed semigroups the condition in Definition 1.5 reads as $B \cap W = B \cap \exp^{-1} S$, by Theorem 1.4).

1.7. Lemma. *Let S be a closed subsemigroup of a connected Lie group G. If the tangent wedge $W = \mathfrak{L}(S)$ of S satisfies $B \cap W = B \cap \exp^{-1} S$ for some 0-neighborhood B then S is locally divisible.*

Proof. Note that the relation $B \cap W = B \cap \exp^{-1} S$ holds for any smaller neighborhood as well, so we may assume that B is a Campbell-Hausdorff neighborhood on which $\exp$ is injective. Write $\exp_B$ for the restriction of the exponential function to B. Then

$$\begin{aligned}\exp_B^{-1}(\exp B \cap S) &= B \cap \exp_B^{-1}(S) = \{b \in B \mid \exp_B(b) \in S\} = B \cap \exp^{-1} S = \\ &= B \cap W,\end{aligned}$$

thus $(\exp B) \cap S = \exp(B \cap W)$ is divisible. Since the exponential images of the Campbell-Hausdorff neighborhoods form a neighborhood basis at $\mathbf{1}$ this implies the assertion. ■

However, the general case is still unsettled, and it has turned out in the meantime that the problems involved are surprizingly hard. This fact seems to indicate that at the heart of this problem there are important structural features of 'Lie semigroups' which, at the time being, are not even touched by the existing theory.

It seems highly probable that the solution of Problem 1 will also require (or at least furnish) the answer to the following questions:

Problem 2:

(i) *Does local divisibility follow from global divisibility without the assumption that the semigroup S in question is closed?*

(ii) *Is a closed and densely exponential subsemigroup of a Lie group locally divisible ?*

In addition to the results by LAWSON and HOFMANN, let us now announce, without proof, a few facts further backing the conjecture that the answer to question (ii) in the above Problem 1 is affirmative. (The proofs of these results will be published later.) Each of the statements (i)–(iii) below is well known to hold for semialgebras W.

1.8. Theorem. *Let S be a densely exponential subsemigroup of a connected Lie group G such that S has interior points with respect to G. Then the following assertions hold:*

(i) *The Lie wedge W of S has interior points with respect to the Lie algebra $\mathfrak{g}$ of G.*

(ii) *The edge $H(W) = W \cap -W$ of the wedge W contains an ideal $\mathfrak{n}$ such that the quotient algebra $\mathfrak{g}/\mathfrak{n}$ has abelain Cartan subalgebras.*

(iii) *If S is reduced, that is, if it does not contain a non-trivial connected normal subgroup of G then for every element $x \in W$ the spectrum of $\mathrm{ad}\, x$ is contained in $\mathbb{R} \cup i\mathbb{R}$.*

(iv) *A connected Lie group G is said to be compact-free if its maximal compact subgroup is the singleton $\{\mathbf{1}\}$; it is called almost compact-free if it contains a compact normal subgroup N such that G/N is compact-free (or, equivalently, if it contains only one maximal compact subgroup). If G is almost compact-free and solvable then the tangent wedge of S is a semialgebra and disperses.* ■

Furthermore, it can be shown that in the situation of Theorem 1.8 the Lie algebra $\mathfrak{g}$ must have a very special structure; the essential points are very close to known features of Lie algebras containing proper semialgebras with interior points. A detailed description of these features can be found in ANSELM EGGERT's dissertation [2].

2. Examples

We close our discussion with some examples.

2.1. Example. The abelian case. Let G be a connected abelian Lie group. Then the exponential function $\mathfrak{g} \to G$ is a covering homomorphism. Obviously, *every* wedge in $\mathfrak{g}$ is a semialgebra. But it is not clear whether every closed divisible subsemigroup is locally divisible. It is not difficult to see that the exponential image of a wedge $W \subseteq \mathfrak{g}$ is a closed subsemigroup of G if and only if $\operatorname{span}(\ker \exp) \cap W = \operatorname{span}\big((\ker \exp) \cap W \cap -W\big)$ ¿From this fact it can be deduced that every closed divisible subsemigroup of G is locally divisible; but the proof is definitely not trivial, even in this 'easy' case. After the reduction to the case where S does not contain a non-trivial compact subgroup, the proof of this fact amounts to showing that if W is a wedge in some real vector space V and Γ is a lattice in V, whose span meets W only in the zero element, then we can always find a zero neighborhood B in V such that $B \cap W + \Gamma = B \cap W$ This neighborhood depends on the mesh size of Γ as well as on the geometry of the wedge W and its position relative to Γ. Thus, even for lattices with wide interstices the neighborhood B might have to be choosen very small.

2.2. Example. Subsemigroups of the Heisenberg group. Let $\mathfrak{g}$ be the three dimensional Heisenberg algebra, i.e. the real vector space $\mathbb{R}^3$, endowed with the Lie bracket $[(a,b,c),(a',b',c')] = (0,0,ab'-a'b)$. We consider the Heisenberg group G as the space $\mathfrak{g}$, endowed with the Campbell-Hausdorff multiplication $x * y = x + y + \frac{1}{2}[x,y]$. Every semialgebra in $\mathfrak{g}$ either contains the center $\mathfrak{g}' = \{(0,0,c) \mid c \in \mathbb{R}\}$ of $\mathfrak{g}$ or has the form $\mathbb{R}^+ \cdot x$ or $\mathbb{R} \cdot x$, with $x \in \mathfrak{g}$. With respect to the $*$-multiplication each semialgebra in $\mathfrak{g}$ is a closed divisible subsemigroup of G, and there are no others (cf. the assertion of Theorem 1.8(iii)). It is an easy exercise to show that the familiar 'Heisenberg beak', the infinitesimally generated closed semigroup $\{(a,b,c) \in G \mid |c| \leq \frac{1}{2}ab\}$, is not divisible.

2.3. Example. The simply connected covering of the motion group. For our next set of examples we choose $\mathfrak{g}$ to be the semidirect product $\mathfrak{g} = \mathbb{C} \rtimes \mathbb{R}$, where $\mathbb{C}$ denotes the set of complex numbers, considered as a real vector space with commutative Lie brackets, and $[(a,b),(c,d)] = (i(ad-bc),0)$. This algebra is isomorphic with the motion algebra, the Lie algebra belonging to the group of all Euclidean motions of the plane. We let G be the associated simply connected Lie group, $G = \{(z,t) \mid z \in \mathbb{C},\, t \in \mathbb{R}\}$, with the multiplication rule $(z,t)(z',t') = (z + e^{it}z', t+t')$. The group G is densely exponential but not exponential; in fact, a point $p \in G$ lies on a one-parameter subgroup if and only if it is not contained in the set $\{(z,k\pi) \in G \mid z \neq 0, k \in \mathbb{Z} \setminus \{0\}\}$.

The half space $W = \{(x,y) \in \mathfrak{g} \mid x \in \mathbb{C},\, y \geq 0\}$ is an invariant wedge, hence a semialgebra, and its exponential image $\exp W$ is a non-closed exponential subset of G. The semigroup $S = \overline{\exp W} = \{(z,t) \mid t \geq 0\}$ is a densely exponential subsemigroup of G. Thus 'densely exponential' does not imply 'exponential' (cf. Fig.1).

All other semialgebras in $\mathfrak{g}$ are of dimension one or zero; the group G does not contain closed divisible subsemigroups of dimension greater than one. The only densely exponential subsemigroups of G are $S, -S$ and the exponential subsemigroups.

2.4. Example. An invariant solvable example. Let G be the semidirect product $\mathbb{C} \rtimes (\mathbb{R} \times \mathbb{T})$, where $\mathbb{T} := \{t \in \mathbb{C} \mid \|t\| = 1\}$, and multiplication follows the rule

$$(z,r,t)(z',r',t') = (z + e^{-ir}tz', r+r', tt').$$

The Lie algebra of G can be defined on the real vector space $\mathbb{C}\oplus\mathbb{R}\oplus\mathbb{R}$ with Lie brackets

$$[(\zeta,\alpha,\beta),(\zeta',\alpha',\beta')]=(i(-\alpha+\beta)\zeta'-i(-\alpha'+\beta')\zeta,0,0)$$

and exponential function

$$\exp(\zeta,\alpha,\beta)=\begin{cases}(-i\zeta\frac{e^{i(\beta-\alpha)}-1}{\beta-\alpha},\alpha,-ie^{i\beta}) & \text{if } \alpha\neq\beta\\ (\zeta,\alpha,-ie^{i\alpha}) & \text{if } \alpha=\beta.\end{cases}$$

We let A be the closed normal subgroup $\{(z,r,0)\mid z\in\mathbb{C},r\in\mathbb{R}\}$ and write $\mathfrak{a}$ for its Lie algebra. Then A is a copy of the simply connected covering group of the motion group, hence is not exponential. However the group G is exponential (it is not simply connected).

We define $S=\{(z,r,t)\mid z\in\mathbb{C},r\geq 0,t\in\mathbb{T}\}$ and note that $S=\exp W$ with $W=\mathfrak{L}(S)=\{(\zeta,\alpha,\beta)\mid \zeta\in\mathbb{C},r\geq 0,\beta\in\mathbb{R}\}$. Thus S is a closed divisible subsemigroup of G. Moreover the wedge W is a half space whose bounding hyperplane is an ideal, hence is invariant, and so is the intersection $W\cap\mathfrak{a}$. Note, however, that the intersection $S\cap A$ is not divisible: indeed, the points $(z,2k\pi,0)$ with $z\neq 0$, $k\in\mathbb{Z}\setminus\{0\}$ are not in the exponential image of $\mathfrak{a}$.

2.5. Example. Examples connected with $\mathfrak{sl}(2,\mathbb{R})$. In this set of examples we consider a connected Lie group G with Lie algebra $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})$. Let us write k for the Cartan-Killing form on $\mathfrak{g}$.

According to the 'Second Classification Theorem for Low Dimensional Semialgebras' ([3], Theorem II.3.7, p. 109, cf. also Proposition V.4.21), p. 418) every generating Lie semialgebra is the intersection of half spaces bounded by a conjugate of the solvable subalgebra $\{w\in\mathfrak{g}\mid k(x,w)=0\}$, where x denotes the nilpotent matrix $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$. The invariant wedges in $\mathfrak{g}$ are precisely the two halves of the standard Lorentzian double cone $C=\{x\in\mathfrak{g}\mid k(x,x)\leq 0\}$ and the trivial ones, $\mathfrak{g}$ and $\{0\}$. We single out one of the two halves of C by defining $\mathcal{K}=\{\left(\begin{smallmatrix}a&b\\c&-a\end{smallmatrix}\right)\in C\mid b\geq 0\}$.

By a result of KARL H. NEEB [8] we know that a generating Lie wedge in $\mathfrak{g}$ is the tangent wedge of a subsemigroup of the simply connected covering group $\widetilde{\mathrm{Sl}(2,\mathbb{R})}$ if and only if it lies in a half space bounded by a tangent plane of $\mathcal{K}$.

Furthermore, the calculations of [3], p. 415ff, show that

$$G\setminus\exp\mathfrak{g}=\bigcup_{z\in Z\setminus\{1\}} z\exp(\mathfrak{g}\setminus C)$$

where Z denotes the centre of G. It follows that no proper generating wedge in $\mathfrak{g}$ which meets the interior of either $\mathcal{K}$ or $-\mathcal{K}$ can be the Lie wedge of a densely exponential subsemigroup. Thus if W is a generating wedge in $\mathfrak{g}$ with $W=\mathfrak{L}(S)$, for some densely exponential subsemigroup S of G, then $W\subseteq\overline{\mathfrak{g}\setminus C}$, so W is conjugate to a subset of the wedge $\mathfrak{sl}(2)^+:=\{\left(\begin{smallmatrix}a&b\\c&-a\end{smallmatrix}\right)\mid a\in\mathbb{R},b\geq 0,\ c\geq 0\}$. Note that $\mathfrak{sl}(2)^+$ is the tangent wedge of the closed semigroup of all matrices in $\mathrm{Sl}(2,\mathbb{R})$ with non-negative entries.

If G is simply connected then $\exp \mathcal{K}$ generates a closed invariant subsemigroup with inner points in G, but this subsemigroup is not divisible.

Furthermore, if $\mathbf{1} \neq g \in \exp(\overline{\mathfrak{g} \setminus C})$ then there is exactly one element $x \in \mathfrak{g}$ with $g = \exp x$. Thus, if S is a densely exponential proper subsemigroup of G with interior points then to every point $s \in S$ there is exactly one element $w \in \mathfrak{g}$ with $\exp w = s$. Since exp is regular at all points of $\overline{\mathfrak{g} \setminus C}$ this implies that for such semigroups S the map $\mathfrak{L}(S) \to G, w \mapsto \exp w$, is an imbedding. It follows that every closed divisible subsemigroup of G is locally divisible and that every generating Lie semialgebra not meeting the interior of the standard double cone is the tangent wedge of a closed divisible subsemigroup of G.

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Fachbereich Mathematik
Technische Hochschule
Darmstadt
Schloßgartenstraße 7
D-6100 Darmstadt
Germany
xmatda4l@ddathd21.bitnet

Institut für Mathematik
u. Angewandte Statistik
Universität f. Bodenkultur
Gregor Mendelstraße 33
A-1180 Wien
Austria/Europe
h506t0@awibok01.bitnet

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