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Enumeration of Concave Integer Partitions

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Abstract

An integer partition $\lambda \vdash n$ corresponds, via its Ferrers diagram, to an artinian monomial ideal $I \subset \mathbb{C}[x,y]$ with $\dim_{\mathbb{C}} \mathbb{C}[x,y]/I = n$. If λ corresponds to an integrally closed ideal we call it *concave* . We study generating functions for the number of concave partitions, unrestricted or with at most r parts.

1. CONCAVE PARTITIONS

By an *integer partition* $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ we mean a weakly decreasing sequence of non-negative integers, all but finitely many of which are zero. The non-zero elements are called the *parts* of the partition. When writing a partition, we often will only write the parts; thus $(2, 1, 1, 0, 0, 0, \dots)$ may be written as $(2, 1, 1)$.

We write $r = \langle \lambda \rangle$ for the number of parts of λ, and $n = |\lambda| = \sum_i \lambda_i$; equivalently, we write $\lambda \vdash n$ if $n = |\lambda|$. The set of all partitions is denoted by $\mathcal{P}$, and the set of partitions of n by $\mathcal{P}(n)$. We put $|\mathcal{P}(n)| = p(n)$. By subscripting any of the above with r we restrict to partitions with at most r parts.

We will use the fact that $\mathcal{P}$ forms a monoid under component-wise addition.

For an integer partition $\lambda \vdash n$ we define its *Ferrer's diagram* $F(\lambda) = \{(i,j) \in \mathbb{N}^2 \,|\, i < \lambda_{j+1}\}$. In figure 1 the black dots comprise the Ferrer's diagram of the partition $\mu = (4, 4, 2, 2)$.

Then $F(\lambda)$ is a finite *order ideal* in the partially ordered set $(\mathbb{N}^2, \leq)$, where $(a,b) \leq (c,d)$ iff $a \leq c$ and $b \leq d$. In fact, integer partitions correspond precisely to finite order ideals in this poset.

The complement $I(\lambda) = \mathbb{N}^2 \setminus F(\lambda)$ is a monoid ideal in the additive monoid $\mathbb{N}^2$. Recall that for a monoid ideal I the *integral closure* $\bar{I}$ is

$$\{\boldsymbol{a} | \ell \boldsymbol{a} \in I \text{ for some } \ell > 0\} \tag{1}$$

and that I is *integrally closed* iff it is equal to its integral closure.

Definition 1. The integer partition λ is *concave* iff $I(\lambda)$ is integrally closed. We denote by $\bar{\lambda}$ the unique partition such that $I(\bar{\lambda}) = \overline{I(\lambda)}$.

Now let R be the complex monoid ring of $\mathbb{N}^2$. We identify $\mathbb{N}^2$ with the set of commutative monomials in the variables x, y, so that $R \simeq \mathbb{C}[x, y]$. Then a monoid ideal $I \subset \mathbb{N}^2$ corresponds to the monomial ideal J in R generated by the monomials $\{x^i y^j \,|\, (i, j) \in I\}$. Furthermore, since the monoid ideals of the form $I(\lambda)$ are precisely those with finite complement to $\mathbb{N}^2$, those monoid ideals will correspond to monomial ideals $J \subset R$ such that R/J has a finite $\mathbb{C}$-vector space basis (consisting of images of those monomials not in J). By abuse of notation, such monomial ideals are called *artinian*, and the $\mathbb{C}$-vector space dimension of R/J is called the *colength* of J.

We get in this way a bijection between

(1) integer partitions of n,
(2) order ideals in $(\mathbb{N}^2, \leq)$ of cardinality n,
(3) monoid ideals in $\mathbb{N}^2$ whose complement has cardinality n, and
(4) monomial ideals in R of colength n.

Recall that if $\mathfrak{a}$ is an ideal in the commutative unitary ring S, then the *integral closure* $\bar{\mathfrak{a}}$ consists of all $u \in S$ that fulfill some equation of the form

$$s^n + b_1 s^{n-1} + \cdots + b_n = 0, \qquad b_i \in \mathfrak{a}^i \tag{2}$$

Then $\mathfrak{a}$ is always contained in its integral closure, which is an ideal. The ideal $\mathfrak{a}$ is said to be *integrally closed* if it coincides with its integral closure.

Note that this notion is **different** from the integral closure of $\mathfrak{a}$ as a **subring** of S. On the other hand, one can show[1] that the integral closure of the Rees algebra $S[\mathfrak{a}t] \subseteq S[t]$ is equal to the graded subring

$$S + \bar{\mathfrak{a}}t + \cdots + \overline{\mathfrak{a}^n}t^n + \cdots$$

For the special case $S = R$, we have that the integral closure of a monomial ideal is again a monomial ideal, and that the latter monomial ideal corresponds to the integral closure of the monoid ideal corresponding to the former monomial ideal [9, 14]. Hence, we have a bijection between

(1) concave integer partitions of n,
(2) integrally closed monoid ideals in $\mathbb{N}^2$ whose complements have cardinality n, and
(3) integrally closed monomial ideals in R of colength n.

Fröberg and Barucci [3] studied the growth of the number of ideals of colength n in certain rings, among them local noetherian rings of dimension 1. Studying the growth of the number of monomial ideals of colength n in R is, by the above, the same as studying the partition function $p(n)$. In this article, we will instead study the growth of the number of integrally closed monomial ideals in R, that is, the number of concave partitions of n.

2. Inequalities defining concave partitions

It is in general a hard problem to compute the integral closure of an ideal in a commutative ring. However, for monomial ideals in a polynomial ring, the following theorem, which can be found in [7, Exercise 4.23] or in [9, 14], makes the problem feasible.

[1] See for instance [13, chapter 6.6]

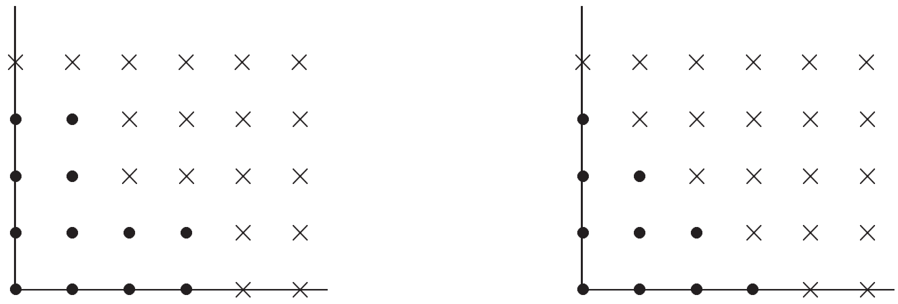

FIGURE 1. μ and $\bar{\mu}$

Theorem 2.1. *Let $I \subset \mathbb{N}^2$ be a monoid ideal, and regard $\mathbb{N}^2$ as a subset of $\mathbb{Q}^2$ in the natural way. Let $\mathrm{conv}_{\mathbb{Q}}(I)$ denote the convex hull of I inside $\mathbb{Q}^2$. Then the integral closure of I is given by*

$$\mathrm{conv}_{\mathbb{Q}}(I) \cap \mathbb{N}^2 \tag{3}$$

Example 2. The partition $\mu = (4, 4, 2, 2)$ corresponds to the monoid ideal

$$((0, 4), (2, 2), (4, 0)),$$

which has integral closure

$$((0, 4), (1, 3), (2, 2), (3, 1), (4, 0)).$$

It follows that $\overline{\mu} = (4, 3, 2, 1)$. In figure 1 we have drawn the lattice points belonging to $F(\mu)$ as dots, and the lattice points belonging to $I(\lambda)$ as crosses.

The above theorem gives the following characterization of concave partitions:

Lemma 2.1. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ be a partition. Then λ is concave iff for all positive integers $i < j < k$,*

$$\lambda_j < 1 + \lambda_i \frac{k - j}{k - i} + \lambda_k \frac{j - i}{k - i} \tag{4}$$

or, equivalently, if

$$\lambda_i(j - k) + \lambda_j(k - i) + \lambda_k(i - j) < k - i \tag{5}$$

Since all quantities involved are integers, (5) is equivalent to

$$\lambda_i(k - j) + \lambda_j(i - k) + \lambda_k(j - i) \geq i - k + 1 \tag{6}$$

3. Generating functions for super-concave partitions

We will enumerate concave partitions by considering another class of partitions which is more amenable to enumeration, yet is close to that of concave partitions.

Definition 3. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ be a partition. Then λ is *super-concave* iff for all positive integers $i < j < k$,

$$\lambda_i(k - j) + \lambda_j(i - k) + \lambda_k(j - i) \geq 0 \tag{7}$$

The reader should note that it is actually a *stronger* property to be super-concave than to be concave. Unlike the latter property, it is not necessarily preserved by conjugation: the partition (2) is super-concave, hence concave, but its conjugate $(1, 1)$ is concave but not super-concave.

Theorem 3.1. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ be a partition, and let $\mu = (\mu_1, \mu_2, \mu_3, \dots)$ be its conjugate, so that $|\{j \,|\, \mu_j = i\}| = \lambda_i - \lambda_{i+1}$ for all i. Then the following are equivalent:*

(i) λ is super-concave,

(ii) for all positive ℓ,

$$-\lambda_\ell + 2\lambda_{\ell+1} - \lambda_{\ell+2} \leq 0 \tag{8}$$

(iii) for all positive ℓ,

$$\lambda_{\ell+1} - \lambda_\ell \geq \lambda_{\ell+2} - \lambda_{\ell+1} \tag{9}$$

(iv) $|\{k|\mu_k = i\}| \geq |\{k|\mu_k = j\}|$ whenever $i \leq j$.

Proof. (i) $\iff$ (ii): Let $\boldsymbol{e}_i$ be the vector with 1 in the i'th coordinate and zeros elsewhere, let $\boldsymbol{f}_j = -\boldsymbol{e}_j + 2\boldsymbol{e}_{j+1} - \boldsymbol{e}_{j+2}$, and let $\boldsymbol{t}_{i,j,k} = (j-k)\boldsymbol{e}_i + (k-i)\boldsymbol{e}_j + (j-i)\boldsymbol{e}_k$. Clearly, (7) is equivalent with $\boldsymbol{t}_{i,j,k} \cdot \lambda \leq 0$, and (8) is equivalent with $\boldsymbol{f}_j \cdot \lambda \leq 0$. We have that $\boldsymbol{f}_\ell = \boldsymbol{t}_{\ell,\ell+1,\ell+2}$. Conversely, we claim that $\boldsymbol{t}_{i,j,k}$ is a positive linear combination of different $\boldsymbol{f}_\ell$. From this claim, it follows that if λ fulfills (8) for all ℓ then λ is super-concave.

We can without loss of generality assume that $i = 1$. Then it is easy to verify that

$$\boldsymbol{t}_{1,j,k} = \sum_{\ell=1}^{j-2} \ell(k-j)\boldsymbol{f}_\ell + \sum_{\ell=j-1}^{k-2} \ell(j-1)(k-\ell-1)\boldsymbol{f}_\ell \tag{10}$$

(ii) $\iff$ (iii) $\iff$ (iv) : This is obvious. $\square$

The *difference operator* Δ is defined on partitions by

$$\Delta(\lambda_1, \lambda_2, \lambda_3, \dots) = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - \lambda_4, \dots) \tag{11}$$

We get that the *second order difference operator* Δ^2 is given by

$$\begin{aligned}\Delta^2(\lambda_1, \lambda_2, \lambda_3, \dots) &= \Delta(\Delta(\lambda_1, \lambda_2, \lambda_3, \dots)) = \\ &= (\lambda_1 - 2\lambda_2 + \lambda_3, \lambda_2 - 2\lambda_3 + \lambda_4, \lambda_3 - 2\lambda_4 + \lambda_5, \dots)\end{aligned} \tag{12}$$

Corollary 3.1. *The super-concave partitions are precisely those with non-negative second differences.*

Definition 4. Let $p_{sc}(n)$ denote the number of super-concave partitions of n, and $p_{sc}(n,r)$ denote the number of super-concave partitions of n with at most r parts. Let similarly $p_c(n)$ and $p_c(n,r)$ denote the number of concave partitions of n, and the number of concave partitions of n with at most r parts, respectively. For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ let $\boldsymbol{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$, and define

$$\begin{aligned}
PS(\boldsymbol{x}) &= \sum_{\lambda \text{ super-concave}} \boldsymbol{x}^\lambda \\
PS_r(x_1, \dots, x_r) &= PS(x_1, x_2, \dots, x_r, 0, 0, 0, \dots) = \sum_{\substack{\lambda \text{ super-concave} \\ \lambda_{r+1} = 0}} \boldsymbol{x}^\lambda \\
PC(\boldsymbol{x}) &= \sum_{\lambda \text{ concave}} \boldsymbol{x}^\lambda \\
PC_r(x_1, \dots, x_r) &= PC(x_1, x_2, \dots, x_r, 0, 0, 0, \dots) = \sum_{\substack{\lambda \text{ concave} \\ \lambda_{r+1} = 0}} \boldsymbol{x}^\lambda
\end{aligned} \tag{13}$$

Partitions with non-negative second differences have been studied by Andrews [2], who proved that there are as many such partitions of n as there are partitions of n into triangular numbers.

Canfield et al [5] have studied partitions with non-negative m'th differences. Specializing their results to the case $m = 2$, we conclude:

Theorem 3.2. *Let n, r be denote positive integers.*

(i) There is a bijection between partitions of n into triangular numbers and super-concave partitions.

(ii) The multi-generating function for super-concave partitions is given by

$$\begin{aligned} PS(\boldsymbol{x}) &= \frac{1}{\prod_{i=1}^{\infty}\left(1 - \prod_{j=1}^{i} x_j^{1+i-j}\right)} \\ &= 1 + x_1 + {x_1}^2 + {x_1}^3 + {x_1}^4 + {x_1}^2 x_2 + {x_1}^5 + {x_1}^4 x_2 + {x_1}^3 x_2 + \dots \end{aligned} \tag{14}$$

(iii) The multi-generating function for super-concave partitions with at most r parts is given by

$$PS_r(x_1, x_2, \dots, x_r) = \frac{1}{\prod_{i=1}^{r}\left(1 - \prod_{j=1}^{i} x_j^{1+i-j}\right)} \tag{15}$$

(iv) The generating function for super-concave partitions is

$$PS(t) = \sum_{n=0}^{\infty} p_{sc}(n) t^n = \prod_{i=1}^{\infty} \frac{1}{1 - t^{\frac{i(i+1)}{2}}} \tag{16}$$

and the one for super-concave partitions with at most r parts is

$$PS_r(t) = \sum_{n=0}^{\infty} p_{sc}(n, r) t^n = \prod_{i=1}^{r} \frac{1}{1 - t^{\frac{i(i+1)}{2}}} \tag{17}$$

(v) The proportion of super-concave partitions with at most r parts among all partitions with at most r parts is

$$\frac{r!}{\prod_{i=1}^{r} \frac{i(i+1)}{2}}. \tag{18}$$

(vi) As $n \to \infty$,

$$\begin{gathered} p_{sc}(n) \sim c n^{-3/2} \exp(3 C n^{1/3}) \\ C = 2^{-1/3} \left[\zeta(3/2)\Gamma(3/2)\right]^{2/3}, \quad c = \frac{\sqrt{3}}{12}\left(\frac{C}{\pi}\right)^{3/2} \end{gathered} \tag{19}$$

The sequence $\left(p_{sc}(n)\right)_{n=0}^{\infty}$ is identical to sequence A007294 in OEIS [10]. We have submitted the sequences $\left(p_{sc}(n, r)\right)_{n=0}^{\infty}$, for $r = 3, 4$, in OEIS [10], as A086159 and A086160. The sequence for $r = 2$ was already in the database, as A008620.

3.1. **Other appearances of super-concave partitions in the literature.** The bijection between partitions into triangular numbers and partitions with non-negative second difference is mentioned in A007294 in OEIS [10], together with a reference to Andrews [2]. That sequence has been contributed by Mira Bernstein and Roland Bacher; we thank Philippe Flajolet for drawing our attention to it.

Gert Almkvist [1] gives an asymptotic analysis of $p_{sc}(n)$ which is finer than (19).

Another derivation of the generating functions above can found in a forthcoming paper "Partition Bijections, a Survey" [8] by Igor Pak. He observes that the set of super-concave partitions with at most r parts consists of the lattice points of the unimodular cone spanned by the vectors $v_0 = (1, \dots, 1)$ and $v_i = (i-1, i-2, \dots, 1, 0, 0, \dots)$ for $1 \le i \le r$.

Corteel and Savage [6] calculate rational generating functions for classes of partitions defined by linear homogeneous inequalities. This applies to super-concave partitions, but not directly to concave partitions, since the inequalities (5) defining them are inhomogeneous.

4. Generating functions for concave partitions

Recall that a concave partition $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$ fulfills (6), and that conversely, every sequence of non-negative integers which is eventually zero and fulfills (6) gives a concave partition. If we fix a positive integer r, then we need only finitely many of the inequalities in (6): we can take those indexed by $i < j < k < r+2$, together with the non-negativity conditions $\lambda_i \ge 0$. Hence, there is a matrix A with r columns, and whose rows are indexed by tuples (i, j, k) with $k \le r+1$, so that a concave partition with at most r parts corresponds to a solution to

$$A\boldsymbol{\lambda} \ge \boldsymbol{b}, \qquad \boldsymbol{\lambda} \in \mathbb{N}^r, \tag{20}$$

whereas a super-concave partition with at most r parts corresponds to a solution to

$$A\boldsymbol{\lambda} \ge \boldsymbol{0}, \qquad \boldsymbol{\lambda} \in \mathbb{N}^r. \tag{21}$$

We let $\mathcal{K} = \{\alpha \in \mathbb{R}^r | A\alpha \ge \boldsymbol{b}, \boldsymbol{\alpha} \ge \boldsymbol{0}\}$, $\mathcal{P} = \{\alpha \in \mathbb{R}^r | A\alpha \ge \boldsymbol{0}, \boldsymbol{\alpha} \ge \boldsymbol{0}\}$. Then $\mathcal{P}$ is a rational polyhedron in the positive orthant. Since the RHS vector $\boldsymbol{b}$ is non-positive, $\mathcal{P}$ contains its recession cone $\mathcal{K}$. The solutions to (21) and (20) are precisely $\mathcal{KI} = \mathcal{K} \cap \mathbb{N}^r$ and $\mathcal{PI} = \mathcal{P} \cap \mathbb{N}^r$, and the generating functions of these two sets of lattice points are precisely PS_r and PC_r.

Example 5. If $r = 3$ and if we order the 3-subsets of $\{1, 2, 3, 4\}$ as 123, 124, 134, 234 then

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{pmatrix}, \qquad \boldsymbol{b} = (-1, -2, -2, -1)^t.$$

$\mathcal{K}$ is the cone generated by the rays $(1, 0, 0)$, $(2, 1, 0)$, and $(3, 2, 1)$, whereas $\mathcal{P}$ is the Minkowski sum of $\mathcal{K}$ and the polytope which is the convex hull of $(0, 0, 0)$, $(0, 0, 1/2$, $(0, 1/3, 2/3)$, $(0, 1/2, 0)$, $(0, 2/3, 1/3)$, $(0, 2/3, 2/3)$. So $\mathcal{P}$ is a rational polyhedron but not a lattice polyhedron.

Lemma 4.1. *The generating function $PC_r(x_1, \dots, x_r)$ is a rational function with the same denominator as $PS_r(x_1, \dots, x_r)$, and with a numerator which evaluates to 1 at $(1, \dots, 1)$. In* *other words,*

$$PC_r(x_1,\dots,x_r) = \frac{Q_r(x_1,\dots,x_r)}{\prod_{i=1}^{r}\left(1-\prod_{j=1}^{i} x_j^{1+i-j}\right)}, \quad Q_r(1,\dots,1)=1. \tag{22}$$

Proof. This can be obtained from the corresponding result for linear diophantine *equalities*[2] by adding slack-variables and then specializing the corresponding formal variables to 1. We give the outline of a self-contained proof.

By Gordan's lemma[3], $\mathcal{KI}$ is a finitely generated affine semigroup. In fact, it has a unique finite minimal generating set, called its *Hilbert basis*. Furthermore, $\mathcal{PI}$ is a module over $\mathcal{KI}$, by which we mean that $\mathcal{KI}+\mathcal{PI}\subseteq\mathcal{PI}$. Now let $R=\mathbb{C}[\mathcal{KI}]$ be the semigroup ring on $\mathcal{KI}$, i.e. the $\mathbb{C}$-vector space spanned by all monomials $\{\boldsymbol{x}^{\boldsymbol{\alpha}} | \boldsymbol{\alpha}\in\mathcal{KI}\}$. We define

$$\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\beta}} = \boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}, \tag{23}$$

and extend this multiplication by linearity to all of R, turning it into a r-multigraded, noetherian $\mathbb{C}$-algebra. Similarly, we define M to be the $\mathbb{C}$-linear span of monomials corresponding to points in $\mathcal{PI}$. (20). The multiplication (23) gives M the structure of r-multigraded R-module.

Since $\mathcal{KI}$ is a finitely generated affine semigroup, R is a finitely generated $\mathbb{C}$-algebra. Since it is a subring of $\mathbb{C}[x_1,\dots,x_r]$, it is an integral domain. The Hilbert series of R is PS_r, and the Hilbert series of M is PC_r.

Now note that since there is some $\gamma\in\mathbb{N}^r$ such that $\mathcal{PI}+\gamma\subseteq\mathcal{KI}$, it follows that M is isomorphic as an R-module to the ideal $\boldsymbol{x}^{\gamma}M\subseteq R$. Consequently, M is a finitely generated torsion-free module over R, of rank 1. Its annihilator is zero, so M has the same dimension as R.

It follows from standard commutative algebra[4] that the Hilbert series of R and M are rational, of the form

$$\frac{N_R(x_1,\dots,x_r)}{\prod_{i=1}^{s}(1-\boldsymbol{x}^{\boldsymbol{\alpha}_i})} \quad \text{and} \quad \frac{N_M(x_1,\dots,x_r)}{\prod_{i=1}^{s}(1-\boldsymbol{x}^{\boldsymbol{\alpha}_i})},$$

where the $\boldsymbol{\alpha}_i$'s are the elements of a basis of $\mathcal{KI}$, and the polynomials N_R and N_M have rational coefficients. Since we know the Hilbert series of R, we conclude that the vectors $(i,i-1,\dots,0,\dots,0)$, for $1\le i\le r$, form a basis for $\mathcal{KI}$.

Furthermore[5], $N_R(1,\dots,1)=1$, and $N_M(1,\dots,1)=\operatorname{rank}(M)=1$. The ring R is Cohen-Macaulay, hence[6] all coefficients of $N_R(t,\dots,t)$ are non-negative. As calculated in (28), the polynomials $N_M(t,\dots,t)$ have some negative coefficients for $r=2,3,4$, so M is not Cohen-Macaulay in general.

□

We can say something more about the numerators:

Theorem 4.1. *Let r be a fixed positive integer. Then*

[2] See [11, Corollary 3.8] and the paragraph immediately following it

[3] Gordan's lemma says the lattice points in a finitely generated rational cone in the positive orthant constitute a normal affine semigroup, see [4, Proposition 6.1.2]

[4] See [11, Theorem 2.3], and note that M is $\mathbb{N}^r$-graded rather than $\mathbb{Z}^r$-graded

[5] See [4] , exercise 4.4.12

[6] See again [4] , exercise 4.4.12

(A) The multigenerating function of concave partitions with at most r parts is given by (22), where $Q_r(x_1,\dots,x_r)$ is a polynomial with integer coefficients such that all exponent vectors of the monomials that occur in Q_r are weakly decreasing.

(B) The generating function for concave partitions with at most r parts is given by

$$PC_r(t) = \sum_{n=0}^{\infty} p_c(n,r)t^n = \frac{Q_r(t)}{\prod_{i=1}^{r}\left(1-t^{\frac{i(i+1)}{2}}\right)} \tag{24}$$

where $Q_r(1)=1$, and the numerator has degree strictly smaller than $r^3/6+r^2/2+r/3$.

(C) $p_c(n,r) \sim p_{sc}(n,r)$ as $n \to \infty$.

(D) The proportion of concave partitions with at most r parts among all partitions with at most r parts is the same as the proportion of super-concave partitions with at most r parts among all partitions with at most r parts, namely

$$\frac{r!}{\prod_{i=1}^{r}\frac{i(i+1)}{2}}. \tag{25}$$

(E) $Q_r(x_1,\dots,x_r) = Q_{r+1}(x_1,\dots,x_r,0)$.

(F)

$$PC(\boldsymbol{x}) = \frac{Q(\boldsymbol{x})}{\prod_{i=1}^{\infty}\left(1-\prod_{j=1}^{i} x_j^{1+i-j}\right)} \tag{26}$$

where $Q(\boldsymbol{x})$ is a formal power series with the property that for each ℓ, $Q(x_1,\dots,x_\ell,0,0,\dots) = Q_\ell(x_1,\dots,x_\ell)$; in other words,

$$Q = 1 + \sum_{i=1}^{\infty}(Q_i - Q_{i-1})$$

Proof. All monomials in

$$\prod_{i=1}^{r}\left(1-\prod_{j=1}^{i} x_j^{1+i-j}\right)$$

have weakly decreasing exponent vectors, as have all monomials in the power series $PC_r(x_1,\dots,x_r)$. Summing weakly decreasing exponent vectors gives weakly decreasing exponent vectors, so all exponent vectors in $Q_r(x_1,\dots,x_r)$ are weakly decreasing.

If we specialize $x_1 = x_2 = \cdots = x_r = t$ we get

$$PC_r(t) = \frac{Q_r(t)}{\prod_{i=1}^{r}\left(1-t^{\frac{i(i+1)}{2}}\right)}, \qquad PS_r(t) = \frac{1}{\prod_{i=1}^{r}\left(1-t^{\frac{i(i+1)}{2}}\right)}.$$

Thus $Q_r(1)=1$, and we conclude that $p_c(n,r) \sim p_{sc}(n,r)$ as $n \to \infty$.

Furthermore, from Stanley's "grey book" [12, Theorem 4.6.25] we have that the rational function $PC_r(t,\dots,t)$ is of degree < 0. The degree of the denominator is

$$\sum_{i=1}^{r}\frac{i(i+1)}{2} = \frac{r^3}{6}+\frac{r^2}{2}+\frac{r}{3}$$

so $Q_r(t)$ have smaller degree than that.

If $(\lambda_1,\dots,\lambda_r,\lambda_{r+1})$ is a concave partition, then so is $(\lambda_1,\dots,\lambda_r,0)$; it follows that $Q_{r+1}(x_1,\dots,x_r,0) = Q_r(x_1,\dots,x_r)$. The assertion about $PC(\boldsymbol{x})$ follows by passing to the limit. ☐

By generating all concave partitions of n with at most r parts, up to a large n, we have calculated that

$$\begin{aligned} Q_1(\boldsymbol{x}) &= 1 \\ Q_2(\boldsymbol{x}) &= 1 + x_1x_2 - x_1^2x_2 \\ Q_3(\boldsymbol{x}) &= Q_2(\boldsymbol{x}) + x_3\left({x_1}^5{x_2}^3 - {x_1}^4{x_2}^3 - 2\,{x_1}^3{x_2}^2 + {x_1}^2{x_2}^2 + x_1x_2\right) \end{aligned} \tag{27}$$

and that

$$\begin{aligned} Q_1(t) &= 1 \\ Q_2(t) &= 1 + t^2 - t^3 \\ Q_3(t) &= 1 + t^2 + t^5 - 2t^6 - t^8 + t^9 \\ Q_4(t) &= 1 + t^2 + t^4 + t^5 - t^6 - t^7 + 2t^9 - 2t^{10} - t^{11} - 2t^{12} + \\ &\quad + 2t^{13} - t^{14} - t^{15} + t^{16} + t^{17} + t^{18} - t^{19} \end{aligned} \tag{28}$$

We have also used the package LinDiophanthus [15] by Doron Zeilberger to verify our results.

By generating all concave partitions of n for $n \leq 20$ we have calculated that

$$\begin{aligned} PC(t) = \sum_{n=0}^{\infty} p_c(n)t^n &= 1 + t + 2t^2 + 3t^3 + 4t^4 + 7t^5 + 9t^6 + 11t^7 + \\ &+ 17t^8 + 23t^9 + 28t^{10} + 39t^{11} + 48t^{12} + 59t^{13} + 79t^{14} + \\ &+ 100t^{15} + 121t^{16} + 152t^{17} + 185t^{18} + 225t^{19} + 280t^{20} + O(t^{21}) \end{aligned} \tag{29}$$

Based on (28), we conjecture that

$$PC(t) = \frac{1 + t^2 + O(t^3)}{\prod_{i=1}^{\infty}\left(1 - t^{\frac{i(i+1)}{2}}\right)} \tag{30}$$

We also conjecture that $\log p_c(n)$ grows as $n^{1/3}$, i.e. approximately as fast as super-concave partitions.

The sequences $\left(p_c(n)\right)_{n=0}^{\infty}$ are in the OEIS [10] as A084913. The sequences $\left(p_c(n,r)\right)_{n=0}^{\infty}$ are A086161, A086162, and A086163 for $r = 2, 3, 4$.

References

[1] Gert Almkvist. Asymptotics of various partitions. manuscript.

[2] George E. Andrews. MacMahon's partition analysis. II. Fundamental theorems. *Ann. Comb.*, 4(3-4):327–338, 2000. Conference on Combinatorics and Physics (Los Alamos, NM, 1998).

[3] Valentina Barucci and Ralf Fröberg. On the number of ideals of finite colength. In *Geometric and combinatorial aspects of commutative algebra (Messina, 1999)*, volume 217 of *Lecture Notes in Pure and Appl. Math.*, pages 11–19. Dekker, New York, 2001.

[4] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*. Cambridge University Press, Cambridge, 1993.

[5] Rod Canfield, Sylvie Corteel, and Pawel Hitczenko. Random partitions with non-negative r-th differences. *Adv. in Appl. Math.*, 27(2-3):298–317, 2001. Special issue in honor of Dominique Foata's 65th birthday (Philadelphia, PA, 2000).

[6] S. Corteel and C. D. Savage. Partitions and compositions defined by inequalities. math.CO/0309110.

[7] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer Verlag, 1995.

[8] Igor Pak. Partition bijections, a survey. *Ramanujan Journal*, to appear.

[9] Veronica Crispin Quinonez. Integrally closed monomial ideals and powers of ideals. Technical report, Stockholm University, 2002.

[10] Neil J. A. Sloane. The on-line encyclopedia of integer sequences. `http://www.research.att.com/~njas/sequences/index.html`.

[11] Richard P. Stanley. *Combinatorics and Commutative Algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser, 2 edition, 1996.

[12] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[13] Wolmer V. Vasconcelos. *Computational Methods in Commutative Algebra and Algebraic Geometry*. Algorithms and Computations in Mathematics. Springer Verlag, 1998.

[14] R. Villarreal. *Monomial Algebras*. Marcel Dekker, 2001.

[15] Doron Zeilberger. Lindiophantus. `http://www.math.temple.edu/~zeilberg/`, 2001. Program for calculating multi-generating functions for solution sets of systems of diophantine equations.

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