Journal of Integer Sequences, Vol. 9 (2006),
Article 06.3.6

# Periodicity and Parity Theorems for a Statistic on $r$-Mino Arrangements

Mark A. Shattuck and Carl G. Wagner
Mathematics Department
University of Tennessee
Knoxville, TN 37996-1300
USA
shattuck@math.utk.edu
wagner@math.utk.edu

**Abstract**

We study polynomial generalizations of the $r$-Fibonacci and $r$-Lucas sequences which arise in connection with a certain statistic on linear and circular $r$-mino arrangements, respectively. By considering special values of these polynomials, we derive periodicity and parity theorems for this statistic on the respective structures.

## 1 Introduction

If $r \geqslant 2$, the $r$-Fibonacci numbers $F_n^{(r)}$ are defined by $F_0^{(r)} = F_1^{(r)} = \cdots = F_{r-1}^{(r)} = 1$, with $F_n^{(r)} = F_{n-1}^{(r)} + F_{n-r}^{(r)}$ if $n \geqslant r$. The $r$-Lucas numbers $L_n^{(r)}$ are defined by $L_1^{(r)} = L_2^{(r)} = \cdots = L_{r-1}^{(r)} = 1$ and $L_r^{(r)} = r+1$, with $L_n^{(r)} = L_{n-1}^{(r)} + L_{n-r}^{(r)}$ if $n \geqslant r+1$. If $r = 2$, the $F_n^{(r)}$ and $L_n^{(r)}$ reduce, respectively, to the classical Fibonacci and Lucas numbers (parametrized as in Wilf [12], by $F_0 = F_1 = 1$, etc., and $L_1 = 1$, $L_2 = 3$, etc.).

Polynomial generalizations of $F_n$ and/or $L_n$ have arisen as generating functions for statistics on binary words [1], lattice paths [5], and linear and circular domino arrangements [8]. Generalizations of $F_n^{(r)}$ and/or $L_n^{(r)}$ have arisen similarly in connection with statistics on Morse code sequences [4] as well as on linear and circular $r$-mino arrangements [9].

Cigler [3] introduces and studies a new class of $q$-Fibonacci polynomials, generalizing the classical sequence, which arise in connection with a certain statistic on Morse code sequences in which the dashes have length 2. The same statistic, which we'll denote by $\pi$, applied more generally to linear $r$-mino arrangements, leads to the polynomial generalization

$$F_n^{(r)}(q,t) := \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} q^{\binom{k+1}{2}} \binom{n-(r-1)k}{k}_q t^k \tag{1.1}$$

of $F_n^{(r)}$. A natural extension of this $\pi$ statistic to circular $r$-mino arrangements leads to the new polynomial generalization

$$L_n^{(r)}(q,t) := \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} q^{\binom{k+1}{2}} \left[\frac{(r-1)k_q + (n-(r-1)k)_q}{(n-(r-1)k)_q}\right] \binom{n-(r-1)k}{k}_q t^k \tag{1.2}$$

of $L_n^{(r)}$.

In addition to deriving the above closed forms for $F_n^{(r)}(q,t)$ and $L_n^{(r)}(q,t)$, we present both algebraic and combinatorial evaluations of $F_n^{(r)}(-1,t)$ and $L_n^{(r)}(-1,t)$, as well as determine when the sequences $F_n^{(r)}(-1,1)$ and $L_n^{(r)}(-1,1)$ are periodic. Our algebraic proofs make frequent use of the identity [11, pp. 201–202]

$$\sum_{n\geqslant 0} \binom{n}{k}_q x^n = \frac{x^k}{(1-x)(1-qx)\cdots(1-q^kx)}, \qquad k\in\mathbb{N}. \tag{1.3}$$

Our combinatorial proofs are based on the fact that $F_n^{(r)}(q,t)$ and $L_n^{(r)}(q,t)$ are bivariate generating functions for a pair of statistics defined, respectively, on linear and circular arrangements of $r$-minos. We also describe some variants of the $\pi$ statistic on circular domino arrangements which lead to additional polynomial generalizations of the Lucas sequence.

In what follows, $\mathbb{N}$ and $\mathbb{P}$ denote, respectively, the nonnegative and positive integers. Empty sums take the value 0 and empty products the value 1, with $0^0 := 1$. If $q$ is an indeterminate, then $0_q := 0$, $n_q := 1+q+\cdots+q^{n-1}$ for $n\in\mathbb{P}$, $0_q^! := 1$, $n_q^! := 1_q2_q\cdots n_q$ for $n\in\mathbb{P}$, and

$$\binom{n}{k}_q := \begin{cases} \frac{n_q^!}{k_q^!(n-k)_q^!}, & \text{if } 0\leqslant k\leqslant n;\\ \\ 0, & \text{if } k<0 \text{ or } 0\leqslant n<k. \end{cases} \tag{1.4}$$

A useful variation of (1.4) is the well known formula [10, p. 29]

$$\binom{n}{k}_q = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k\\ d_i\in\mathbb{N}}} q^{0d_0+1d_1+\cdots+kd_k} = \sum_{t\geqslant 0} p(k,n-k,t)q^t, \tag{1.5}$$

where $p(k,n-k,t)$ denotes the number of partitions of the integer $t$ with at most $n-k$ parts, each no larger than $k$.

# 2 Linear $r$-Mino Arrangements

Let $\mathcal{R}_{n,k}^{(r)}$ denote the set of coverings of the numbers $1,2,\ldots,n$ arranged in a row by $k$ indistinguishable $r$-minos and $n-rk$ indistinguishable squares, where pieces do not overlap, an *$r$-mino*, $r\geqslant 2$, is a rectangular piece covering $r$ numbers, and a *square* is a piece covering a single number. Each such covering corresponds uniquely to a word in the alphabet $\{r, s\}$ comprising $k$ $r$'s and $n - rk$ $s$'s so that

$$|\mathcal{R}^{(r)}_{n,k}| = \binom{n-(r-1)k}{k}, \qquad 0 \leqslant k \leqslant \lfloor n/r \rfloor, \tag{2.1}$$

for all $n \in \mathbb{P}$. (If we set $\mathcal{R}^{(r)}_{0,0} = \{\emptyset\}$, the "empty covering," then (2.1) holds for $n = 0$ as well.) In what follows, we will identify coverings $c$ with such words $c_1c_2\cdots$ in $\{r, s\}$. With

$$\mathcal{R}^{(r)}_n := \bigcup_{0\leqslant k\leqslant \lfloor n/r\rfloor} \mathcal{R}^{(r)}_{n,k}, \qquad n \in \mathbb{N}, \tag{2.2}$$

it follows that

$$|\mathcal{R}^{(r)}_n| = \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} \binom{n-(r-1)k}{k} = F^{(r)}_n, \tag{2.3}$$

where $F^{(r)}_0 = F^{(r)}_1 = \cdots = F^{(r)}_{r-1} = 1$, with $F^{(r)}_n = F^{(r)}_{n-1} + F^{(r)}_{n-r}$ if $n \geqslant r$. Note that

$$\sum_{n\geqslant 0} F^{(r)}_n x^n = \frac{1}{1 - x - x^r}. \tag{2.4}$$

Given a covering $c = c_1c_2\cdots$, let

$$\pi(c) := \sum_{i:c_i=r} i; \tag{2.5}$$

note that $\pi(c)$ gives the total resulting when one counts the number of pieces preceding each $r$-mino, inclusive, and adds up these numbers.

Let

$$F^{(r)}_n(q, t) := \sum_{c\in\mathcal{R}^{(r)}_n} q^{\pi(c)} t^{v(c)}, \qquad n \in \mathbb{N}, \tag{2.6}$$

where $v(c) :=$ the number of $r$-minos in the covering $c$.

Categorizing linear covers of $1, 2, \ldots, n$ according to whether the piece covering $n$ is a square or $r$-mino yields the recurrence relation

$$F^{(r)}_n(q, t) = F^{(r)}_{n-1}(q, t) + q^{n-r+1} t F^{(r)}_{n-r}\left(q, t/q^{r-1}\right), \qquad n \geqslant r, \tag{2.7}$$

with $F^{(r)}_i(q, t) = 1$ if $0 \leqslant i \leqslant r - 1$, since the total number of pieces in $c \in \mathcal{R}^{(r)}_m$ is $m - (r - 1)v(c)$. Categorizing covers of 1,2,...,$n$ according to whether the piece covering 1 is a square or $r$-mino yields

$$F^{(r)}_n(q, t) = F^{(r)}_{n-1}(q, qt) + qtF^{(r)}_{n-r}(q, qt), \qquad n \geqslant r. \tag{2.8}$$

By combining relations (2.7) and (2.8), one gets a recurrence for $F^{(r)}_n(q, t)$ for each number $q$ and $t$. For example when $r = 3$, this is

$$\begin{aligned} F^{(3)}_n(q, t) = F^{(3)}_{n-1}(q, t) + q^{n-2} t F^{(3)}_{n-5}(q, t) \quad &+ \quad q^{n-3}(1 + q)t^2 F^{(3)}_{n-7}(q, t) \\ &+ \quad q^{n-3} t^3 F^{(3)}_{n-9}(q, t). \end{aligned} \tag{2.9}$$

The $F^{(r)}_n(q, t)$ have the following explicit formula.

**Theorem 2.1.** *For all $n \in \mathbb{N}$,*

$$F_n^{(r)}(q,t) = \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} q^{\binom{k+1}{2}} \binom{n-(r-1)k}{k}_q t^k. \tag{2.10}$$

*Proof.* It clearly suffices to show that

$$\sum_{c \in \mathcal{R}_{n,k}^{(r)}} q^{\pi(c)} = q^{\binom{k+1}{2}} \binom{n-(r-1)k}{k}_q.$$

Each $c \in \mathcal{R}_{n,k}^{(r)}$ corresponds uniquely to a sequence $(d_0, d_1, \ldots, d_k)$, where $d_0$ is the number of squares following the $k^{th}$ $r$-mino (counting from left to right) in the covering $c$, $d_k$ is the number of squares preceding the first $r$-mino, and, for $0 < i < k$, $d_{k-i}$ is the number of squares between the $i^{th}$ and $(i+1)^{st}$ $r$-mino. Then $\pi(c) = (d_k + 1) + (d_k + d_{k-1} + 2) + \cdots + (d_k + d_{k-1} + \cdots + d_1 + k) = \binom{k+1}{2} + kd_k + (k-1)d_{k-1} + \cdots + 1d_1$ so that

$$\begin{aligned}\sum_{c \in \mathcal{R}_{n,k}^{(r)}} q^{\pi(c)} &= q^{\binom{k+1}{2}} \sum_{\substack{d_0+d_1+\cdots+d_k = n-rk \\ d_i \in \mathbb{N}}} q^{0d_0+1d_1+\cdots+kd_k} \\ &= q^{\binom{k+1}{2}} \binom{n-(r-1)k}{k}_q,\end{aligned}$$

by (1.5). □

**Theorem 2.2.** *The ordinary generating function of the sequence $(F_n^{(r)}(q,t))_{n \geqslant 0}$ is given by*

$$\sum_{n \geqslant 0} F_n^{(r)}(q,t) x^n = \sum_{k \geqslant 0} \frac{q^{\binom{k+1}{2}} t^k x^{rk}}{(1-x)(1-qx)\cdots(1-q^k x)}. \tag{2.11}$$

*Proof.* By (2.10) and (1.3),

$$\begin{aligned}\sum_{n \geqslant 0} F_n^{(r)}(q,t) x^n &= \sum_{n \geqslant 0} \left( \sum_{0 \leqslant k \leqslant \lfloor n/r \rfloor} q^{\binom{k+1}{2}} \binom{n-(r-1)k}{k}_q t^k \right) x^n \\ &= \sum_{k \geqslant 0} q^{\binom{k+1}{2}} t^k x^{(r-1)k} \sum_{n \geqslant kr} \binom{n-(r-1)k}{k}_q x^{n-(r-1)k} \\ &= \sum_{k \geqslant 0} q^{\binom{k+1}{2}} t^k x^{(r-1)k} \cdot \frac{x^k}{(1-x)(1-qx)\cdots(1-q^k x)}.\end{aligned}$$

□

Note that $F_n^{(r)}(1,1) = F_n^{(r)}$, whence (2.11) generalizes (2.4). Setting $q = 1$ and $q = -1$ in (2.11) yields

**Corollary 2.2.1.** *The ordinary generating function of the sequence $(F_n^{(r)}(1,t))_{n\geqslant 0}$ is given by*

$$\sum_{n\geqslant 0} F_n^{(r)}(1,t)x^n = \frac{1}{1-x-tx^r}. \tag{2.12}$$

and

**Corollary 2.2.2.** *The ordinary generating function of the sequence $(F_n^{(r)}(-1,t))_{n\geqslant 0}$ is given by*

$$\sum_{n\geqslant 0} F_n^{(r)}(-1,t)x^n = \frac{1+x-tx^r}{1-x^2+t^2x^{2r}}. \tag{2.13}$$

When $r = 2$ and $t = 1$ in (2.13), we get

$$\sum_{n\geqslant 0} F_n^{(2)}(-1,1)x^n = \frac{1+x-x^2}{1-x^2+x^4} = \frac{(1+x+x^3-x^4)(1-x^6)}{1-x^{12}}, \tag{2.14}$$

which implies

**Corollary 2.2.3.** *The sequence $(F_n^{(2)}(-1,1))_{n\geqslant 0}$ is periodic with period 12; namely, if $a_n := F_n^{(2)}(-1,1)$ for $n \geqslant 0$, then $a_0 = 1$, $a_1 = 1$, $a_2 = 0$, $a_3 = 1$, $a_4 = -1$, and $a_5 = 0$ with $a_{n+6} = -a_n$, $n \geqslant 0$.*

(We call a sequence $(b_n)_{n\geqslant 0}$ *periodic with period* $d$ if $b_{n+d} = b_n$ for all $n \geqslant m$ for some $m \in \mathbb{N}$.)

*Remark.* Corollary 2.2.3 is the $q = -1$ case of the well known formula

$$\sum_{0\leqslant k\leqslant \lfloor n/2\rfloor} (-1)^k q^{\binom{k}{2}} \binom{n-k}{k}_q = \begin{cases} (-1)^{\lfloor n/3\rfloor} q^{n(n-1)/6}, & \text{if } n \equiv 0,\ 1 \pmod 3; \\ 0, & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

See, e.g., Cigler [5], Ekhad and Zeilberger [6], and Kupershmidt [7].

We now show that the periodic behavior of $F_n^{(r)}(-1,1)$ seen when $r = 2$ is restricted to that case. The following lemma is established in [9]. We include its proof here for completeness.

**Lemma 2.3.** *If $r \geqslant 3$, then $g_r(x) := 1 - x + x^r$ does not divide any polynomial of the form $1 - x^m$, where $m \in \mathbb{P}$.*

*Proof.* We first describe the roots of unity that are zeros of $g_r(x)$, where $r \geqslant 2$. If $z$ is such a root of unity, let $y = z^{r-1}$. Since $z(1-z^{r-1}) = 1$ and $z$ is a root of unity, it follows that both $y$

and $1-y$ are roots of unity. In particular, $|y| = |1-y| = 1$. Therefore, $1-2\mathrm{Re}(y)+|y|^2 = 1$, so $\mathrm{Re}(y) = \frac{1}{2}$. This forces $y$, and hence $1-y$, to be primitive $6^{th}$ roots of unity. But $1-y = \frac{1}{z}$, so $z$ is also a primitive $6^{th}$ root of unity.

This implies that the only possible roots of unity which are zeros of $g_r$ are the primitive $6^{th}$ roots of unity. Since the derivative of $g_r$ has no roots of unity as zeros, these $6^{th}$ roots of unity can only be simple zeros of $g_r$. In particular, if every root of $g_r$ is a root of unity, then $r = 2$. □

**Theorem 2.4.** *The sequence $(F_n^{(r)}(-1,1))_{n\geqslant 0}$ is never periodic for $r \geqslant 3$.*

*Proof.* By (2.13) at $t = 1$, we must show that $1 - x^2 + x^{2r}$ does not divide the product $(1-x^m)(1+x-x^r)$ for any $m \in \mathbb{P}$ whenever $r \geqslant 3$. First note that the polynomials $1-x^2+x^{2r}$ and $1 + x - x^r$ cannot share a zero; for if $t_0$ is a common zero, then $t_0^2 - 1 = t_0^{2r} = (t_0 + 1)^2$, i.e, $t_0 = -1$, which isn't a zero of either polynomial. Observe next that $1-x^2+x^{2r} = g_r(x^2)$, where $g_r(x)$ is as in Lemma 2.3, so that $1 - x^2 + x^{2r}$ fails to divide $1 - x^m$ for any $m \in \mathbb{P}$, since $g_r(x)$ fails to, which completes the proof. □

Iterating (2.7) or (2.8) yields $F_{-i}^{(r)}(q,t) = 0$ if $1 \leqslant i \leqslant r-1$, which we'll take as a convention.

**Theorem 2.5.** *Let $m \in \mathbb{N}$. If $m$ and $r$ have the same parity, then*

$$F_m^{(r)}(-1,t) = F_{\lfloor m/2\rfloor}^{(r)}(1,-t^2) - tF_{(m-r)/2}^{(r)}(1,-t^2), \tag{2.15}$$

*and if $m$ and $r$ have different parity, then*

$$F_m^{(r)}(-1,t) = F_{\lfloor m/2\rfloor}^{(r)}(1,-t^2). \tag{2.16}$$

*Proof.* Taking the even and odd parts of both sides of (2.13) followed by replacing $x$ with $x^{1/2}$ yields

$$\sum_{n\geqslant 0} F_{2n}^{(r)}(-1,t)x^n = \frac{1-tx^{r/2}}{1-x+t^2x^r}$$

and

$$\sum_{n\geqslant 0} F_{2n+1}^{(r)}(-1,t)x^n = \frac{1}{1-x+t^2x^r},$$

when $r$ is even, and

$$\sum_{n\geqslant 0} F_{2n}^{(r)}(-1,t)x^n = \frac{1}{1-x+t^2x^r}$$

and

$$\sum_{n\geqslant 0} F_{2n+1}^{(r)}(-1,t)x^n = \frac{1-tx^{(r-1)/2}}{1-x+t^2x^r},$$

when $r$ is odd, from which (2.15) and (2.16) now follow from (2.12) upon putting together cases.

For a combinatorial proof of (2.15) and (2.16), we first assign to each $r$-mino arrangement $c \in \mathcal{R}_m^{(r)}$ the weight $w_c := (-1)^{\pi(c)} t^{v(c)}$, where $t$ is an indeterminate. Let $\mathcal{R}_m^{(r)'}$ consist of those $c = c_1 c_2 \cdots$ in $\mathcal{R}_m^{(r)}$ satisfying the conditions $c_{2i-1} = c_{2i}$, $i \geqslant 1$. Suppose $c \in \mathcal{R}_m^{(r)} - \mathcal{R}_m^{(r)'}$, with $i_0$ being the smallest value of $i$ for which $c_{2i-1} \neq c_{2i}$. Exchanging the positions of the $(2i_0 - 1)^{st}$ and $(2i_0)^{th}$ pieces within $c$ produces a $\pi$-parity changing involution of $\mathcal{R}_m^{(r)} - \mathcal{R}_m^{(r)'}$ which preserves $v$.

If $m$ and $r$ have the same parity, then

$$\begin{aligned}
F_m^{(r)}(-1,t) &= \sum_{c \in \mathcal{R}_m^{(r)}} w_c = \sum_{c \in \mathcal{R}_m^{(r)'}} w_c = \sum_{\substack{c \in \mathcal{R}_m^{(r)'} \\ v(c) \text{ even}}} w_c \; + \sum_{\substack{c \in \mathcal{R}_m^{(r)'} \\ v(c) \text{ odd}}} w_c \\
&= \sum_{\substack{c \in \mathcal{R}_m^{(r)'} \\ v(c) \text{ even}}} (-1)^{v(c)/2} t^{v(c)} \; - t \sum_{\substack{c \in \mathcal{R}_{m-r}^{(r)'} \\ v(c) \text{ even}}} (-1)^{v(c)/2} t^{v(c)} \\
&= \sum_{z \in \mathcal{R}_{\lfloor m/2 \rfloor}^{(r)}} (-1)^{v(z)} t^{2v(z)} - t \sum_{z \in \mathcal{R}_{(m-r)/2}^{(r)}} (-1)^{v(z)} t^{2v(z)} \\
&= F_{\lfloor m/2 \rfloor}^{(r)}(1, -t^2) - t F_{(m-r)/2}^{(r)}(1, -t^2),
\end{aligned}$$

which gives (2.15), since each pair of consecutive $r$-minos in $c \in \mathcal{R}_m^{(r)'}$ contributes a factor of $-1$ towards the sign $(-1)^{\pi(c)}$ and since members of $\mathcal{R}_m^{(r)'}$ for which $v(c)$ is odd end in a single $r$-mino. If $m$ and $r$ differ in parity, then

$$F_m^{(r)}(-1,t) = \sum_{c \in \mathcal{R}_m^{(r)}} w_c = \sum_{c \in \mathcal{R}_m^{(r)'}} w_c = \sum_{z \in \mathcal{R}_{\lfloor m/2 \rfloor}^{(r)}} (-1)^{v(z)} t^{2v(z)} = F_{\lfloor m/2 \rfloor}^{(r)}(1, -t^2),$$

which gives (2.16), since members of $\mathcal{R}_m^{(r)'}$ must contain an even number of $r$-minos. $\square$

The involution of the previous theorem in the case $r = 2$ can be extended to account for the periodicity in Corollary 2.2.3 as follows. If $n \geqslant 6$, let $\mathcal{R}_n^{(2)*} \subseteq \mathcal{R}_n^{(2)'}$ consist of those domino arrangements $c = c_1 c_2 \cdots$ that contain at least $4\lfloor n/6 \rfloor$ pieces satisfying the conditions

$$c_{4i-3} c_{4i-2} c_{4i-1} c_{4i} = ssdd, \qquad 1 \leqslant i \leqslant \lfloor n/6 \rfloor; \tag{2.17}$$

if $0 \leqslant n \leqslant 5$, then let $\mathcal{R}_n^{(2)*} = \mathcal{R}_n^{(2)'}$.

A $\pi$-parity changing involution of $\mathcal{R}_n^{(2)'} - \mathcal{R}_n^{(2)*}$ when $n \geqslant 6$ is given by the pairing

$$(ssdd)^k ssssu \longleftrightarrow (ssdd)^k ddu,$$

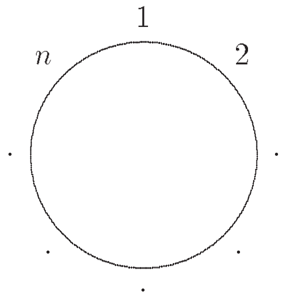
where $k \geqslant 0$ and $u$ is some (non-empty) word in $\{d, s\}$. If $n = 6m + i$, where $m \geqslant 1$ and $0 \leqslant i \leqslant 5$, then

$$\begin{aligned}
F_n^{(2)}(-1,1) &= \sum_{c \in \mathcal{R}_n^{(2)}} (-1)^{\pi(c)} = \sum_{c \in \mathcal{R}_n^{(2)'}} (-1)^{\pi(c)} = \sum_{c \in \mathcal{R}_n^{(2)*}} (-1)^{\pi(c)} \\
&= (-1)^m \sum_{c \in \mathcal{R}_i^{(2)*}} (-1)^{\pi(c)} = (-1)^m F_i^{(2)}(-1,1),
\end{aligned}$$

which implies Corollary 2.2.3, upon checking directly the cases $0 \leqslant n \leqslant 5$, as each $ssdd$ unit in $c \in \mathcal{R}_n^{(2)*}$ contributes a factor of $-1$ towards the sign $(-1)^{\pi(c)}$.

# 3 Circular $r$-Mino Arrangements

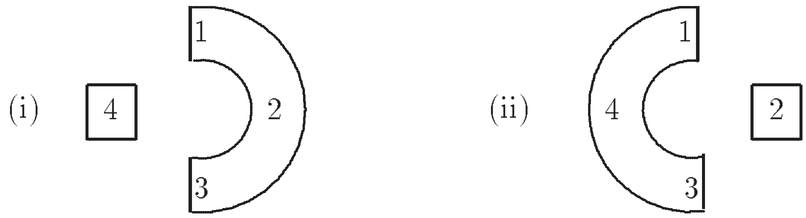
If $n \in \mathbb{P}$ and $0 \leqslant k \leqslant \lfloor n/r \rfloor$, let $\mathcal{C}_{n,k}^{(r)}$ denote the set of coverings by $k$ $r$-minos and $n - rk$ squares of the numbers $1, 2, \ldots, n$ arranged clockwise around a circle:




By the *initial segment* of an $r$-mino occurring in such a cover, we mean the segment first encountered as the circle is traversed clockwise. Classifying members of $\mathcal{C}_{n,k}^{(r)}$ according as (i) 1 is covered by one of $r$ segments of an $r$-mino or (ii) 1 is covered by a square, and applying (2.1), yields

$$\begin{aligned} \left|\mathcal{C}_{n,k}^{(r)}\right| &= r\binom{n-(r-1)k-1}{k-1} + \binom{n-(r-1)k-1}{k} \\ &= \frac{n}{n-(r-1)k}\binom{n-(r-1)k}{k}, \qquad 0 \leqslant k \leqslant \lfloor n/r \rfloor. \end{aligned} \tag{3.1}$$

Below we illustrate two members of $\mathcal{C}_{4,1}^{(3)}$:




In covering (i), the initial segment of the 3-mino covers 1, and in covering (ii), the initial segment covers 3.

With

$$\mathcal{C}_n^{(r)} := \bigcup_{0 \leqslant k \leqslant \lfloor n/r \rfloor} \mathcal{C}_{n,k}^{(r)}, \qquad n \in \mathbb{P}, \tag{3.2}$$

it follows that

$$\left|\mathcal{C}_n^{(r)}\right| = \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} \frac{n}{n-(r-1)k}\binom{n-(r-1)k}{k} = L_n^{(r)}, \tag{3.3}$$

where $L_1^{(r)} = L_2^{(r)} = \cdots = L_{r-1}^{(r)} = 1$, $L_r^{(r)} = r+1$, and $L_n^{(r)} = L_{n-1}^{(r)} + L_{n-r}^{(r)}$ if $n \geqslant r+1$. Note that

$$\sum_{n\geqslant 1} L_n^{(r)} x^n = \frac{x + rx^r}{1-x-x^r}. \tag{3.4}$$

We'll associate to each $c \in \mathcal{C}_n^{(r)}$ a word $u_c = u_1u_2\cdots$ in the alphabet $\{r, s\}$, where

$$u_i := \begin{cases} r, & \text{if the } i^{th} \text{ piece of } c \text{ is an } r\text{-mino;} \\ s, & \text{if the } i^{th} \text{ piece of } c \text{ is a square,} \end{cases}$$

and one determines the $i^{th}$ piece of $c$ by starting with the piece covering 1 and proceeding clockwise from that piece. Note that for each word starting with $r$, there are exactly $r$ associated members of $\mathcal{C}_n^{(r)}$, while for each word starting with $s$, there is only one associated member.

Given $c \in \mathcal{C}_n^{(r)}$ and its associated word $u_c = u_1u_2\cdots$, let

$$\pi(c) := \sum_{i:u_i=r} i; \tag{3.5}$$

note that $\pi(c)$ gives the sum of the numbers gotten by counting the number of pieces preceding each $r$-mino, inclusive (counting back each time counterclockwise to the piece covering 1).

Let

$$L_n^{(r)}(q,t) := \sum_{c\in\mathcal{C}_n^{(r)}} q^{\pi(c)} t^{v(c)}, \qquad n \in \mathbb{P}, \tag{3.6}$$

where $v(c) :=$ the number of $r$-minos in the covering $c$.

Categorizing circular covers $c$ of $1, 2, \ldots, n$ according to whether the last letter in $u_c$ is an $s$ or $r$ yields the recurrence relation

$$L_n^{(r)}(q,t) = L_{n-1}^{(r)}(q,t) + q^{n-r+1} t L_{n-r}^{(r)}(q, t/q^{r-1}), \quad n \geqslant r+1, \tag{3.7}$$

with $L_i^{(r)}(q,t) = 1$ if $1 \leqslant i \leqslant r-1$ and $L_r^{(r)}(q,t) = 1 + rqt$, as seen upon removing the final piece of $c$, sliding the remaining pieces together to form a circle, and renumbering (if necessary) so that 1 corresponds to the same position as before. The $L_n^{(r)}(q,t)$, though, do not seem to satisfy a recurrence like (2.8). The following theorem gives an explicit formula for $L_n^{(r)}(q,t)$.

**Theorem 3.1.** *For all $n \in \mathbb{P}$,*

$$L_n^{(r)}(q,t) = \sum_{0\leqslant k\leqslant\lfloor n/r\rfloor} q^{\binom{k+1}{2}} \left[\frac{(r-1)k_q + (n-(r-1)k)_q}{(n-(r-1)k)_q}\right] \binom{n-(r-1)k}{k}_q t^k. \tag{3.8}$$

*Proof.* It suffices to show that

$$\sum_{c\in\mathcal{C}_{n,k}^{(r)}} q^{\pi(c)} = q^{\binom{k+1}{2}} \left[\frac{(r-1)k_q + (n-(r-1)k)_q}{(n-(r-1)k)_q}\right] \binom{n-(r-1)k}{k}_q.$$

Partitioning $\mathcal{C}_{n,k}^{(r)}$ into the categories employed above in deriving (3.1), and applying (2.10), yields

$$\begin{aligned}\sum_{c\in\mathcal{C}_{n,k}^{(r)}} q^{\pi(c)} &= rq^{k-1+1}\cdot q^{\binom{k}{2}}\binom{n-(r-1)k-1}{k-1}_q \\ &\quad + q^k\cdot q^{\binom{k+1}{2}}\binom{n-(r-1)k-1}{k}_q \\ &= q^{\binom{k+1}{2}}\left[\frac{rk_q+q^k(n-rk)_q}{(n-(r-1)k)_q}\right]\binom{n-(r-1)k}{k}_q \\ &= q^{\binom{k+1}{2}}\left[\frac{(r-1)k_q+(n-(r-1)k)_q}{(n-(r-1)k)_q}\right]\binom{n-(r-1)k}{k}_q,\end{aligned} \tag{3.9}$$

which completes the proof. □

Note that $L_n^{(r)}(1,1) = L_n^{(r)}$. By (3.8) and (2.10), the $L_n^{(r)}(q,t)$ are related to the $F_n^{(r)}(q,t)$ by the formula

$$L_n^{(r)}(q,t) = F_n^{(r)}(q,t) + (r-1)qtF_{n-r}^{(r)}(q,qt), \qquad n \geqslant 1, \tag{3.10}$$

which reduces to

$$L_n^{(r)} = F_n^{(r)} + (r-1)F_{n-r}^{(r)}, \qquad n \geqslant 1, \tag{3.11}$$

when $q=t=1$. Formula (3.10) can also be realized by considering the way in which 1 is covered in $c \in \mathcal{C}_n^{(r)}$, the first term representing those $c$ for which 1 is covered by a square or an initial segment of an $r$-mino and the second term representing the remaining $r-1$ possibilities.

**Theorem 3.2.** *The ordinary generating function of the sequence* $(L_n^{(r)}(q,t))_{n\geqslant 1}$ *is given by*

$$\sum_{n\geqslant 1} L_n^{(r)}(q,t)x^n = \frac{x}{1-x} + \sum_{k\geqslant 1}\frac{q^{\binom{k+1}{2}}t^k x^{rk}\left[r-(r-1)q^k x\right]}{(1-x)(1-qx)\cdots(1-q^k x)}. \tag{3.12}$$

*Proof.* From (3.9),

$$
\begin{aligned}
\sum_{n\geqslant 1} L_n^{(r)}(q,t)x^n &= \sum_{n\geqslant 1} x^n \sum_{0\leqslant k\leqslant \lfloor n/r\rfloor} \Bigg( q^{k+\binom{k+1}{2}} t^k \binom{n-(r-1)k-1}{k}_q \\
&\quad + r q^{\binom{k+1}{2}} t^k \binom{n-(r-1)k-1}{k-1}_q \Bigg) \\
&= \frac{x}{1-x} + \sum_{k\geqslant 1} q^{k+\binom{k+1}{2}} t^k \sum_{n\geqslant rk+1} \binom{n-(r-1)k-1}{k}_q x^n \\
&\quad + r\sum_{k\geqslant 1} q^{\binom{k+1}{2}} t^k \sum_{n\geqslant rk} \binom{n-(r-1)k-1}{k-1}_q x^n \\
&= \frac{x}{1-x} + \sum_{k\geqslant 1} q^{k+\binom{k+1}{2}} t^k x^{(r-1)k+1} \cdot \frac{x^k}{(1-x)(1-qx)\cdots(1-q^k x)} \\
&\quad + r\sum_{k\geqslant 1} q^{\binom{k+1}{2}} t^k x^{(r-1)k+1} \cdot \frac{x^{k-1}}{(1-x)(1-qx)\cdots(1-q^{k-1}x)} \\
&= \frac{x}{1-x} + \sum_{k\geqslant 1} q^{\binom{k+1}{2}} t^k \cdot \frac{x^{rk}\left[q^k x + r(1-q^k x)\right]}{(1-x)(1-qx)\cdots(1-q^k x)},
\end{aligned}
$$

by (1.3). □

Note that (3.12) reduces to (3.4) when $q = t = 1$. Setting $q = 1$ and $q = -1$ in (3.12) yields

**Corollary 3.2.1.** *The ordinary generating function of the sequence $(L_n^{(r)}(1,t))_{n\geqslant 1}$ is given by*

$$
\sum_{n\geqslant 1} L_n^{(r)}(1,t)x^n = \frac{x + rtx^r}{1 - x - tx^r}. \tag{3.13}
$$

and

**Corollary 3.2.2.** *The ordinary generating function of the sequence $(L_n^{(r)}(-1,t))_{n\geqslant 1}$ is given by*

$$
\sum_{n\geqslant 1} L_n^{(r)}(-1,t)x^n = \frac{x + x^2 - rtx^r - (r-1)tx^{r+1} - rt^2x^{2r}}{1 - x^2 + t^2x^{2r}}. \tag{3.14}
$$

When $r = 2$ and $t = 1$ in (3.14), we get

$$
\sum_{n\geqslant 1} L_n^{(2)}(-1,1)x^n = \frac{x - x^2 - x^3 - 2x^4}{1 - x^2 + x^4} = \frac{(x - x^2 - 3x^4 - x^5 - 2x^6)(1 - x^6)}{1 - x^{12}}, \tag{3.15}
$$

which implies

**Corollary 3.2.3.** *The sequence* $(L_n^{(2)}(-1,1))_{n\geqslant 1}$ *is periodic with period* 12*; namely, if* $a_n := L_n^{(2)}(-1,1)$ *for* $n \geqslant 1$*, then* $a_1 = 1$*,* $a_2 = -1$*,* $a_3 = 0$*,* $a_4 = -3$*,* $a_5 = -1$*, and* $a_6 = -2$ *with* $a_{n+6} = -a_n$*,* $n \geqslant 1$*.*

This periodic behavior is again restricted to the case $r = 2$.

**Theorem 3.3.** *The sequence* $(L_n^{(r)}(-1,1))_{n\geqslant 1}$ *is never periodic for* $r \geqslant 3$*.*

*Proof.* By (3.14) at $t = 1$, we must show that $f(x) := 1-x^2+x^{2r}$ does not divide the product $(1-x^m)h(x)$, where $h(x) := x+x^2-rx^r-(r-1)x^{r+1}-rx^{2r}$, for any $m \in \mathbb{P}$ whenever $r \geqslant 3$. By the proof of Theorem 2.4, it suffices to show that $f$ and $h$ are relatively prime. Suppose, to the contrary, that $t_0$ is a common zero of $f$ and $h$ so that $t_0(1+t_0)+r(1-t_0^2) = t_0(1+t_0)-rt_0^{2r} = t_0^r[r+(r-1)t_0]$. Squaring, substituting $t_0^{2r} = t_0^2-1$, and noting $t_0 \neq -1$ implies that $t_0$ must then be a root of the equation $(x+1)\left[(r-1)x-r\right]^2 = (x-1)[(r-1)x+r]^2$, which reduces to $(r^2-1)x^2 = r^2$. But $t_0 = \pm\frac{r}{\sqrt{r^2-1}}$ is a zero of neither $f$ nor $h$ after all, which implies $f$ and $h$ are relatively prime and completes the proof. $\square$

Recall that $F_{-i}^{(r)}(q,t) = 0$ if $1 \leqslant i \leqslant r-1$, by convention.

**Theorem 3.4.** *Let* $m \in \mathbb{P}$*. If* $r$ *is even, then*

$$L_{2m}^{(r)}(-1,t) = L_m^{(r)}(1,-t^2) - rtF_{m-\frac{r}{2}}^{(r)}(1,-t^2) \tag{3.16}$$

*and*

$$L_{2m-1}^{(r)}(-1,t) = F_{m-1}^{(r)}(1,-t^2) - (r-1)tF_{m-\frac{r}{2}-1}^{(r)}(1,-t^2), \tag{3.17}$$

*and if* $r$ *is odd, then*

$$L_{2m}^{(r)}(-1,t) = L_m^{(r)}(1,-t^2) - (r-1)tF_{m-\left(\frac{r+1}{2}\right)}^{(r)}(1,-t^2) \tag{3.18}$$

*and*

$$L_{2m-1}^{(r)}(-1,t) = F_{m-1}^{(r)}(1,-t^2) - rtF_{m-\left(\frac{r+1}{2}\right)}^{(r)}(1,-t^2). \tag{3.19}$$

*Proof.* Taking the even and odd parts of both sides of (3.14) followed by replacing $x$ with $x^{1/2}$ yields

$$\sum_{m\geqslant 1} L_{2m}^{(r)}(-1,t)x^m = \frac{x - rtx^{\frac{r}{2}} - rt^2x^r}{1-x+t^2x^r}$$

and

$$\sum_{m\geqslant 1} L_{2m-1}^{(r)}(-1,t)x^m = \frac{x-(r-1)tx^{\frac{r}{2}+1}}{1-x+t^2x^r},$$

when $r$ is even, and

$$\sum_{m\geqslant 1} L_{2m}^{(r)}(-1,t)x^m = \frac{x-(r-1)tx^{\frac{(r+1)}{2}} - rt^2x^r}{1-x+t^2x^r}$$

and

$$\sum_{m\geqslant 1} L_{2m-1}^{(r)}(-1,t)x^m = \frac{x - rtx^{\frac{(r+1)}{2}}}{1-x+t^2x^r},$$

when $r$ is odd, from which (3.16)–(3.19) now follow from (3.13) and (2.12).

For a combinatorial proof of (3.16)–(3.19), we first assign to each covering $c \in \mathcal{C}_n^{(r)}$ the weight $w_c := (-1)^{\pi(c)}t^{v(c)}$, where $t$ is an indeterminate. Let $\mathcal{C}_n^{(r)'}$ consist of those $c$ in $\mathcal{C}_n^{(r)}$ whose associated word $u_c = u_1u_2\cdots$ satisfies the conditions $u_{2i} = u_{2i+1}$, $i \geqslant 1$. Suppose $c \in \mathcal{C}_n^{(r)} - \mathcal{C}_n^{(r)'}$, with $i_0$ being the smallest value of $i$ for which $u_{2i} \neq u_{2i+1}$. Exchanging the positions of the $(2i_0)^{th}$ and $(2i_0+1)^{st}$ pieces within $c$ produces a $\pi$-parity changing, $v$-preserving involution of $\mathcal{C}_n^{(r)} - \mathcal{C}_n^{(r)'}$.

If $r$ is even and $n = 2m$, then

$$\begin{aligned}
L_{2m}^{(r)}(-1,t) &= \sum_{c\in\mathcal{C}_{2m}^{(r)}} w_c = \sum_{c\in\mathcal{C}_{2m}^{(r)'}} w_c = \sum_{\substack{c\in\mathcal{C}_{2m}^{(r)'}\\ v(c)\text{ even}}} w_c + \sum_{\substack{c\in\mathcal{C}_{2m}^{(r)'}\\ v(c)\text{ odd}}} w_c\\
&= \sum_{\substack{c\in\mathcal{C}_{2m}^{(r)'}\\ v(c)\text{ even}}} (-1)^{v(c)/2}t^{v(c)} - rt\sum_{\substack{c\in\mathcal{R}_{2m-r}^{(r)'}\\ v(c)\text{ even}}} (-1)^{v(c)/2}t^{v(c)}\\
&= \sum_{z\in\mathcal{C}_m^{(r)}} (-1)^{v(z)}t^{2v(z)} - rt\sum_{z\in\mathcal{R}_{m-\frac{r}{2}}^{(r)}} (-1)^{v(z)}t^{2v(z)}\\
&= L_m^{(r)}(1,-t^2) - rtF_{m-\frac{r}{2}}^{(r)}(1,-t^2),
\end{aligned}$$

which gives (3.16), where $\mathcal{R}_n^{(r)'}$ is as in the proof of Theorem 2.5, since members of $\mathcal{C}_{2m}^{(r)'}$ with $v(c)$ even must begin and end with the same type of piece, while members with $v(c)$ odd must have $u_1 = r$ in $u_c$ with $r$ possibilities for the position of its initial segment. Similarly, if $r$ is odd and $n = 2m-1$, then

$$\begin{aligned}
L_{2m-1}^{(r)}(-1,t) &= \sum_{c\in\mathcal{C}_{2m-1}^{(r)'}} w_c = \sum_{\substack{c\in\mathcal{C}_{2m-1}^{(r)'}\\ u_1=s}} w_c + \sum_{\substack{c\in\mathcal{C}_{2m-1}^{(r)'}\\ u_1=r}} w_c\\
&= \sum_{c\in\mathcal{R}_{2m-2}^{(r)'}} w_c - rt\sum_{c\in\mathcal{R}_{2m-r-1}^{(r)'}} w_c\\
&= F_{m-1}^{(r)}(1,-t^2) - rtF_{m-(\frac{r+1}{2})}^{(r)}(1,-t^2),
\end{aligned}$$

which gives (3.19).

For the cases that remain, let $\mathcal{C}_n^{(r)*} \subseteq \mathcal{C}_n^{(r)'}$ such that $\mathcal{C}_n^{(r)'} - \mathcal{C}_n^{(r)*}$ comprises those $c$ which satisfy the following additional conditions:

(i) $c$ contains an even number of pieces in all;

(ii) $u_1 \neq u_p$ in $u_c = u_1u_2\cdots u_p$;

(iii) if $u_1 = r$, then 1 corresponds to the initial segment of the $r$-mino covering it.

Pair members of $\mathcal{C}_n^{(r)'} - \mathcal{C}_n^{(r)*}$ of opposite $\pi$-parity as follows: given $c \in \mathcal{C}_n^{(r)'} - \mathcal{C}_n^{(r)*}$, let $c'$ be the covering resulting when $u_c = u_1u_2\cdots u_p$ is read backwards.

If $r$ is even and $n = 2m-1$, then

$$\begin{aligned} L_{2m-1}^{(r)}(-1,t) &= \sum_{c\in\mathcal{C}_{2m-1}^{(r)}} w_c = \sum_{c\in\mathcal{C}_{2m-1}^{(r)*}} w_c = \sum_{\substack{c\in\mathcal{C}_{2m-1}^{(r)*}\\ u_1=s}} w_c + \sum_{\substack{c\in\mathcal{C}_{2m-1}^{(r)*}\\ u_1=r}} w_c \\ &= \sum_{\substack{c\in\mathcal{R}_{2m-2}^{(r)'}\\ v(c)\text{ even}}} w_c - (r-1)t \sum_{\substack{c\in\mathcal{R}_{2m-r-2}^{(r)'}\\ v(c)\text{ even}}} w_c \\ &= F_{m-1}^{(r)}(1,-t^2) - (r-1)tF_{m-\frac{r}{2}-1}^{(r)}(1,-t^2), \end{aligned}$$

which gives (3.17), since members of $\mathcal{C}_{2m-1}^{(r)*}$ with $u_1 = s$ must end in a double letter, while those with $u_1 = r$ must end in a single $s$ with 1 not corresponding to the initial segment of the $r$-mino covering it. Similarly, if $r$ is odd and $n = 2m$, then

$$\begin{aligned} L_{2m}^{(r)}(-1,t) &= \sum_{c\in\mathcal{C}_{2m}^{(r)*}} w_c = \sum_{\substack{c\in\mathcal{C}_{2m}^{(r)*}\\ u_1=u_p}} w_c + \sum_{\substack{c\in\mathcal{C}_{2m}^{(r)*}\\ u_1\neq u_p}} w_c \\ &= \sum_{\substack{c\in\mathcal{C}_{2m}^{(r)*}\\ u_1=u_p}} (-1)^{v(c)/2}t^{v(c)} - (r-1)t \sum_{c\in\mathcal{R}_{2m-r-1}^{(r)'}} (-1)^{v(c)/2}t^{v(c)} \\ &= \sum_{z\in\mathcal{C}_m^{(r)}} (-1)^{v(z)}t^{2v(z)} - (r-1)t \sum_{z\in\mathcal{R}_{m-\left(\frac{r+1}{2}\right)}^{(r)}} (-1)^{v(z)}t^{2v(z)} \\ &= L_m^{(r)}(1,-t^2) - (r-1)tF_{m-\left(\frac{r+1}{2}\right)}^{(r)}(1,-t^2), \end{aligned}$$

which gives (3.18). □

# 4 Variants of the $\pi$ Statistic

Modifying the $\pi$ statistic of the previous section in different ways yields additional polynomial generalizations of $L_n^{(r)}$. In this section, we look at some specific variants of the $\pi$ statistic on circular $r$-mino arrangements, taking $r = 2$ for simplicity. We'll use the notation $\mathcal{C}_n = \mathcal{C}_n^{(2)}$, $\mathcal{C}_{n,k} = \mathcal{C}_{n,k}^{(2)}$, and $F_n(q,t) = F_n^{(2)}(q,t)$.

We first partition $\mathcal{C}_n$ as follows: let $\overrightarrow{\mathcal{C}}_n$ comprise those coverings in which 1 is covered by a square or by an initial segment of a domino and let $\overleftarrow{\mathcal{C}}_n$ comprise those coverings in which 1 is covered by the second segment of a domino.

Define the statistic $\pi_1$ on $\mathcal{C}_n$ by

$$\pi_1(c) = \begin{cases} \pi(c), & \textit{if } c \in \overrightarrow{\mathcal{C}}_n; \\ \pi(c) - 2v(c) + n, & \textit{if } c \in \overleftarrow{\mathcal{C}}_n. \end{cases} \tag{4.1}$$

Note that $\pi_1(c)$ gives the sum of the numbers obtained by counting back counterclockwise the pieces from each domino to the piece covering 2 whenever $c \in \overleftarrow{\mathcal{C}}_n$.

**Theorem 4.1.** *For all* $n \in \mathbb{P}$,

$$\sum_{c \in \mathcal{C}_n} q^{\pi_1(c)} t^{v(c)} = \sum_{0 \leqslant k \leqslant \lfloor n/2 \rfloor} q^{\binom{k+1}{2}} \left[ \frac{(n-k)_q + q^{n-2k} k_q}{(n-k)_q} \right] \binom{n-k}{k}_q t^k. \tag{4.2}$$

*Proof.* By (2.10) when $r = 2$,

$$\begin{aligned} \sum_{c \in \mathcal{C}_{n,k}} q^{\pi_1(c)} &= q^{\binom{k+1}{2}} \binom{n-k}{k}_q + q^{\binom{k}{2}+k+(n-2k)} \binom{n-k-1}{k-1}_q \\ &= q^{\binom{k+1}{2}} \left[ \binom{n-k}{k}_q + q^{n-2k} \frac{k_q}{(n-k)_q} \binom{n-k}{k}_q \right] \\ &= q^{\binom{k+1}{2}} \left[ \frac{(n-k)_q + q^{n-2k} k_q}{(n-k)_q} \right] \binom{n-k}{k}_q . \end{aligned}$$

□

If $\hat{L}_n(q,t)$ denotes the distribution polynomial in (4.2), then

$$\hat{L}_n(q,t) = F_n(q,t) + q^{n-1} t F_{n-2}(q, t/q), \quad n \geqslant 1, \tag{4.3}$$

by (4.2) and (2.10), or by considering whether or not $c$ belongs to $\overrightarrow{\mathcal{C}}_n$. The $\hat{L}_n(q,t)$ satisfy the nice recurrence

$$\hat{L}_n(q,t) = \hat{L}_{n-1}(q, qt) + qt \hat{L}_{n-2}(q, qt), \quad n \geqslant 3, \tag{4.4}$$

with $\hat{L}_1(q,t) = 1$ and $\hat{L}_2(q,t) = 1 + 2qt$, the first term of (4.4) accounting for those $c \in \overrightarrow{\mathcal{C}}_n$ where 1 is covered by a square as well as those $c \in \overleftarrow{\mathcal{C}}_n$ where 2 is covered by a square and the second term accounting for the cases that remain.

Next define $\pi_2$ on $\mathcal{C}_n$ by

$$\pi_2(c) = \begin{cases} \pi(c), & \textit{if } c \in \overrightarrow{\mathcal{C}}_n; \\ \pi(c) - v(c), & \textit{if } c \in \overleftarrow{\mathcal{C}}_n. \end{cases} \tag{4.5}$$

**Theorem 4.2.** *For all* $n \in \mathbb{P}$,

$$\sum_{c \in \mathcal{C}_n} q^{\pi_2(c)} t^{v(c)} = \sum_{0 \leqslant k \leqslant \lfloor n/2 \rfloor} q^{\binom{k}{2}} \frac{n_q}{(n-k)_q} \binom{n-k}{k}_q t^k. \tag{4.6}$$

*Proof.* By (2.10) when $r=2$,

$$\begin{aligned}\sum_{c\in\mathcal{C}_{n,k}} q^{\pi_2(c)} &= q^{\binom{k+1}{2}}\binom{n-k}{k}_q + q^{\binom{k}{2}+k-k}\binom{n-k-1}{k-1}_q \\ &= q^{\binom{k}{2}}\left[\frac{q^k(n-k)_q+k_q}{(n-k)_q}\right]\binom{n-k}{k}_q \\ &= q^{\binom{k}{2}}\frac{n_q}{(n-k)_q}\binom{n-k}{k}_q.\end{aligned}$$

□

Theorem 4.2 provides a combinatorial interpretation of the generalized Lucas polynomials

$$Luc_n(x,t) := \sum_{0\leqslant k\leqslant\lfloor n/2\rfloor} q^{\binom{k}{2}}\frac{n_q}{(n-k)_q}\binom{n-k}{k}_q x^{n-2k}t^k, \tag{4.7}$$

studied by Cigler [2, 3]. Note that the joint distribution of $\pi_2$ and $v$ on $\mathcal{C}_n$ is $Luc_n(1,t)$, with the $x$ variable of $Luc_n(x,t)$ recording the number of squares in $c\in\mathcal{C}_n$. Considering whether or not $c$ belongs to $\overrightarrow{\mathcal{C}}_n$ leads directly to the relation (cf. [3])

$$Luc_n(1,t) = F_n(q,t)+tF_{n-2}(q,t), \quad n\geqslant 1. \tag{4.8}$$

The $Luc_n(1,t)$ do not seem to satisfy a two-term recurrence like (3.7) or (4.4).

Similar reasoning shows that $Luc_n(1,t)$ is also the joint distribution of the statistics $\pi_3$ and $v$ on $\mathcal{C}_n$, where

$$\pi_3(c) = \begin{cases}\pi(c)-v(c), & \text{if } c\in\overrightarrow{\mathcal{C}}_n;\\ \pi(c)-2v(c)+n, & \text{if } c\in\overleftarrow{\mathcal{C}}_n,\end{cases} \tag{4.9}$$

which yields the relation

$$Luc_n(1,t) = F_n(q,t/q)+q^{n-1}tF_{n-2}(q,t/q), \quad n\geqslant 1. \tag{4.10}$$

The $\pi_2$ statistic on $\mathcal{C}_n$ can be generalized to $\mathcal{C}_n^{(r)}$ by letting $\pi_2(c)=\pi(c)$, if the number 1 is covered by a square or an initial segment of an $r$-mino, and letting $\pi_2(c)=\pi(c)-v(c)$, otherwise. Reasoning as in Theorem 4.2 with $\pi_2$ on $\mathcal{C}_n^{(r)}$ leads to

$$Luc_n^{(r)}(x,t) := \sum_{0\leqslant k\leqslant\lfloor n/r\rfloor} q^{\binom{k}{2}}\left[\frac{(r-2)k_q+(n-(r-2)k)_q}{(n-(r-1)k)_q}\right]\binom{n-(r-1)k}{k}_q x^{n-rk}t^k, \tag{4.11}$$

which generalizes $Luc_n(x,t)$. The $Luc_n^{(r)}(x,t)$ are connected with the $F_n^{(r)}(q,t)$ by the simple relation

$$Luc_n^{(r)}(1,t) = F_n^{(r)}(q,t)+(r-1)tF_{n-r}^{(r)}(q,t), \qquad n\geqslant 1, \tag{4.12}$$

which generalizes (4.8).

## Acknowledgment

The authors would like to thank the anonymous referee for formulas (2.8), (2.9), (4.11), and (4.12).

## References

[1] L. Carlitz, Fibonacci notes 3: $q$-Fibonacci polynomials, *Fibonacci Quart.* **12** (1974), 317–322.

[2] J. Cigler, Einige $q$-Analoga der Lucas- and Fibonacci-Polynome, *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II* **211** (2002), 3–20.

[3] J. Cigler, A new class of $q$-Fibonacci polynomials, *Electron. J. Combin.* **10** (2003), #R19.

[4] J. Cigler, Some algebraic aspects of Morse code sequences, *Discrete Math. Theor. Comput. Sci.* **6** (2003), 55–68.

[5] J. Cigler, $q$-Fibonacci polynomials and the Rogers-Ramanujan identities, *Ann. Combin.* **8** (2004), 269–285.

[6] S. Ekhad and D. Zeilberger, The number of solutions of $X^2 = 0$ in triangular matrices over GF($q$), *Electron. J. Combin.* **3** (1) (1996), #R2.

[7] B. Kupershmidt, $q$-Analogues of classical 6-periodicity, *J. Nonlinear Math. Physics* **10** (2003), 318–339.

[8] M. Shattuck and C. Wagner, Parity theorems for statistics on domino arrangements, *Electron. J. Combin.* **12** (2005), #N10.

[9] M. Shattuck and C. Wagner, A new statistic on linear and circular $r$-mino arrangements, *Electron. J. Combin.* **13** (1) (2006), #R42.

[10] R. Stanley, *Enumerative Combinatorics, Vol. I*, Wadsworth and Brooks/Cole, 1986.

[11] C. Wagner, Generalized Stirling and Lah numbers, *Discrete Math.* **160** (1996), 199–218.

[12] H. Wilf, *generatingfunctionology*, Academic Press, 1990.

---

2000 *Mathematics Subject Classification:* Primary 11B39; Secondary 05A15.
*Keywords:* $r$-mino arrangement, polynomial generalization, Fibonacci numbers, Lucas numbers.

---

(Concerned with sequences A000045 and A000204.)

---

Received April 4 2006; revised version received August 17 2006. Published in *Journal of Integer Sequences*, August 18 2006.

---

Return to Journal of Integer Sequences home page.