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Zeroing the baseball indicator and the chirality of triples

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Abstract

Starting with a common baseball umpire indicator, we consider the zeroing number for two-wheel indicators with states (a, b) and three-wheel indicators with states (a, b, c). Elementary number theory yields formulae for the zeroing number. The solution in the three-wheel case involves a curiously nontrivial minimization problem whose solution determines the chirality of the ordered triple (a, b, c) of pairwise relatively prime numbers. We prove that chirality is in fact an invariant of the unordered triple $\{a, b, c\}$. We also show that the chirality of Fibonacci triples alternates between 1 and 2.

1. Introduction

A standard three-wheel baseball umpire indicator consists of a first wheel (for strikes) with 4 cyclic states $(0, 1, 2, 3)$, a second wheel (for balls) with 5 cyclic states $(0, 1, 2, 3, 4)$, and a third wheel (for outs) with 4 cyclic states $(0, 1, 2, 3)$. The daughter of the second author asked what is the least number of clicks required, if one alternates between advancing the strike wheel and the ball wheel one click cyclically, to return these two wheels to their original states. We call this the (two-wheel) *zeroing number*. After her father gave her the wrong answer of $2\text{lcm}(4, 5) = 40$, she asked why the correct answer was in fact 31. With a total of 40 clicks both wheels advance 20 times, and since $20 \equiv 0 \pmod 4$ and $20 \equiv 0 \pmod 5$ both wheels are returned to their original state. However we can do better. A total of 31 clicks advances the strike wheel 16 times and advances the ball wheel only 15 times. Since $16 \equiv 0 \pmod 4$ and $15 \equiv 0 \pmod 5$ both wheels are therefore zeroed. The zeroing number is 31, as 31 is the minimal such number. However if instead one starts with balls and then moves onto strikes, one finds that a total of only 9 clicks are required to zero the indicator.

We note that zeroing number does depend on the ordering of the wheels and that the the sum of these two zeroing numbers gives the incorrect solution: $31 + 9 = 40$.

Using elementary number theory we find a complete solution for the general two-wheel indicator with cyclic states $a, b \geq 2$. The problem of the general three-wheel indicator, with cyclic states $a, b, c \geq 2$, yields a much more interesting result. In both cases all wheels should have at least 2 cyclic states, since otherwise we would have wheels that could not be clicked. For the three-wheel indicator, when a, b, c are not pairwise relatively prime a satisfactory solution is easily obtained. However when they are pairwise relatively prime the solution involves a nontrivial minimization problem. Of interest is whether the zeroing number modulo 3 is 1 (so that the final click is on the first wheel) or 2 (so that the final click is on the second wheel). We call this number the *chirality* of the triple (a, b, c). While the zeroing number is highly dependent on the ordering of the wheels, we prove that the chirality does not depend on the ordering and is an invariant of the unordered triple $\{a, b, c\}$. The chirality is a mysterious quantity demonstrating many interesting patterns. We prove one of these for the Fibonacci sequence, but we still seek more general explanations.

2. Two wheels

If we let n be the number of times the first wheel is advanced, then in order to obtain the zeroing number for two wheels with a and b states we must find the minimum positive solution of

$$\begin{aligned} n &\equiv 0 \pmod{a} \\ n &\equiv 0 \pmod{b} \end{aligned}$$

or of

$$\begin{aligned} n &\equiv 0 \pmod{a} \\ n-1 &\equiv 0 \pmod{b} \end{aligned}$$

depending on whether we stop after moving the second or first wheel. The first set of equations implies that $n = \mathrm{lcm}(a, b)$. The second implies

$$\begin{aligned} n &= ak \\ ak &= 1 \pmod{b} \end{aligned}$$

for the smallest positive number $k < b$. So $n = aa_b^{-1}$ where a_b^{-1} is the multiplicative inverse of a in $\mathbb{Z}_b^*$.

Only when a and b are relatively prime does the second system have solutions and then it provides the zeroing number. In general, the zeroing number is

$$f(a,b) = \begin{cases} 2\mathrm{lcm}(a,b), & \text{if } \gcd(a,b) \neq 1; \\ 2aa_b^{-1} - 1, & \text{if } \gcd(a,b) = 1. \end{cases} \tag{1}$$

When a and b are relatively prime the zeroing of the wheels first occurs when the final move is on the first wheel, as the second quantity above is obviously smaller than the first. In fact, there is the following statement about how much smaller it is.

Theorem 2.1. *If a and b are relatively prime then*

$$f(a,b) + f(b,a) = 2ab$$

or equivalently
$$aa_b^{-1} + bb_a^{-1} - 1 = ab.$$

Proof. After $f(a,b)$ advances the (a,b)-indicator has been zeroed for the first time. If one continues advancing the wheels, starting with the b-wheel, zeroing will first occur after another $f(b,a)$ advances. Thus the sum represents the smallest number of advances zeroing both wheels in which they are both advanced an equal number of times. We note that $\mathrm{lcm}(a,b) = ab$ since a, b are relatively prime. □

As an aside we remark that if a and b are relatively prime we have $f(a,b) \neq f(b,a)$! Otherwise by Theorem 2.1 $f(a,b) = ab$ and by Equation (1) $f(a,b) = 2aa_b^{-1} - 1$. The first is a multiple of a, while the second is not.

We can make explicit machine computations using Mathematica [3], by using the Euler phi function [1, 2] when $\gcd(a,b) = 1$: $a_b^{-1} \equiv a^{\phi(b)-1} \pmod b$.

3. Three wheels

The case of 3 or more wheels can be treated similarly, but now the minimization problem becomes nontrivial. Again let n be the number of times the first wheel is advanced. If all three wheels are advanced the same number of times, we must solve

$$\begin{aligned} n &\equiv 0 \pmod a, \\ n &\equiv 0 \pmod b, \\ n &\equiv 0 \pmod c. \end{aligned}$$

Therefore a total of
$$f_0(a,b,c) = 3\mathrm{lcm}(a,b,c)$$
advances are necessary. The other possible solutions involve unequal numbers of advances with one or two of the wheels advanced one less than the first. If the final move is on the first wheel then we must solve

$$\begin{aligned} n &\equiv 0 \pmod a, \\ n-1 &\equiv 0 \pmod b, \\ n-1 &\equiv 0 \pmod c. \end{aligned}$$

Therefore a total of
$$f_1(a,b,c) = 3aa_{\mathrm{lcm}(b,c)}^{-1} - 2 \qquad (2)$$
advances are necessary. If the final move is on the second wheel then we must solve

$$\begin{aligned} n &\equiv 0 \pmod a, \\ n &\equiv 0 \pmod b, \\ n-1 &\equiv 0 \pmod c. \end{aligned}$$

Therefore a total of
$$f_2(a,b,c) = 3\mathrm{lcm}(a,b)\mathrm{lcm}(a,b)_c^{-1} - 1 \qquad (3)$$
advances are necessary.

(a,b,c)	$f_1(a,b,c)$	$f_2(a,b,c)$	$f(a,b,c)$	$\chi(a,b,c)$
$(2,3,5)$	46	17	17	2
$(3,5,2)$	61	44	44	2
$(5,2,3)$	73	29	29	2
$(2,5,3)$	46	29	29	2
$(3,2,5)$	61	17	17	2
$(5,3,2)$	73	44	44	2

TABLE 1. Some numerical data

We conclude that the zeroing number is

$$f(a,b,c) = \min \begin{cases} 3\mathrm{lcm}(a,b,c), & \text{Case 0;} \\ 3aa^{-1}_{\mathrm{lcm}(b,c)} - 2, & \text{Case 1;} \\ 3\mathrm{lcm}(a,b)\mathrm{lcm}(a,b)^{-1}_{c} - 1, & \text{Case 2} \end{cases} \tag{4}$$

where the minimization takes place over all existing cases.

Clearly $f_0(a,b,c)$ is always defined, but is larger than either of the two remaining cases (if they exist). So this minimization problem is nontrivial only when both $f_1(a,b,c)$ and $f_2(a,b,c)$ are defined. For Case 1 to exist means that a and $\mathrm{lcm}(b,c)$ are relatively prime. For Case 2 to exist means that $\mathrm{lcm}(a,b)$ and c are relatively prime. Taken together, this means that only when a,b,c are pairwise relatively prime is the minimization nontrivial.

As a source of examples we provide some numerical data in Table 1.

4. CHIRALITY

In chemistry a molecule is said to be chiral if it is not superimposable on its mirror image. Therefore such a molecule has two distinct chiralities, left and right handedness. However in the following definition it is more natural to denote these chiralities by 1 and 2.

Definition 4.1. The chirality of an ordered triple of pairwise relatively prime natural numbers ≥ 2 is the triple's zeroing number modulo 3. We write $\chi(a,b,c) = f(a,b,c) \bmod 3$. We note that the chirality of such a triple will always be 1 or 2.

The chirality therefore corresponds to the case number from Section 3 that provides the zeroing number.

Theorem 4.1. *The chirality of the ordered triple (a,b,c) (of pairwise relatively prime natural numbers ≥ 2) is invariant under any permutation of the triple. It is thus an invariant of the set $\{a,b,c\}$.*

Recalling the origins of the problem, it can be said that the chirality depends on the team, not on the lineup!

In order to prove the theorem we make use of some umpire indicator identities which are valid whenever f_1 and f_2 are defined on the given arguments.

Lemma 4.1.

$$f_1(a,b,c) + f_1(b,c,a) + f_1(c,a,b) = 3\mathrm{lcm}(a,b,c)k_1 \tag{5}$$

$$f_2(a,b,c) + f_2(c,a,b) + f_2(b,c,a) = 3\mathrm{lcm}(a,b,c)k_2 \tag{6}$$

$$f_2(a,b,c) + f_1(c,a,b) = 3\mathrm{lcm}(a,b,c) \tag{7}$$

where $k_1, k_2 \in \{1, 2\}$

Proof. By following the advances on a three-wheel baseball umpire indicator, the left hand sides of all three identities are easily seen to be multiples of $3\mathrm{lcm}(a, b, c)$. And since f_1 and f_2 are always strictly less than $3\mathrm{lcm}(a, b, c)$ the multiples must be as indicated. □

We now proceed to prove the theorem, so we assume that a, b, c are pairwise relatively prime numbers ≥ 2. We note then that $\mathrm{lcm}(a, b, c) = abc$. Adding equations (5) and (6) we get

$$\begin{aligned} f_2(a, b, c) + f_2(c, a, b) + f_2(b, c, a)+ & \\ f_1(a, b, c) + f_1(b, c, a) + f_1(c, b, a) & = 3abc(k_1 + k_2). \end{aligned}$$

Applying the following three identities of the form of Equation (7)

$$\begin{aligned} f_2(a, b, c) + f_1(c, a, b) &= 3abc \\ f_2(c, a, b) + f_1(b, c, a) &= 3abc \\ f_2(b, c, a) + f_1(a, b, c) &= 3abc \end{aligned} \tag{8}$$

we find that $3abc(k_1 + k_2) = 9abc$, so $k_1 + k_2 = 3$ and $\{k_1, k_2\} = \{1, 2\}$.

If $k_1 = 1$ then when we subtract Equation (7) from Equation (5) we get $f_1(a, b, c) + f_1(b, c, a) - f_2(a, b, c) = 0$. Therefore $f_2(a, b, c) > f_1(a, b, c)$. So we get chirality 1.

Similarly if $k_2 = 1$ we get chirality 2. It follows that chirality 1 is equivalent to $k_1 = 1$ (and $k_2 = 2$), while chirality 2 is equivalent to $k_1 = 2$ (and $k_2 = 1$). We also note that

$$\chi(a, b, c) = \frac{f_1(a, b, c) + f_1(b, c, a) + f_1(c, b, a)}{3\mathrm{lcm}(a, b, c)}. \tag{9}$$

Looking at Equation (9) we see that cyclic permutations of the triple do not change the chirality!

As an aside we note that we now have a three-wheel analog to Theorem 2.1:

$$f(a, b, c) + f(b, c, a) + f(c, a, b) = 3abc$$

We must now prove that chirality is invariant under transposition of two wheels of the triple. Due to the invariance under cyclic permutations, all such transpositions are equivalent, so it suffices to check just one such transposition. We start with the three identities in Equation (8). Switching the second and third arguments of f_1 preserves the identities. So we have:

$$\begin{aligned} f_2(a, b, c) + f_1(c, b, a) &= 3abc \\ f_2(c, a, b) + f_1(b, a, c) &= 3abc \\ f_2(b, c, a) + f_1(a, c, b) &= 3abc \end{aligned}$$

Summing these three identities we get

$$\begin{aligned} f_2(a, b, c) + f_2(c, a, b) + f_2(b, c, a)+ & \\ f_1(a, c, b) + f_1(c, b, a) + f_1(b, a, c) & = 9abc. \end{aligned}$$

Using Equation (6) we find

$$3abck_2 + f_1(a, c, b) + f_1(c, b, a) + f_1(b, a, c) = 9abc$$

which along with $k_1 + k_2 = 3$ implies that

$$f_1(a,c,b) + f_1(c,b,a) + f_1(b,a,c) = 3abck_1.$$

Applying Equation (9) we see that $\chi(a,c,b) = k_1 = \chi(a,b,c)$ and we have proven Theorem 4.1.

5. Fibonacci triples

Looking at the data for the three-wheel umpire indicator one notices many intriguing patterns for the chirality of a triple of pairwise relatively prime numbers ≥ 2. It is difficult to analyze the data in general since one is dealing with a 3-dimentional array, where even the existence of chirality depends on the distribution of primes.

As an example we investigate one such pattern in detail. Let F_n be the n-th Fibonacci number, where $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$. Chirality is defined for $\{F_n, F_{n+1}, F_{n+2}\}$ whenever $n \geq 3$. Using Equation (4) we found, using Mathematica [3] that $\chi(2,3,5) = 2$, $\chi(3,5,8) = 1$, $\chi(5,8,13) = 2$, $\chi(8,13,21) = 1$, $\ldots$. We conjectured that this sequence continues to alternate, and then we proved:

Theorem 5.1. *For $n \geq 3$*

$$\chi(F_n, F_{n+1}, F_{n+2}) = \begin{cases} 1, & \text{if } n \text{ is even;} \\ 2, & \text{if } n \text{ is odd.} \end{cases}$$

In order to prove this theorem we use the following Fibonacci identities:

Lemma 5.1 (Fibonacci identities)**.**

$$\begin{aligned} F_{n+1}^2 - F_n F_{n+2} &= (-1)^n \\ F_{n+1}F_{n+2} - F_n F_{n+3} &= (-1)^n \end{aligned}$$

Proof. The first identity is easily proven by induction. It is a standard exercise in elementary number theory courses.

The second identity can proven using a similar induction. However we note that it is just a reformulation of the first identity:

$$\begin{aligned} F_{n+1}F_{n+2} - F_n F_{n+3} &= F_{n+1}F_{n+2} - F_n F_{n+2} - F_n F_{n+1} \\ &= F_{n+1}(F_{n+2} - F_n) - F_n F_{n+2} \\ &= F_{n+1}^2 - F_n F_{n+2} \\ &= (-1)^n. \end{aligned}$$

□

We now prove the theorem itself. First we assume that n is odd. We then must show that $\chi(F_n, F_{n+1}, F_{n+2}) = 2$, but by the invariance of chirality this is equivalent to showing that $\chi(F_n, F_{n+2}, F_{n+1}) = 2$.

From Equation (2)

$$f_1(F_n, F_{n+2}, F_{n+1}) = 3F_n {F_n}^{-1}_{F_{n+2}F_{n+1}} - 2$$

But by the second Fibonacci identity $F_n F_{n+3} \equiv 1 \pmod{F_{n+2}F_{n+1}}$, therefore

$$f_1(F_n, F_{n+2}, F_{n+1}) = 3F_n F_{n+3} - 2 \tag{10}$$

From Equation (3)

$$f_2(F_n, F_{n+2}, F_{n+1}) = 3F_nF_{n+2}(F_nF_{n+2})^{-1}_{F_{n+1}} - 1$$

But by the first Fibonacci identity $F_nF_{n+2} \equiv 1 \pmod{F_{n+1}}$, therefore

$$f_2(F_n, F_{n+2}, F_{n+1}) = 3F_nF_{n+2} - 1. \tag{11}$$

Comparing Equation (10) and Equation (11) we find that f_2 is smaller and therefore, when n is odd, $\chi(F_n, F_{n+1}, F_{n+2}) = 2$ as desired.

We now assume that n is even. We must show that $\chi(F_n, F_{n+1}, F_{n+2}) = 1$, but by the invariance of chirality this is equivalent to showing that $\chi(F_{n+1}, F_{n+2}, F_n) = 1$.

From Equation (2)

$$f_1(F_{n+1}, F_{n+2}, F_n) = 3F_{n+1}F_{n+1}{}^{-1}_{F_{n+2}F_n} - 2$$

But by the first Fibonacci identity $F_{n+1}F_{n+1} \equiv 1 \pmod{F_{n+2}F_n}$, therefore

$$f_1(F_{n+1}, F_{n+2}, F_n) = 3F_{n+1}F_{n+1} - 2 \tag{12}$$

From Equation (3)

$$f_2(F_{n+1}, F_{n+2}, F_n) = 3F_{n+1}F_{n+2}(F_{n+1}F_{n+2})^{-1}_{F_n} - 1$$

But by the second Fibonacci identity $F_{n+1}F_{n+2} \equiv 1 \pmod{F_n}$, therefore

$$f_2(F_{n+1}, F_{n+2}, F_n) = 3F_{n+1}F_{n+2} - 1. \tag{13}$$

Comparing Equation (12) and Equation (13) we find that f_1 is smaller and therefore, when n is even, $\chi(F_n, F_{n+1}, F_{n+2}) = 1$ as desired.

This completes the proof of Theorem 5.1.

Note that we have also proven:

Theorem 5.2. *If n is odd*

$$f(F_n, F_{n+2}, F_{n+1}) = 3F_nF_{n+2} - 1.$$

If n is even

$$f(F_{n+1}, F_{n+2}, F_n) = 3F_{n+1}^2 - 2.$$

The order of the arguments of f is crucial, since while chirality is an invariant of unordered triples, the zeroing number is not.

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