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Asymptotic Estimate for the Multinomial Coefficients

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Abstract

The multinomial coefficient $\binom{n,q}{k}$ is defined to be the coefficient of x^k in $(1+x+x^2+\cdots+x^{q-1})^n$. It is conjectured that for given $n>2$, $T(n,q) := \binom{n,q}{cn} - \binom{n,q-1}{cn}$ is unimodal and the maximum occurs at $q = \lfloor \log_{1+\frac{1}{c}} n \rfloor$ or $q = \lfloor \log_{1+\frac{1}{c}} n \rfloor + 1$. As an attempt to prove this conjecture, we give an asymptotic estimate for $\binom{n,q}{cn}$ as n tends to infinity, where c is a positive integer.

1 Introduction

The multinomial coefficient $\binom{n,q}{k}$ is defined by

$$\sum_{k=0}^{\infty} \binom{n,q}{k} x^k = (1+x+x^2+\cdots+x^{q-1})^n.$$

Clearly $\binom{n,q}{k}$ is a natural generalization of the well-known binomial and trinomial coefficients and thus belongs to a large class of fundamental combinatorial numbers. It was studied extensively by many mathematicians since Euler. For details related to this number the readers are referred to [1, 2, 4, 5, 9, 14]. Some applications in coding theory and communication theory can be found in [8, 10].

Multinomial coefficients count the numbers of certain compositions. For a positive integer k, a *composition* (also called an ordered partition) is a finite sequence of positive integers $x_1, x_2, \ldots, x_r$ such that $x_1 + x_2 + \cdots + x_r = k$. The x_i's are called *parts* of the composition. A composition with n parts is called a *n-part composition*.

Let $b(k, n, q)$ be the number of n-part compositions of k such that each part is bounded by q. Obviously $b(k, n, q)$ equals $\binom{n,q}{k-n}$, which is the coefficient of x^k in the expansion of $(x + x^2 + \cdots + x^q)^n = x^n(1 + x + \cdots + x^{q-1})^n$. It also equals the number of different ways putting k identical balls into n distinct boxes with each one nonempty and containing at most $q-1$ balls, or equivalently, the number of k-multisets in $\{1, 2, \ldots, n\}$ such that each number appears and is repeated at most $q-1$ times. In this note we will mainly focus on the study of $\binom{n,q}{k}$.

From the multinomial theorem one has

$$\binom{n,q}{k} = \sum_{\substack{i_1+i_2+\cdots+i_q=n,\\ i_2+2i_3+\cdots+(q-1)i_q=k}} \binom{n}{i_1, i_2, \ldots, i_q}.$$

However, it was proved in [15] that when $q > 2$, $\binom{n,q}{k}$ has no closed form, that is, it cannot be written as a sum of finite hypergeometric terms. A natural question is thus to ask if there are any nice asymptotic estimates for $\binom{n,q}{k}$ for suitable parameters n, q, k.

We are also interested in the unimodality of the multinomial coefficients.

Definition 1. A sequence $a_0, a_1, \ldots, a_n$ of real numbers is *unimodal* if for some $0 \le k \le n$ one has

$$a_0 \le a_1 \le \cdots \le a_{k-1} \le a_k \ge a_{k+1} \ge \cdots \ge a_n.$$

For instance, the sequence $\binom{n}{i}, 0 \le i \le n$ is unimodal. Unimodality plays an important role in combinatorics, number theory and representation theory. Many interesting and important examples are surveyed by Stanley [16, 17].

It is well known that for given n, q, $\binom{n,q}{k}$ is unimodal (see, for example, [1]).

Proposition 2. *For given positive integers n, q, $\binom{n,q}{k}$ is unimodal on k and reaches its maximum at $k = \lfloor \frac{qn}{2} \rfloor$.*

Recall that $b(k, n, q)$ is the number of n-part compositions of k such that each part is bounded by q. Let $a(k, n, q)$ be the number of compositions of k with n parts such that the largest part is q. Then $a(k, n, q) = b(k, n, q) - b(k, n, q-1)$ and in particular $a(2n, n, q) = \binom{n,q}{n} - \binom{n,q-1}{n}$.

Let $b(k, q) = \sum_{n=1}^{k} b(k, n, q)$ (respectively, $a(k, q) = \sum_{n=1}^{k} a(k, n, q)$) be the number of compositions of a positive integer k with parts bounded by q (respectively, the largest part is q). It is well known that [12]

$$\sum_{k=0}^{\infty} b(k, q) x^k = \frac{1-x}{1 - 2x + x^{q+1}}.$$

Based on this formula and analytical tools, Odlyzko and Richmond [14] proved the next statement.

Lemma 3 (Odlyzko and Richmond)**.** *Let $a(k,q)$ be defined as above. Then $a(k,q)$ is unimodal for any k and the maximum value occurs for $q = \lfloor \log_2 k \rfloor$ infinitely often and $q = \lfloor \log_2 k \rfloor + 1$ infinitely often and always at one of these two values and no other.*

Based on numerical results, an improved conjectured is proposed.

Conjecture 4. Let $a(k,q)$ be defined as above. Let c be a positive integer. Then for any n, $a((c+1)n, n, q) = \binom{n,q}{cn} - \binom{n,q-1}{cn}$ is unimodular on q and the maximum value occurs for $q = \lfloor \log_{1+\frac{1}{c}} n \rfloor + 1$ or $q = \lfloor \log_{1+\frac{1}{c}} n \rfloor + 1$.

In particular, $a(2n, n, q) = \binom{n,q}{n} - \binom{n,q-1}{n}$ is unimodular on q and the maximum value occurs for $q = \lfloor \log_2 n \rfloor$ or $q = \lfloor \log_2 n \rfloor + 1$.

Our attempt to establish this conjecture starts with an investigation to the asymptotic behaviors of $\binom{n,q}{k}$ when k is linear of n. We will first review some classical results.

For the simplest case $q = 2$, it is well known that the binomial coefficient $\binom{n,2}{cn} = \binom{n}{cn}$ has asymptotic estimate

$$\binom{n,2}{cn} \sim \frac{1}{\sqrt{2\pi(c-c^2)n}} (c^{-c}(c-1)^{-c+1})^n,$$

where $0 < c < 1$ is a constant.

For the case $q = 3$, it is known for $k = n$, the central trinomial coefficient has asymptotic estimate

$$\binom{n,3}{n} \sim \frac{3^{n+1/2}}{2\sqrt{\pi n}}.$$

For large q and general k, based on the integral representation

$$\binom{n,q}{k} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left(\frac{\sin q\theta}{\sin \theta} \right)^n \cos(((q-1)n - 2k)\theta) d\theta,$$

André [3] proved that

$$\sup_k \binom{n,q}{k} \sim \frac{\sqrt{6} q^n}{\sqrt{(q^2-1)\pi n}}, \quad n \to \infty.$$

This estimate has several other proofs; see, for example, a recent one by Eger [6, 7], by representing it as the distribution of sums of independent discrete random variables. An asymptotic distribution in this case was given by Neuschel [13].

Star [18] generalized the result of André. Write $k = \frac{1}{2}(n-s)(q+1)$, where $s = Kn^\theta, 0 \le \theta \le 1/2$ and $K > 0$ is a constant. Star proved that

$$\binom{n,q}{k-n} = \frac{\sqrt{6} q^n}{\sqrt{(q^2-1)\pi n}} \cdot \left(1 + \frac{\sum_{j=0}^{1} h_{1,j}(q) s^{2j}}{n^1} + \cdots + \frac{\sum_{j=0}^{m-1} h_{m-1,j}(q) s^{2j}}{n^{m-1}} + O(\frac{1+s^{2m}}{n^m}) \right),$$

where $h_{i,j}(q)$ are some rational functions in the function field $\mathbb{R}(q)$.

The main result of this note is an asymptotic estimate for $\binom{n,q}{k}$ for large $q > 3$ and for general $k = cn$, where c is fixed positive integer. The proof uses simple analysis based on Hayman's method.

Lemma 5. *Suppose $q > 3$ and $k = cn$, where $c < q$ is an absolute positive integer. Then we have*

$$\binom{n,q}{cn} \sim \frac{\phi(r)}{\sqrt{2\pi n}} \left(\frac{1-r^q}{r-r^2}\right)^n,$$

as $n \to \infty$, where

$$\phi(r) = \left(\frac{r}{(1-r)^2} - \frac{q^2 r^q}{(1-r^q)^2}\right)^{-1/2}, \quad r = \frac{1}{d} + \frac{q}{c^2 d^{q+2}} + \theta \frac{q^3}{d^{2q}},$$

$|\theta| \le 1$ and $d = 1 + \frac{1}{c}$. In particular, when $c = 1$ we have

$$r = \frac{1}{2} + \frac{q}{2^{q+2}} + \theta \frac{q^3}{2^{2q}}.$$

The proof of Theorem 5 is given in Section 2. For simplicity of the computations, we only gives details of the proof for the special case $k = n$, i.e., $c = 1$. The proof of case $c > 1$ is essentially the same as the case $c = 1$.

2 Proof of Theorem 5

Definition 6. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a complex analytic function for $|z| < R$, where $0 < R \le \infty$. Define

$$M(r) = \max_{|z|=r} |f(z)|. \tag{1}$$

If for large enough r, we have $M(r) = f(r)$, then $f(z)$ is called an admissible function. The references [11, 19] present a discussion on admissible functions.

Hayman [11] showed that such good functions have nice asymptotic estimates for their coefficients.

Lemma 7 (Hayman)**.** *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an admissible function, which is analytic in the disk $|z| < R$. Denote*

$$a(r) = r\frac{f'(r)}{f(r)}, \quad b(r) = ra'(r),$$

and suppose $0 < r_n < R$ is a positive real zero satisfying

$$a(r_n) = n, \quad \forall n \in N.$$

Then

$$a_n \sim \frac{f(r_n)}{r_n^n \sqrt{2\pi b(r_n)}}, \quad n \to \infty.$$

Lemma 8. *[2] The function $f(z) = (1 + z + z^2 + \cdots + z^{q-1})^n$ is an admissible function analytical in the disk $|z| < 1$.*

Lemma 9. *For $q \geq 3$ the equation*

$$(q-2)x^{q+1} - (q-1)x^q + 2x - 1 = 0, \quad q \in N$$

has only two positive real roots including 1 being one of them. The second root r satisfies

$$\left| r - \frac{1}{2} - \frac{q}{2^{q+2}} \right| \leq \frac{q^3}{2^{2q}}.$$

Proof. Since the cases $q = 3, 4$ can be verified directly, we may assume $q > 4$. Suppose $f(x) = (q-2)x^{q+1} - (q-1)x^q + 2x - 1$. Then $f''(x) = qx^{q-2}(xq^2 - 2x - xq - q^2 + 2q - 1) = 0$ gives two inflection points $\frac{(q-1)^2}{(q+1)(q-2)}, 0$. This proves that there are only two positive real roots including 1.

Now suppose that $r = \frac{1}{2} + \frac{c}{2^{q+2}}$ is a positive real zero of $f(x)$, where c is regarded as a variable depending on q and will be specified. Then

$$\begin{aligned} f(r) &= (q-2)(\frac{1}{2} + \frac{c}{2^{q+2}})^{q+1} - (q-1)(\frac{1}{2} + \frac{c}{2^{q+2}})^q + 2(\frac{1}{2} + \frac{c}{2^{q+2}}) - 1 \\ &= (\frac{-q}{2^{q+1}} + \frac{(q-2)c}{2^{2q+2}})(1 + \frac{c}{2^{q+1}})^q + \frac{c}{2^{q+1}} \\ &= 0. \end{aligned}$$

Assume without of generality that $0 \leq c \leq q^{3/2}$. By Taylor's theorem one has

$$\left| (1 + \frac{c}{2^{q+1}})^q - 1 - \frac{cq}{2^{q+1}} \right| \leq \frac{c^2 q^2}{2^{2q}} \leq \frac{q^5}{2^{2q}}.$$

And hence

$$\left| f(r) + \frac{q-c}{2^{q+1}} + \frac{c(q^2 - q + 2)}{2^{2q+2}} \right| \leq \frac{q^6}{2^{3q+1}}.$$

Since $f(r) = 0$, we then have

$$q - c + \frac{c(q^2 - q + 2)}{2^{q+1}} + \frac{\theta q^6}{2^{2q}} = 0,$$

where $0 \leq |\theta| \leq 1$.

Let $c = q + c'$. Putting it into the above equality, one has

$$c'(1 - \frac{q^2 - q + 2}{2^{q+1}}) = \frac{q^3 - q^2 + 2q}{2^{q+1}} + \frac{\theta q^6}{2^{2q}}.$$

This implies

$$\left| c - q - \frac{q^3 - q^2 + 2q}{2^{q+1} - q^2 + q - 2} \right| \leq \frac{2q^6}{2^{2q}}.$$

Replace in $r = \frac{1}{2} + \frac{c}{2^{q+2}}$ to obtain, for $q > 16$,

$$\frac{q^3 - q^2}{2^{2q+2}} \le r - \frac{1}{2} - \frac{q}{2^{q+2}} \le \frac{q^3}{2^{2q+2}}.$$

The remaining cases of q can be easily verified. $\square$

Lemma 10. *Assume $q > 3$. Then we have the asymptotic estimate*

$$\binom{n,q}{n} \sim \frac{\phi(r)}{\sqrt{2\pi n}} \left(\frac{1-r^q}{r-r^2}\right)^n, \quad n \to \infty$$

where $\phi(r) = \left(\frac{r}{(1-r)^2} - \frac{q^2 r^q}{(1-r^q)^2}\right)^{-1/2}$, *and* $\left|r - \frac{1}{2} - \frac{q}{2^{q+2}}\right| \le \frac{q^3}{2^{2q}}$.

Proof. Let $f(z) = (1 + z + z^2 + \cdots + z^{q-1})^n = (\frac{1-z^q}{1-z})^n$. By Lemma 8, $f(z)$ is an admissible analytical function on $\mathbb{C} - \{\infty\}$. Applying Hayman's theorem (Lemma 7), we have

$$\begin{aligned} a(x) &= x\frac{f'(x)}{f(x)} = \frac{-nx(qx^{q-1} - qx^q - 1 + x^q)}{(1-x^q)(1-x)}, \\ b(x) &= xa'(x) = \frac{nx(1 - x^{q-1}q^2 - 2x^q + x^{2q} + 2q^2x^q - x^{q+1}q^2)}{(-1+x^q)^2(x-1)^2} \\ &= nx\left(\frac{1}{(1-x)^2} - \frac{q^2x^{q-1}}{(x^q-1)^2}\right). \end{aligned}$$

The equation $a(x_n) = n$ yields

$$\frac{-nx_n(qx_n^{q-1} - qx_n^q - 1 + x_n^q)}{(1-x_n^q)(1-x_n)} = n,$$

and thus

$$(q-2)x_n^{q+1} - (q-1)x_n^q + 2x_n - 1 = 0.$$

The result now follows from Lemma 9. $\square$

Corollary 11. *When $q > 3$, for large n we have the estimate*

$$\binom{n,q}{n} \sim \frac{(1 + \frac{q^2-6q}{2^q} + \theta_1\frac{q^2}{2^{2q}})2^n}{\sqrt{\pi n}}(1 - \frac{1}{2^{q-2}} + \theta_2\frac{q^2}{2^{2q}})^n, \ n \to \infty,$$

where $|\theta_i| \le 1$ for $i = 1, 2$.

Proof. Since

$$\left|r - \frac{1}{2} - \frac{q}{2^{q+2}}\right| \le \frac{q^3}{2^{2q}},$$

it follows that

$$\left|\frac{r}{(1-r)^2}-\frac{1}{2}-\frac{3q}{2^{q-1}}\right| \le \frac{q}{2^{2q}},$$

and

$$\left|\frac{q^2r^q}{(1-r^q)^2}-\frac{q^2}{2^q}\right| \le \frac{q^4}{2^{2q}}.$$

Thus

$$\left|\phi(r)-\sqrt{2}(1+\frac{q^2-6q}{2^q})\right| \le \frac{q^2}{2^{2q}}.$$

Similarly

$$\left|\frac{1-r^q}{r-r^2}-2+\frac{1}{2^{q-1}}\right| \le \frac{q^2}{2^{2q}},$$

and the result follows from Theorem 10. □

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