Journal of Integer Sequences, Vol. 22 (2019),
Article 19.2.4

On Free Numerical Semigroups and the Construction of Minimal Telescopic Sequences

Caleb M. Shor
Department of Mathematics
Western New England University
Springfield, MA 01119
USA
cshor@wne.edu

Abstract

A free numerical semigroup is a submonoid of the non-negative integers with finite complement that is additively generated by the terms in a telescopic sequence with gcd 1. However, such a sequence need not be minimal, which is to say that some proper subsequence may generate the same numerical semigroup, and that subsequence need not be telescopic. In this paper, we will see that for a telescopic sequence with any gcd, there is a minimal telescopic sequence that generates the same submonoid. In particular, given a free numerical semigroup we can construct a telescopic generating sequence that is minimal. In the process, we will examine some operations on and constructions of telescopic sequences in general.

1 Introduction

Let $\mathbb{N}$ denote the set of positive integers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a set $A \subseteq \mathbb{N}_0$, let $\langle A \rangle$ denote the set of all (finite) $\mathbb{N}_0$-linear combinations of elements of A. Let S be a submonoid of $\mathbb{N}_0$. All submonoids of $\mathbb{N}_0$ are finitely generated, so $S = \langle A \rangle$ for some finite $A = \{a_1, \ldots, a_k\} \subset \mathbb{N}_0$. We also write this as $S = \langle a_1, \ldots, a_k \rangle$. We say a set A is *minimal* if $\langle A' \rangle \neq \langle A \rangle$ for all $A' \subsetneq A$. It is well known that any set has a unique minimal subset that generates the same submonoid, and hence there exists a bijection between submonoids of $\mathbb{N}_0$ and minimal subsets of $\mathbb{N}_0$. (See Rosales and García-Sánchez [19, Cor. 2.8] for details.)

In this paper, we are interested in generating sets that are ordered in some way. We therefore consider (finite) sequences $G = (g_1, \ldots, g_k)$ for some $k \in \mathbb{N}$ with $g_i \in \mathbb{N}_0$, which we denote by $G \in \mathbb{N}_0^k$.[1] We similarly define the submonoid generated by G as $\langle G \rangle = \langle g_1, \ldots, g_k \rangle$. As with sets, a sequence G is *minimal* if $\langle G' \rangle \neq \langle G \rangle$ for all proper subsequences G' of G. Any sequence then has a unique minimal subsequence that generates the same submonoid. Observe that any permutation of a minimal sequence is also necessarily minimal.

For $G \in \mathbb{N}_0^k$ a sequence with $g_1 + g_2 > 0$, let $G_i = (g_1, \ldots, g_i)$ and $d_i = \gcd(G_i)$ for $1 \leq i \leq k$. Let $c(G) = (c_2, \ldots, c_k)$ where $c_j = d_{j-1}/d_j$ for $2 \leq j \leq k$.[2] If $c_j g_j \in \langle G_{j-1} \rangle$ for all $j \geq 2$, then we say G is a *telescopic* (or *smooth*) sequence. Some authors include the additional properties where $\gcd(G) = 1$ and/or G is an increasing sequence. In this paper, we require neither.

The reader can confirm that all sequences $G \in \mathbb{N}_0^k$ with $g_1 + g_2 > 0$ for $k = 1$ and $k = 2$ are telescopic. There are longer sequences that are not telescopic, though some permutation is. One example is $G = (4, 5, 6)$, which is not telescopic, though $G' = (4, 6, 5)$ is telescopic. And finally, there are sequences for which no permutation is telescopic, such as $G = (3, 4, 5)$.

Telescopic sequences arise naturally in the world of numerical semigroups. They generate so-called *free numerical semigroups*, which have some nice properties that we will see along with references for further reading in Section 2. Unfortunately, the usage of the word "free" here does not coincide with the categorical idea of free objects.

The motivation for this paper is how the definitions of "telescopic" and "minimal" interact. Consider the following example.

Example 1. Let $S = \langle G \rangle$ for $G = (660, 550, 352, 50, 201) \in \mathbb{N}_0^5$. We see that $c(G) = (6, 5, 11, 2)$ and that $c_j g_j \in \langle G_{j-1} \rangle$ for $2 \leq j \leq 5$, so G is a telescopic sequence. Since $\gcd(G) = 1$, S is a free numerical semigroup. However, G is not minimal because $550 = 0 \cdot 660 + 0 \cdot 352 + 11 \cdot 50 + 0 \cdot 201$. We eliminate 550 from G to obtain the proper subsequence $G' = (660, 352, 50, 201)$ with $\langle G' \rangle = \langle G \rangle = S$. We can see that G' is minimal. However, G' is not telescopic.

This leads to the following question.

Question 2. Given a telescopic sequence G, does there exist a telescopic sequence G' that is minimal and has $\langle G' \rangle = \langle G \rangle$?

We can ask this more generally.

Question 3. Suppose G and H are sequences such that $\langle G \rangle = \langle H \rangle$. If G is telescopic, must some permutation of H be telescopic?

[1] By taking $G \in \mathbb{N}_0^k$, we allow for the possibility that $g_i = g_j$ for $i \neq j$. We can also have $g_i = 0$ for some i. Clearly, if we remove any repeats or 0s, the resulting subsequence will still generate the same submonoid. However, our results hold in the more general case of allowing repeats and/or zeros, so we choose to allow them to give slightly more general results.

[2] Observe that c_j is a well-defined integer when $d_j \neq 0$. We have $d_j \neq 0$ for all j precisely when $g_1 + g_2 > 0$.

As Example 1 illustrates, for G telescopic and G' its unique minimal subsequence, G' need not be telescopic. In this particular case, the permutation $(660, 50, 352, 201)$ of G' is telescopic. We will show that this always happens – i.e., that the answer to Question 2 is "yes," as is the answer to the more general Question 3. In the context of numerical semigroups, this means that any free numerical semigroup is generated by a telescopic minimal sequence, and we will give a procedure to compute it.

1.1 Organization

This paper is organized as follows. In Section 2, we will give some background material on free numerical semigroups. Following that, given a sequence $(c_2, \ldots, c_k)$ and some $d \in \mathbb{N}$, we provide an explicit method (Remark 19) in Section 3 to produce any telescopic sequence $G = (g_1, \ldots, g_k)$ such that $c(G) = (c_2, \ldots, c_k)$ and $\gcd(G) = d$. In Section 4, we describe two functions that map telescopic sequences to telescopic sequences. As we will see, any function that maps telescopic sequences to telescopic sequences (with the same gcd) must be a composition of these functions (with certain parameters).

Then, in Section 5 we present a construction that takes a telescopic sequence as input and outputs a minimal telescopic sequence that generates the same submonoid, answering Question 2 affirmatively. The main result is Theorem 52. As a corollary, we find that every free numerical semigroup is generated by a minimal telescopic sequence. We also answer the more general Question 3 affirmatively.

Finally, in Section 6 we combine two results (Remark 19 and Proposition 43) to give an explicit method to construct any minimal telescopic sequence G with $c(G) = (c_2, \ldots, c_k)$. As an application, we have Corollary 58, which says that any non-decreasing telescopic sequence $G = (g_1, \ldots, g_k)$ with $c_j > 1$ for all j is necessarily minimal.

2 Background

Free numerical semigroups, which are generated by telescopic sequences G with $\gcd(G) = 1$, have been studied in various contexts by Brauer and Shockley [5], Herzog [10], Bertin and Carbonne [3], Rödseth [17], Kirfel and Pellikaan [12], Rosales and García-Sánchez [18, 19], Leher [13], Ayano [2], Robles-Pérez and Rosales [16], Gassert and Shor [9], and others. We will highlight some of their properties.

2.1 Numerical semigroups

We begin with numerical semigroups. For a comprehensive reference on the subject, see the work of Rosales and García-Sánchez [19].

A *numerical semigroup* S is a submonoid of $\mathbb{N}_0$ with finite complement. It is well-known that every numerical semigroup S is given by $S = \langle A \rangle$ for some finite $A \subset \mathbb{N}_0$ with $\gcd(A) = 1$. Since we are interested in generating sequences, we equivalently have that every numerical semigroup S is given by $S = \langle G \rangle$ for some $k \in \mathbb{N}$ and $G \in \mathbb{N}_0^k$ with $\gcd(G) = 1$. Elements of the complement of S are known as *gaps* of S, and we denote the set of gaps by $H(S)$. The *genus* of S, denoted $g(S)$, is the number of gaps of S. The *Frobenius element* of S, denoted $F(S)$, is the largest integer not in S, which is $\max(H(S))$ when $S \neq \mathbb{N}_0$. The *embedding dimension* of S, denoted $e(S)$, is the cardinality of the unique minimal generating set of S (which is the cardinality of any minimal generating sequence of S).

Given a numerical semigroup S, it can be difficult to compute $g(S)$, $F(S)$, and other properties of the set of gaps. Curtis [6] showed that there cannot be a polynomial formula to compute the Frobenius number of S as a function of the generating elements of S when $e(S) > 2$. Ramírez Alfonsín [14] proved that the problem of computing the Frobenius number of S, when $e(S) > 2$, is NP-hard. For more on the Frobenius problem, see Ramírez Alfonsín's book [15].

A very helpful tool in understanding the gaps of a numerical semigroup is called the Apéry set [1]. For any nonzero $t \in S$, the *Apéry set of S relative to t* is

$$\mathrm{Ap}(S;t) = \{s \in S : s - t \notin S\}. \tag{1}$$

Equivalently, $\mathrm{Ap}(S;t)$ is the set of elements in S that are minimal in their congruence class modulo t.

If we know $\mathrm{Ap}(S;t)$, then we immediately know the genus of S (from Selmer [20]) and the Frobenius number of S (from Brauer and Shockley [5]).

Theorem 4 ([5, Lem. 3], [20, Eqn. 2.3])**.** *For S a numerical semigroup and any nonzero $t \in S$,*

$$F(S) = \max(\mathrm{Ap}(S;t)) - t, \tag{2}$$

and

$$g(S) = \frac{1-t}{2} + \frac{1}{t} \sum_{n \in \mathrm{Ap}(S;t)} n. \tag{3}$$

We can also deduce other properties of the gaps of S with the following identity and an appropriately chosen function f.

Theorem 5 ([9, Thm. 2.3])**.** *For S a numerical semigroup with set of gaps $H(S)$, any nonzero $t \in S$, and any function f defined on $\mathbb{N}_0$,*

$$\sum_{n \in H(S)} [f(n+t) - f(n)] = \sum_{n \in \mathrm{Ap}(S;t)} f(n) - \sum_{n=0}^{t-1} f(n). \tag{4}$$

For instance, with $f(n) = n$ in Equation (4), we obtain the genus formula in Equation (3). Tuenter [21] presents additional applications in the case where $S = \langle a, b \rangle$.

2.2 Free numerical semigroups

A numerical semigroup S is a *free numerical semigroup* if $S = \langle G \rangle$ for a telescopic sequence G. The following result from Bertin and Carbonne [3, Sect. 2] motivates our interest in free numerical semigroups. We present it for any value of $\gcd(G)$, whereas it usually appears in the case of $\gcd(G) = 1$.

Theorem 6. *Let $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ be telescopic, let $c(G) = (c_2, \ldots, c_k)$, and let $d = \gcd(G)$. For any $n \in d\mathbb{Z}$, there is a unique representation*

$$n = \sum_{i=1}^{k} n_i g_i, \tag{5}$$

with integers $n_1, \ldots, n_k$ where $0 \leq n_j < c_j$ for $j = 2, \ldots, k$. Furthermore, for such an $n \in d\mathbb{Z}$, we have $n \in \langle G \rangle$ if and only if $n_1 \geq 0$.

This theorem is a generalization of the result that, given relatively prime positive integers a and b, any integer n can be uniquely written as $n = n_1 a + n_2 b$ with $n_1, n_2 \in \mathbb{Z}$ and $0 \leq n_2 < a$.

We then obtain an explicit description of the Apéry set of a free numerical semigroup relative to the first generating element.

Corollary 7 ([9, Prop. 3.6])**.** *Suppose S is a free numerical semigroup, so $S = \langle G \rangle$ for $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ a telescopic sequence with $\gcd(G) = 1$. For $c(G) = (c_2, \ldots, c_k)$, we have*

$$\mathrm{Ap}(S; g_1) = \left\{ \sum_{j=2}^{k} n_j g_j : 0 \leq n_j < c_j \right\}. \tag{6}$$

Using Corollary 7 with Theorem 4 we obtain the formulas for the Frobenius number and genus of a free numerical semigroup. The formula for $F(S)$ in this case was known to Brauer [4].

Corollary 8. *For S a free numerical semigroup generated by $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$, a telescopic sequence with $\gcd(G) = 1$ and $c(G) = (c_2, \ldots, c_k)$,*

$$F(S) = -g_1 + \sum_{j=2}^{k} (c_j - 1) g_j, \tag{7}$$

and

$$g(S) = \frac{1}{2} \left(1 - g_1 + \sum_{j=2}^{k} (c_j - 1) g_j \right). \tag{8}$$

Observe that for S a free numerical semigroup we have $F(S) = 2g(S) - 1$. Numerical semigroups with this property are *symmetric*. These semigroups are equivalently characterized as those for which the map $\phi : S \to \mathbb{Z} \setminus S$ given by $\phi(s) = F(S) - s$ is a bijection.

We can combine Corollary 7 with Theorem 5 to obtain an explicit identity for the gaps of a free numerical semigroup.

Corollary 9 ([9, Cor. 3.7])**.** *Suppose $S = \langle G \rangle$ for $G = (g_1, \ldots, g_k)$ telescopic and $c(G) = (c_2, \ldots, c_k)$. For $H(S)$ the set of gaps of S, and for any function f defined on $\mathbb{N}_0$,*

$$\sum_{n \in H(S)} [f(n + g_1) - f(n)] = \sum_{n_2=0}^{c_2-1} \cdots \sum_{n_k=0}^{c_k-1} f\left(\sum_{j=2}^{k} n_j g_j\right) - \sum_{n=0}^{g_1-1} f(n). \tag{9}$$

As before, with the function $f(n) = n$ in Equation (9) we recover the genus formula given in Equation (8).

2.3 Prior results

The motivating question for this paper is whether a free numerical semigroup has a minimal telescopic generating sequence. We can answer the question immediately for embedding dimension at most 3.

If $e(S) = 1$, then $S = \mathbb{N}_0 = \langle G \rangle$ for $G = (1)$, which is a minimal telescopic sequence.

If $e(S) = 2$, then $S = \langle G \rangle$ for $G = (a, b)$ with $a, b > 1$ and $\gcd(a, b) = 1$. The sequence $G = (a, b)$ is telescopic, so S is generated by a minimal telescopic sequence.

If $e(S) = 3$, then we need a result of Herzog [10]. (We quote the result as written by Rosales and García-Sánchez [19].)

Theorem 10 ([19, Cor. 9.5])**.** *For S a numerical semigroup with $e(S) = 3$, S is symmetric if and only if $S = \langle G \rangle$, for $G = (am_1, am_2, bm_1 + cm_2)$ where $a, b, c, m_1, m_2 \in \mathbb{N}$ with $\gcd(m_1, m_2) = 1$, $a, m_1, m_2 \geq 2$, $b + c \geq 2$, and $\gcd(a, bm_1 + cm_2) = 1$.*

For our purposes, if S is a free numerical semigroup then S is symmetric. If we also have $e(S) = 3$, then by the above theorem we have $S = \langle G \rangle$ for $G = (am_1, am_2, bm_1 + cm_2)$, which is necessarily telescopic (and of course minimal).

This approach will not work when $e(S) \geq 4$, however, because there are symmetric numerical semigroups that are not free. For instance, the numerical semigroup $S = \langle e + 1, \ldots, e + e \rangle$ of embedding dimension e is symmetric and not free for all $e \geq 4$.

In what follows, we will take a different approach to show that every free numerical semigroup S is generated by a minimal telescopic sequence.

3 Explicit form of telescopic sequences

In this section, our goal is to obtain an explicit form for the terms in a telescopic sequence, which we will need for our main result in Section 5. We will see a method to construct a telescopic sequence G with $\gcd(G) = d$ given a desired sequence $c(G)$ and $d \in \mathbb{N}$. Our method uses the same ideas as in "gluing" of numerical semigroups, described by Watanabe [22] and by Rosales and García-Sánchez [19, Chap. 8].

3.1 Notation and preliminaries

For notation, given $c_{m+1}, \ldots, c_n \in \mathbb{N}_0$, let

$$C_{m,n} = \prod_{j=m+1}^{n} c_j. \tag{10}$$

If $n \leq m$, then this product is empty, so $C_{m,n} = 1$. In particular, we will make use of the fact that $C_{m,m} = 1$. Additionally, as was mentioned in the introduction, for $i = 1, \ldots, k$, let $G_i = (g_1, \ldots, g_i)$. Finally, recall that we will only consider sequences G with $g_1 + g_2 > 0$.

We begin with a few lemmas.

Lemma 11. *Let $G \in \mathbb{N}_0^k$ be any sequence with $c(G) = (c_2, \ldots, c_k)$. Then $\gcd(G_i) = C_{i,k} \gcd(G)$ for all $i = 1, \ldots, k$. In particular, if $\gcd(G) = 1$, then $\gcd(G_i) = C_{i,k}$.*

Proof. Observe that

$$C_{i,k} = c_{i+1}c_{i+2} \cdots c_k = \frac{\gcd(G_i)}{\gcd(G_{i+1})} \frac{\gcd(G_{i+1})}{\gcd(G_{i+2})} \cdots \frac{\gcd(G_{k-1})}{\gcd(G_k)}.$$

With cancellation and the fact that $G_k = G$, we get $C_{i,k} = \gcd(G_i)/\gcd(G)$, as desired. □

For any sequence $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ and any $m \in \mathbb{N}_0$, let $mG = (mg_1, \ldots, mg_k) \in \mathbb{N}_0^k$. If $m \mid g_1, \ldots, g_k$, let $G/m = (g_1/m, \ldots, g_k/m) \in \mathbb{N}_0^k$.

Lemma 12. *For any $G \in \mathbb{N}_0^k$ and any $m \in \mathbb{N}_0$,*

1. *$(mG)_i = m(G_i)$; and*

2. *$\gcd((mG)_i) = m \gcd(G_i)$.*

Proof. Let $H = mG = (mg_1, \ldots, mg_k)$. Then $H_i = (mg_1, \ldots, mg_i) = m(g_1, \ldots, g_i) = m(G_i)$, so $\gcd(H_i) = \gcd(m(G_i)) = m \gcd(G_i)$. □

Lemma 13. *For any $G \in \mathbb{N}_0^k$ and any $m \in \mathbb{N}$,*

1. *$c(mG) = c(G)$; and*

2. *G is telescopic if and only if mG is telescopic.*

Proof. Let $c(G) = (c_2, \dots, c_k)$, $H = mG$, and $c(H) = (e_2, \dots, e_k)$. For $i = 1, \dots, k$, we have $H_i = m(G_i)$. Thus, for $j = 2, \dots, k$,

$$\begin{aligned} e_j &= \gcd(H_{j-1})/\gcd(H_j) \\ &= (m\gcd(G_{j-1}))/(m\gcd(G_j)) \\ &= \gcd(G_{j-1})/\gcd(G_j) \\ &= c_j, \end{aligned}$$

so $c(H) = c(G)$.

Finally, for any $j = 2, \dots, k$, suppose $c_j g_j \in \langle G_{j-1} \rangle$. For $m \in \mathbb{N}$, this occurs exactly when $c_j m g_j \in \langle mG_{j-1} \rangle$. Since $e_j = c_j$ and $h_j = mg_j$, we conclude that $c_j g_j \in \langle G_{j-1} \rangle$ if and only if $e_j h_j \in \langle H_{j-1} \rangle$. This holds for all $j = 2, \dots, k$, so G is telescopic if and only if H is telescopic. $\square$

Let S_k denote the symmetric group on k letters. Elements of S_k act on sequences $G \in \mathbb{N}_0^k$ in a natural way: for $\sigma \in S_k$ and $G = (g_1, \dots, g_k)$, let $\sigma(G) = (g_{\sigma(1)}, \dots, g_{\sigma(k)}) \in \mathbb{N}_0^k$. Since $\sigma(G)$ is a permutation of G, we have $\langle \sigma(G) \rangle = \langle G \rangle$. In the following proposition, we show that the permutation $(1\ 2) \in S_k$ takes telescopic sequences to telescopic sequences.

Proposition 14. *For $k \geq 2$, $G = (g_1, \dots, g_k) \in \mathbb{N}_0^k$, and $\sigma = (1\ 2) \in S_k$, G is telescopic if and only if $\sigma(G)$ is telescopic.*

Proof. Suppose G is telescopic and let $H = \sigma(G)$. Then $H = (h_1, \dots, h_k) = (g_2, g_1, g_3, \dots, g_k)$. Let $c(H) = (e_2, \dots, e_k)$.

We have $e_2 = \gcd(H_1)/\gcd(H_2) = g_2/\gcd(g_2, g_1)$, so

$$e_2 h_2 = \frac{g_2}{\gcd(g_2, g_1)} g_1 = \frac{g_1}{\gcd(g_1, g_2)} g_2.$$

Since $g_1/\gcd(g_1, g_2) \in \mathbb{N}_0$, $e_2 h_2 \in \langle g_2 \rangle = \langle H_1 \rangle$.

For $i > 1$, H_i is a permutation of G_i, so $\langle H_i \rangle = \langle G_i \rangle$ and $\gcd(H_i) = \gcd(G_i)$. For $j > 2$, $e_j = \gcd(H_{j-1})/\gcd(H_j) = \gcd(G_{j-1})/\gcd(G_j) = c_j$, and therefore $e_j h_j = c_j g_j$. Since G is telescopic, $c_j g_j \in \langle G_{j-1} \rangle$, so $e_j h_j \in \langle G_{j-1} \rangle = \langle H_{j-1} \rangle$. Thus $e_j h_j \in \langle H_{j-1} \rangle$ for $j = 2, \dots, k$, so H is a telescopic sequence.

For the reverse implication, since σ has order two, if $\sigma(G)$ is telescopic, then $\sigma(\sigma(G)) = G$ is telescopic as well. $\square$

Remark 15. Since we already have the restriction that $g_1 + g_2 > 0$, and since G is telescopic exactly when $(1\ 2)(G)$ is telescopic, for the rest of this paper we will assume $g_1 > 0$. Therefore $\gcd(g_1, \dots, g_i) > 0$ for all $i \geq 1$, and since $c_j = \gcd(g_1, \dots, g_{j-1})/\gcd(g_1, \dots, g_j)$, this will ensure that $c_j > 0$ for all $j \geq 2$ as well. We can therefore divide by g_1, $\gcd(G_i)$ for all i, and c_j for all j without worrying about possibly dividing by zero.

Finally, we show that the sequence consisting of the first m terms of a telescopic sequence forms a telescopic sequence on its own.

Lemma 16. *For any $k \in \mathbb{N}$, let $G \in \mathbb{N}_0^k$ be a telescopic sequence with $g_1 > 0$ and $c(G) = (c_2, \ldots, c_k)$. For any $m \in \{1, \ldots, k\}$, the sequence $G_m = (g_1, \ldots, g_m)$ is telescopic with $c(G_m) = (c_2, \ldots, c_m)$.*

Proof. Let $m \in \{1, \ldots, k\}$, let $H = G_m$, and let $c(H) = (e_2, \ldots, e_m)$. Then $H = (h_1, \ldots, h_m)$ with $h_i = g_i$ for $i = 1, \ldots, m$. Observe that $h_1 > 0$. We have $e_j = \gcd(H_{j-1})/\gcd(H_j) = \gcd(G_{j-1})/\gcd(G_j) = c_j$ for $2 \le j \le m$. Since $c_j g_j \in \langle G_{j-1} \rangle$ for $j = 2, \ldots, k$, and since $m \le k$, it follows that $e_j h_j \in \langle H_{j-1} \rangle$ for $j = 2, \ldots, m$. Therefore H is telescopic with $c(H) = (c_2, \ldots, c_m)$. □

3.2 Construction of telescopic sequences

We can now give an explicit description for the elements of a telescopic sequence. We do so first in the case where $\gcd(G) = 1$ and then for any value of $\gcd(G)$.

Proposition 17. *Suppose $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ with $\gcd(G) = 1$ and $c(G) = (c_2, \ldots, c_k)$. Let $z_1 = 1$. Then G is a telescopic sequence if and only if, for each $i = 2, \ldots, k$, there exists $z_i \in \mathbb{N}_0$ such that $g_i = z_i C_{i,k}$, $\gcd(z_i, c_i) = 1$, and*

$$z_i \in \langle z_j C_{j,i-1} : 1 \le j < i \rangle .$$

Proof. ($\Longrightarrow$) Suppose G is telescopic. For $i = 2, \ldots, k$, we will use strong induction.

We first show the statement is true for the base case $i = 2$. Since G is telescopic we have $c_2 g_2 \in \langle g_1 \rangle$. Let $z_2 = c_2 g_2 / g_1 \in \langle 1 \rangle = \langle z_1 C_{1,1} \rangle$ since $z_1 = C_{1,1} = 1$. Then, since $g_1 = C_{1,k}$ (by Lemma 11), we get $g_2 = z_2 g_1 / c_2 = z_2 C_{2,k}$. By definition $c_2 = g_1 / \gcd(g_1, g_2)$, so $C_{2,k} = g_1 / c_2 = \gcd(g_1, g_2) = \gcd(C_{1,k}, z_2 C_{2,k}) = \gcd(c_2 C_{2,k}, z_2 C_{2,k}) = C_{2,k} \gcd(c_2, z_2)$. Hence $\gcd(c_2, z_2) = 1$. The conditions are therefore satisfied for $i = 2$.

For strong induction, for $i = 2, \ldots, n$, with $n < k$, we assume $g_i = z_i C_{i,k}$ with $\gcd(z_i, c_i) = 1$ and $z_i \in \langle z_j C_{j,i-1} : 1 \le j < i \rangle$. By Lemma 11, $g_1 = C_{1,k}$ so $C_{n,k} \mid g_1$. Then, since $C_{n,k} \mid C_{j,k}$ for $j = 2, \ldots, n$, by induction $C_{n,k} \mid g_2, \ldots, g_n$. Since G is telescopic, we have $c_{n+1} g_{n+1} \in \langle g_1, \ldots, g_n \rangle$. We can divide through by $C_{n,k}$ (using the fact that $g_j / C_{n,k} = z_j C_{j,n}$) to get

$$\frac{c_{n+1} g_{n+1}}{C_{n,k}} = \frac{g_{n+1}}{C_{n+1,k}} \in \langle z_1 C_{1,n}, \ldots, z_n C_{n,n} \rangle \subseteq \mathbb{N}_0.$$

Let $z_{n+1} = g_{n+1} / C_{n+1,k}$. Then $g_{n+1} = z_{n+1} C_{n+1,k}$, and $z_{n+1} \in \langle z_j C_{j,n} : 1 \le j < n+1 \rangle$.

Next, we verify the gcd condition. By Lemma 11, $\gcd(G_n) = C_{n,k}$, so

$$c_{n+1} = \frac{\gcd(G_n)}{\gcd(G_{n+1})} = \frac{C_{n,k}}{\gcd(C_{n,k}, g_{n+1})}.$$

Therefore

$$\begin{aligned} C_{n+1,k} &= C_{n,k} / c_{n+1} \\ &= \gcd(C_{n,k}, g_{n+1}) \\ &= \gcd(C_{n,k}, z_{n+1} C_{n+1,k}) \\ &= C_{n+1,k} \gcd(c_{n+1}, z_{n+1}), \end{aligned}$$

so $\gcd(c_{n+1}, z_{n+1}) = 1$.

Since we have verified all of the conditions for $i = n+1$, by induction the statement is true for all $i = 2, \ldots, k$.

($\Longleftarrow$) Suppose $G = (g_1, g_2, \ldots, g_k) = (C_{1,k}, z_2C_{2,k}, \ldots, z_kC_{k,k})$ with $\gcd(z_i, c_i) = 1$ and $z_i \in \langle z_jC_{j,i-1} : 1 \leq j < i\rangle$ for $i = 2, \ldots, k$. For G to be telescopic, we need $c_ig_i \in \langle G_{i-1}\rangle$ for $i = 2, \ldots, k$. Since $c_ig_i = c_iz_iC_{i,k} = z_iC_{i-1,k}$ and $z_i \in \langle z_jC_{j,i-1} : 1 \leq j < i\rangle$, we have

$$\begin{aligned} c_ig_i = z_iC_{i-1,k} &\in \langle z_jC_{j,i-1}C_{i-1,k} : 1 \leq j < i\rangle \\ &= \langle z_jC_{j,k} : 1 \leq j < i\rangle \\ &= \langle g_1, \ldots, g_{i-1}\rangle \\ &= \langle G_{i-1}\rangle \end{aligned}$$

for $i = 2, \ldots, k$. Thus, G is telescopic. □

We now give the result for any $\gcd(G)$.

Corollary 18. *Suppose $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ and $c(G) = (c_2, \ldots, c_k)$. Let $z_1 = \gcd(G)$. Then G is a telescopic sequence if and only if, for each $i = 2, \ldots, k$, there exists $z_i \in \langle z_1\rangle$ such that $g_i = z_iC_{i,k}$, $\gcd(z_i/z_1, c_i) = 1$, and*

$$z_i \in \langle z_jC_{j,i-1} : 1 \leq j < i\rangle.$$

Proof. Let $d = \gcd(G)$ and $H = G/d$, so $\gcd(H) = 1$. By Lemma 13, $c(H) = c(G) = (c_2, \ldots, c_k)$. Let $y_1 = 1$. For $H = (h_1, \ldots, h_k)$, by Proposition 17, H is telescopic if and only if, for each $i = 2, \ldots, k$, there exists $y_i \in \mathbb{N}_0$ such that $h_i = y_iC_{i,k}$, $\gcd(y_i, c_i) = 1$, and $y_i \in \langle y_jC_{j,i-1} : 1 \leq j < i\rangle$. Since $G = dH$, $g_i = dh_i$. By Lemma 13, G is telescopic if and only if H is telescopic. Finally, let $z_i = dy_i$.

We rewrite the quoted result of Proposition 17 to now say that with $z_1 = dy_1 = d$, G is telescopic if and only if, for each $i = 2, \ldots, k$, there exists $z_i \in d\mathbb{N}_0$ such that $g_i = z_iC_{i,k}$, $\gcd(z_i/d, c_i) = 1$, and $z_i \in \langle z_jC_{j,i-1} : 1 \leq j < i\rangle$. Since $z_1 = d$, the corollary statement follows. □

Remark 19. For $k \in \mathbb{N}$, given any sequence $(c_2, \ldots, c_k) \in \mathbb{N}^{k-1}$ and any $d \in \mathbb{N}$, we have a process to construct any telescopic sequence $G \in \mathbb{N}_0^k$ for which $\gcd(G) = d$ and $c(G) = (c_2, \ldots, c_k)$. We just need non-negative integers $z_1, \ldots, z_k$ where $z_1 = d$ and, for $i = 2, \ldots, k$, $\gcd(z_i, dc_i) = d$ and $z_i \in \langle z_jC_{j,i-1} : 1 \leq j < i\rangle$. Then, for $i = 1, \ldots, k$, we let $g_i = z_iC_{i,k}$ to produce $G = (g_1, \ldots, g_k)$, a sequence for which $\gcd(G) = d$ and $c(G) = (c_2, \ldots, c_k)$.

Example 20. Suppose we want a telescopic sequence $G \in \mathbb{N}_0^5$ with $\gcd(G) = 4$ and $c(G) = (c_2, c_3, c_4, c_5) = (3, 2, 5, 3)$.

To start, we have $z_1 = \gcd(G) = 4$.

For z_2, z_3, z_4, z_5, we have some choices to make. For z_2, we need $\gcd(z_2, 4 \cdot 3) = 4$ and $z_2 \in \langle z_1\rangle$. For z_3, we need $\gcd(z_3, 4\cdot 2) = 4$ and $z_3 \in \langle 3z_1, z_2\rangle$. For z_4, we need $\gcd(z_4, 4\cdot 5) = 4$ and $z_4 \in \langle 6z_1, 2z_2, z_3\rangle$. For z_5, we need $\gcd(z_5, 4 \cdot 3) = 4$ and $z_5 \in \langle 30z_1, 10z_2, 5z_3, z_4\rangle$.

One option is to take $z_2 = z_3 = z_4 = z_5 = 4$. With $g_i = z_i C_{i,k}$, we get $G = (g_1, g_2, g_3, g_4, g_5) = (360, 120, 60, 12, 4)$.

Another option is to take $z_2 = 8$, $z_3 = 20$, $z_4 = 28$, $z_5 = 44$. With $g_i = z_i C_{i,k}$, we get $G = (360, 240, 300, 84, 44)$.

Both sequences are telescopic with $\gcd(G) = 4$ and $c(G) = (3, 2, 5, 3)$.

Observe that the first sequence in Example 20 is not minimal, whereas the second sequence is. In Section 6, we present Corollary 56, a refinement of Remark 19, which allows us to construct any minimal telescopic sequence. We postpone its appearance because it relies on a result (Proposition 43) from Section 5.

3.3 Specific values for certain telescopic sequences

We describe the sequences $(c_2, \ldots, c_k)$ and $(z_1, \ldots, z_k)$ that occur for some families of telescopic sequences: geometric, supersymmetric, and compound sequences. Gassert and Shor [8] detail some applications of Corollary 9 to these sequences, as well as connections to certain algebraic curves.

A (finite) geometric sequence G of length k with $\gcd(G) = 1$ is a sequence of the form $(g_1, \ldots, g_k)$ where $g_i = a^{k-i} b^{i-1}$ for $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. In this case, $\gcd(g_1, \ldots, g_i) = a^{k-i}$, so $c_i = a$ for $i = 2, \ldots, k$. In the notation of Proposition 17, we have $z_i = b^{i-1}$ for $i = 1, \ldots, k$.

As named and studied by Fröberg, Gottlieb, and Häggkvist [7], a *supersymmetric sequence* G of length k with $\gcd(G) = 1$ is a sequence of the form $(g_1, \ldots, g_k)$ where $g_i = A/a_i$ for pairwise coprime natural numbers $a_1, \ldots, a_k$ and $A = a_1 \cdots a_k$. In this case, $\gcd(g_1, \ldots, g_i) = A/(a_1 \cdots a_i)$, so $c_i = a_i$. In the notation of Proposition 17, we have $z_1 = 1$ and $z_i = a_1 \cdots a_{i-1}$ for $i = 2, \ldots, k$.

Geometric and supersymmetric sequences are special cases of compound sequences, which were named and studied by Kiers, O'Neill, and Ponomarenko [11]. A *compound sequence* G of length k with $\gcd(G) = 1$ is a sequence of the form $(g_1, \ldots, g_k)$ where $g_i = b_1 \cdots b_{i-1} a_i \cdots a_{k-1}$ for $a_i, b_j \in \mathbb{N}$ with $\gcd(a_i, b_j) = 1$ for all $i \geq j$. In this case, $\gcd(g_1, \ldots, g_i) = a_i \cdots a_{k-1}$, so $c_i = a_{i-1}$. In the notation of Proposition 17, we have $z_1 = 1$ and $z_i = b_1 \cdots b_{i-1}$ for $i = 2, \ldots, k$.

4 Operations on telescopic sequences

In this section, we will introduce two operations: ρ_n, which maps a sequence of length k to a sequence of length $k - 1$; and $\tau_{g,m}$, a gluing map as described by Rosales and García-Sánchez [19, Chap. 8], which maps a sequence of length k to a sequence of length $k + 1$. With appropriate parameters, these operations "preserve telescopicness," which is to say they map telescopic sequences to telescopic sequences. We will also show that any function between two telescopic sequences with the same greatest common divisor can be written as a composition of these functions. In particular, ρ_n will be useful in Section 5 where we will use it to eliminate redundant terms from a telescopic sequence that is not minimal.

As a reminder, in Remark 15 we saw that we can assume $g_1 > 0$ for any sequence $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$. We will continue to make this assumption, from which it follows that $\gcd(G_i) > 0$ for all $i \geq 1$ and $c_j > 0$ for all $j \geq 2$.

4.1 Two useful sequence operations

For $k, l \in \mathbb{N}$, let $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ and $H = (h_1, \ldots, h_l) \in \mathbb{N}_0^l$. For integers i and j with $0 \leq i < j \leq k$, let $G_{i,j} = (g_{i+1}, g_{i+2}, \ldots, g_j) \in \mathbb{N}_0^{j-i}$. Let $G \times H = (g_1, \ldots, g_k, h_1, \ldots, h_l) \in \mathbb{N}_0^{k+l}$.

Using this notation, we have the following results.

Proposition 21. *For any* $G \in \mathbb{N}_0^k$,

1. $\gcd(G \times H) = \gcd(\gcd(G), \gcd(H))$ *for any* $H \in \mathbb{N}_0^l$*; and*
2. $G_{i,j} \times G_{j,l} = G_{i,l}$ *for any integers* i, j, l *with* $0 \leq i < j < l \leq k$.

Proof. The first result is a consequence of the fact that for $a, b, c \in \mathbb{Z}$, $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$.

The second result is equivalent to saying $(g_{i+1}, \ldots, g_j) \times (g_{j+1}, \ldots, g_l) = (g_{i+1}, \ldots, g_l)$, which is true when $i < j < l$. $\square$

We now introduce two operations on sequences that preserve telescopicness. The first, ρ_n, produces a sequence with one fewer entry. The second, $\tau_{g,m}$, produces a sequence that has one more entry. As we will see, given any two telescopic sequences G and H with $\gcd(G) = \gcd(H)$, one can compose finitely many of these functions together to transform G into H.

For the definition of ρ_n below, note that for $G \in \mathbb{N}_0^k$, $c(G) = (c_2, \ldots, c_k)$, and $n = 2, \ldots, k$, we have $c_n \mid C_{n-1,k}$. Then, by Lemma 11, $c_n \mid \gcd(G_{n-1})$, so $G_{n-1}/c_n \in \mathbb{N}_0^{n-1}$.

Definition 22. For any $k \geq 2$, $G \in \mathbb{N}_0^k$, $c(G) = (c_2, \ldots, c_k)$, and $n = 2, \ldots, k$, let

$$\rho_n(G) = (G_{n-1}/c_n) \times G_{n,k} \in \mathbb{N}_0^{n-1} \times \mathbb{N}_0^{k-n} = \mathbb{N}_0^{k-1}. \tag{11}$$

In other words, $\rho_n(g_1, \ldots, g_k) = (g_1/c_n, \ldots, g_{n-1}/c_n, g_{n+1}, \ldots, g_k)$.

Remark 23. We specify $k \geq 2$ in the definition because there are no corresponding n for which to define ρ_n when $k = 1$.

Remark 24. The definition of ρ_n allows us to remove the nth entry of G for any $n \in \{2, \ldots, k\}$. If we wish to remove the first entry, we can apply the permutation $\sigma = (1\ 2)$ first and then apply ρ_2.

$$\rho_2(\sigma(G)) = \rho_2(g_2, g_1, g_3, \ldots, g_k) = (\gcd(g_2, g_1), g_3, \ldots, g_k).$$

Since $\gcd(g_2, g_1) = \gcd(g_1, g_2)$, it follows that $\rho_2(\sigma(G)) = \rho_2(G)$. Thus, removal of the first entry is equivalent to removal of the second entry.

Definition 25. For any $k \geq 1$, $G \in \mathbb{N}_0^k$, $g \in \langle G \rangle$, and $m \in \mathbb{N}$ with $\gcd(m, g) = 1$, let

$$\tau_{g,m}(G) = (mG) \times (g) \in \mathbb{N}_0^{k+1}. \tag{12}$$

In other words, $\tau_{g,m}(g_1, \ldots, g_k) = (mg_1, \ldots, mg_k, g)$.

Example 26. As in Example 1, let $G = (660, 550, 352, 50, 201) \in \mathbb{N}_0^5$. We have $c(G) = (6, 5, 11, 2)$.

We can apply ρ_n for any $n \in \{2, 3, 4, 5\}$. Applying ρ_2 (and thereby removing g_2), we get $\rho_2(G) = (G_1/c_2) \times G_{2,5} = (110, 352, 50, 201) \in \mathbb{N}_0^4$. If we wish to remove g_1, we apply the permutation $(1\ 2)$ first. Let $H = (1\ 2)(G) = (550, 660, 352, 50, 201)$. Then $c(H) = (e_2, e_3, e_4, e_5) = (5, 6, 11, 2)$ and

$$\rho_2((1\ 2)(G)) = \rho_2(H) = (H_1/e_2) \times H_{2,5} = (110, 352, 50, 201) \in \mathbb{N}_0^4,$$

which, as expected, is equal to $\rho_2(G)$.

We can apply $\tau_{g,m}$ for any $g \in \langle G \rangle$ and any m with $\gcd(m, g) = 1$. Applying $\tau_{251,3}$, we have $\tau_{251,3}(G) = (3G) \times (251) = (1980, 1650, 1056, 150, 603, 251) \in \mathbb{N}_0^6$.

Our goal with ρ_n and $\tau_{g,m}$ is to determine the conditions under which they send telescopic sequences to telescopic sequences. We begin by investigating their basic properties. In Lemma 27, we see how ρ_n and $\tau_{g,m}$ act on multiples of sequences. In Lemma 28, we compute $\gcd(\rho_n(G))$ and $c(\rho_n(G))$. In Lemma 29, we compute $\gcd(\tau_{g,m}(G))$ and $c(\tau_{g,m}(G))$.

Lemma 27. *Suppose $G \in \mathbb{N}_0^k$ and $d \in \mathbb{N}$. For $n = 2, \ldots, k$, $\rho_n(dG) = d\rho_n(G)$. For any $g \in \langle G \rangle$ and any $m \in \mathbb{N}$ with $\gcd(g, m) = 1$, $\tau_{dg,m}(dG) = d\tau_{g,m}(G)$.*

Proof. Let $c(G) = (c_2, \ldots, c_k)$. By Lemma 13, $c(dG) = c(G)$, so $c(dG) = (c_2, \ldots, c_k)$. Then

$$\begin{aligned}
\rho_n(dG) &= ((dG)_{n-1}/c_n) \times ((dG)_{n,k}) \\
&= (d(G_{n-1}/c_n)) \times (d(G_{n,k})) \\
&= d\left((G_{n-1}/c_n) \times G_{n,k}\right) \\
&= d\rho_n(G).
\end{aligned}$$

Next, $\tau_{dg,m}(dG) = (mdG) \times (dg) = d\left((mG) \times (g)\right) = d\tau_{g,m}(G)$. □

Lemma 28. *Let $G \in \mathbb{N}_0^k$ with $c(G) = (c_2, \ldots, c_k)$. For any $n = 2, \ldots, k$, we have $\gcd(\rho_n(G)) = \gcd(G)$ and $c(\rho_n(G)) = (c_2, \ldots, c_{n-1}, c_{n+1}, \ldots, c_k)$.*

Proof. We begin with the case where $\gcd(G) = 1$.

Let $H = \rho_n(G)$, so $H = (h_1, \ldots, h_{k-1})$ where, for $i = 1, \ldots, n-1$, we have $h_i = g_i/c_n$, and for $i = n, \ldots, k-1$, we have $h_i = g_{i+1}$. Let $c(H) = (e_2, \ldots, e_{k-1})$.

We begin by computing $\gcd(H_i)$. For $i < n$, $H_i = G_i/c_n$. Thus $\gcd(H_i) = \gcd\left(G_i/c_n\right)$. By Lemma 11, $\gcd(G_i) = C_{i,k} = c_{i+1} \cdots c_k$. Since $i < n$, $\gcd(G_i)$ contains c_n as a factor. Therefore $\gcd\left(G_i/c_n\right) = C_{i,k}/c_n$.

For $i \geq n$, we wish to show $\gcd(H_i) = C_{i+1,k}$. We proceed with induction on i. If $i = n$, $H_n = (G_{n-1}/c_n) \times (g_{n+1})$. By the previous paragraph, $\gcd(G_{n-1}/c_n) = C_{n,k}$. By Proposition 17, $g_{n+1} = z_{n+1}C_{n+1,k}$ with $\gcd(c_{n+1}, z_{n+1}) = 1$. Then

$$\begin{aligned}
\gcd(H_n) &= \gcd\left((G_{n-1}/c_n) \times (g_{n+1})\right) \\
&= \gcd\left(\gcd(G_{n-1}/c_n), g_{n+1}\right) \\
&= \gcd(C_{n,k}, g_{n+1}) \\
&= \gcd(c_{n+1}C_{n+1,k}, z_{n+1}C_{n+1,k}) \\
&= C_{n+1,k} \gcd(c_{n+1}, z_{n+1}) \\
&= C_{n+1,k}.
\end{aligned}$$

For $i > n$, by Proposition 17 we have $g_{i+2} = z_{i+2}C_{i+2,k}$ with $\gcd(c_{i+2}, z_{i+2}) = 1$. By induction we assume $\gcd(H_i) = C_{i+1,k}$, so

$$\begin{aligned}
\gcd(H_{i+1}) &= \gcd(h_1, \ldots, h_{i+1}) \\
&= \gcd(\gcd(H_i), g_{i+2}) \\
&= \gcd(C_{i+1,k}, z_{i+2}C_{i+2,k}) \\
&= \gcd(c_{i+2}C_{i+2,k}, z_{i+2}C_{i+2,k}) \\
&= C_{i+2,k}.
\end{aligned}$$

Therefore $\gcd(H_i) = C_{i+1,k}$ for $i > n$. In particular, $\gcd(H) = \gcd(H_{k-1}) = C_{k,k} = 1$.

Now we compute e_j. For $j < n$, $e_j = \gcd(H_{j-1})/\gcd(H_j) = (C_{j-1,k}/c_n)/(C_{j,k}/c_n) = c_j$. For $j = n$, $e_j = \gcd(H_{n-1})/\gcd(H_n) = (C_{n-1,k}/c_n)/C_{n+1,k} = c_{n+1} = c_{j+1}$. For $j > n$, $e_j = \gcd(H_{j-1})/\gcd(H_j) = C_{j,k}/C_{j+1,k} = c_{j+1}$. Therefore $c(H) = (c_2, \ldots, c_{n-1}, c_{n+1}, \ldots, c_k)$, as desired.

Now, suppose $\gcd(G) = d \geq 1$, let $G' = G/d$. Since $c(G) = c(G')$, we have $c(G') = (c_2, \ldots, c_k)$. Since $\gcd(G') = 1$, we apply the above result to obtain $\gcd(\rho_n(G')) = 1$ and $c(\rho_n(G')) = (c_2, \ldots, c_{n-1}, c_{n+1}, \ldots, c_k)$. By Lemma 27, $\rho_n(G) = \rho_n(dG') = d\rho_n(G')$, so $\gcd(\rho_n(G)) = \gcd(d\rho_n(G')) = d\gcd(\rho_n(G')) = d$ and $c(\rho_n(G)) = c(d\rho_n(G')) = c(\rho_n(G')) = (c_2, \ldots, c_{n-1}, c_{n+1}, \ldots, c_k)$. ∎

Lemma 29. *Let $G \in \mathbb{N}_0^k$ with $c(G) = (c_2, \ldots, c_k)$. For any $g \in \langle G \rangle$ and any $m \in \mathbb{N}$ with $\gcd(g, m) = 1$, we have $\gcd(\tau_{g,m}(G)) = \gcd(G)$ and $c(\tau_{g,m}(G)) = c(G) \times (m)$.*

Proof. Let $H = \tau_{g,m}(G)$, and let $c(H) = (e_2, \ldots, e_{k+1})$. We'll compute e_j for $j = 2, \ldots, k$, and for $j = k + 1$.

We have $H_k = mG$. By Lemma 13, $c(mG) = c(G)$. By Lemma 16, $c(H_k) = (e_2, \ldots, e_k)$. Thus $e_j = c_j$ for $j = 2, \ldots, k$.

For $j = k+1$, first note that

$$\begin{aligned}
\gcd(H) &= \gcd((mG) \times (g)) \\
&= \gcd(\gcd(mG), g) \\
&= \gcd(m \gcd(G), g) \\
&= \gcd(\gcd(G), g) \\
&= \gcd(G),
\end{aligned}$$

where the fourth equality follows from $\gcd(g, m) = 1$ and the final equality follows from the fact that $g \in \langle G \rangle$, which implies that $\gcd(G) \mid g$. Then

$$\begin{aligned}
e_{k+1} &= \gcd(H_k)/\gcd(H_{k+1}) \\
&= \gcd(mG_k)/\gcd(H) \\
&= m \gcd(G)/\gcd(G) \\
&= m.
\end{aligned}$$

Combining the results for $j = 2, \ldots, k$ and $j = k+1$, we have $c(H) = (c_2, \ldots, c_k, m) = c(G) \times (m)$, as desired. □

With appropriate parameters, ρ_n and $\tau_{g,m}$ are inverses of each other as the following lemma shows.

Lemma 30. *Let $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ with $c(G) = (c_2, \ldots, c_k)$, $g \in \langle G \rangle$, and $m \in \mathbb{N}$ with $\gcd(g, m) = 1$. Then $\tau_{g_k, c_k}(\rho_k(G)) = G$ and $\rho_{k+1}(\tau_{g,m}(G)) = G$.*

Proof. Since $\rho_k(G) = G_{k-1}/c_k$,

$$\begin{aligned}
\tau_{g_k,c_k}(\rho_k(G)) &= \tau_{g_k,c_k}\left(G_{k-1}/c_k\right) \\
&= (c_k G_{k-1}/c_k) \times (g_k) \\
&= G_{k-1} \times (g_k) \\
&= G.
\end{aligned}$$

By Lemma 29, $c(\tau_{g,m}(G)) = (c_2, \ldots, c_k, m)$, so

$$\begin{aligned}
\rho_{k+1}(\tau_{g,m}(G)) &= \rho_{k+1}\left((mG) \times (g)\right) \\
&= \rho_{k+1}\left((mg_1, \ldots, mg_k, g)\right) \\
&= (mg_1, \ldots, mg_k)/m \\
&= (g_1, \ldots, g_k) \\
&= G.
\end{aligned}$$

□

4.2 Telescopicness-preserving operations

Now that we have developed some of the properties of ρ_n and $\tau_{g,m}$, we will see how they map telescopic sequences to telescopic sequences. The goal is to then construct a map between any two telescopic sequences with the same greatest common divisor using some sequence of these maps composed together.

We first show that ρ_n preserves telescopicness.

Proposition 31. *For $k \geq 2$, let $G \in \mathbb{N}_0^k$. For $n = 2, \ldots, k$, if G is telescopic, then $\rho_n(G)$ is telescopic.*

Proof. Let $H = \rho_n(G)$. Then $H = (h_1, \ldots h_{k-1})$ where $h_i = g_i/c_n$ for $i < n$, and $h_i = g_{i+1}$ for $i \geq n$. Let $c(H) = (e_2, \ldots, e_{k-1})$. By Lemma 28, $c(H) = (c_2, \ldots, c_{n-1}, c_{n+1}, \ldots, c_k)$. Since G is telescopic, $c_j g_j \in \langle G_{j-1} \rangle$ for $j = 2, \ldots, k$. We need to show $e_j h_j \in \langle H_{j-1} \rangle$ for $j = 2, \ldots, k-1$.

For $i = 1, \ldots, n-1$, $H_i = G_i/c_n$, so for $j = 2, \ldots, n-1$, $e_j = c_j$. Since G is telescopic, $c_j g_j \in \langle G_{j-1} \rangle$. Thus $e_j h_j = c_j g_j / c_n \in \langle G_{j-1}/c_n \rangle = \langle H_{j-1} \rangle$.

Next, we have $c_n g_n \in \langle G_{n-1} \rangle$. Since $c_n \mid \gcd(G_{n-1})$, $g_n \in \langle G_{n-1}/c_n \rangle$, so $\langle G_{n-1}/c_n \rangle = \langle (G_{n-1}/c_n) \times (g_n) \rangle$. For $i = n, \ldots, k-1$, $\langle G_i \rangle \subset \langle (G_{n-1}/c_n) \times G_{n-1,i} \rangle = \langle (G_{n-1}/c_n) \times G_{n,i} \rangle = \langle H_{i-1} \rangle$. For $j = n, \ldots, k-1$, we have $e_j h_j = c_{j+1} g_{j+1} \in \langle G_j \rangle \subset \langle H_{j-1} \rangle$.

Since $e_j h_j \in \langle H_{j-1} \rangle$ for $j = 2, \ldots, k-1$, H is telescopic. ☐

Remark 32. Unfortunately, Proposition 31 is not an "if and only if" statement. Since all sequences of length 2 are telescopic, we can illustrate this with a sequence of length 3 that is not telescopic. For example, consider the sequence $G = (3, 4, 5)$. We have that $c(G) = (c_2, c_3) = (3, 1)$. Since $1 \cdot 5 \notin \langle 3, 4 \rangle$, G is not telescopic. However, $\rho_2(G) = (1, 5)$ and $\rho_3(G) = (3, 4)$, and both are telescopic sequences.

Next, we show that, with proper parameters, $\tau_{g,m}$ preserves telescopicness.

Proposition 33. *For $k \in \mathbb{N}$, let $G \in \mathbb{N}_0^k$, $g \in \langle G \rangle$, and $m \in \mathbb{N}$ with $\gcd(g, m) = 1$. Then G is telescopic if and only if $\tau_{g,m}(G)$ is telescopic.*

Proof. ($\implies$) Let $H = \tau_{g,m}(G)$, so $H = (h_1, \ldots, h_{k+1})$ with $h_i = m g_i$ for $i = 1, \ldots, k$, and $h_{k+1} = g$. Let $c(H) = (e_2, \ldots, e_{k+1})$. By Lemma 29, $c(H) = (c_2, \ldots, c_k, m)$. Since G is telescopic, $c_j g_j \in \langle G_{j-1} \rangle$ for $j = 2, \ldots, k$. We need to show $e_j h_j \in \langle H_{j-1} \rangle$ for $j = 2, \ldots, k+1$.

For $j = 2, \ldots, k$, $e_j h_j = c_j g_j \in \langle G_{j-1} \rangle = \langle H_{j-1} \rangle$. For $j = k+1$, $e_j h_j = mg$. Since $g \in \langle G \rangle$, $mg \in \langle mG \rangle = \langle H_k \rangle$. Thus $e_{k+1} h_{k+1} \in \langle H_k \rangle$. Therefore $e_j h_j \in \langle H_{j-1} \rangle$ for $j = 2, \ldots, k+1$, so H is telescopic.

($\impliedby$) For $G = (g_1, \ldots, g_k)$, if $\tau_{g,m}(G) \in \mathbb{N}_0^{k+1}$ is telescopic, then $\rho_{k+1}(\tau_{g,m}(G))$ is also telescopic. Since $\rho_{k+1}(\tau_{g,m})(G) = \rho_{k+1}((mG) \times (g)) = G$, and ρ_n maps telescopic sequences to telescopic sequences, we conclude that G is telescopic. ☐

Now that we know ρ_n and $\tau_{g,m}$ send telescopic sequences to telescopic sequences (using appropriate parameters with $\tau_{g,m}$), in Lemma 34 we will construct a map from $\mathbb{N}_0^k$ to $\mathbb{N}_0$

using a sequence of ρ_n functions that sends G to $\gcd(G)$. Following this, in Lemma 35 we will construct a map from $\mathbb{N}_0$ to $\mathbb{N}_0^k$ using a sequence of $\tau_{g,m}$ functions that sends $\gcd(G)$ to G.

Lemma 34. *For any $G \in \mathbb{N}_0^k$ (telescopic or not), let*

$$R_G = \rho_2 \circ \rho_3 \circ \cdots \circ \rho_k : \mathbb{N}_0^k \to \mathbb{N}_0. \tag{13}$$

Then $R_G(G) = \gcd(G)$.

Proof. If $k = 1$, then R_G is the identity map. Since $\gcd(g_1) = g_1$, $R_G(G) = \gcd(G)$.

Now, suppose $k > 1$. Let $c(G) = (c_2, \ldots, c_k)$. Then $\rho_k(G) = G_{k-1}/c_k$ and $c(\rho_k(G)) = (c_2, \ldots, c_{k-1})$. Then $\rho_{k-1}(\rho_k(G)) = (G_{k-2}/c_k)/c_{k-1}$ and $c(\rho_{k-1}(\rho_k(G))) = (c_2, \ldots, c_{k-2})$. Continuing in this way we find

$$(\rho_2 \circ \rho_3 \circ \cdots \circ \rho_k)(G) = G_1/(c_k c_{k-1} \cdots c_2) = g_1/C_{1,k} = \gcd(G),$$

by Lemma 11. $\square$

Lemma 35. *For $G \in \mathbb{N}_0^k$ a telescopic sequence with $c(G) = (c_2, \ldots, c_k)$ and $z_i = g_i/C_{i,k}$ for $i = 1, \ldots, k$, let*

$$T_G = \tau_{z_k,c_k} \circ \tau_{z_{k-1},c_{k-1}} \circ \cdots \circ \tau_{z_2,c_2} : \mathbb{N}_0 \to \mathbb{N}_0^k. \tag{14}$$

Then $T_G(\gcd(G)) = G$.

Proof. If $k = 1$, then T_G is the identity map. Since $\gcd(g_1) = g_1$, $T_G(\gcd(G)) = G$.

Now, suppose $k > 1$. We first consider the case where $\gcd(G) = 1$. For $j = 2, \ldots, k$, we wish to show $\tau_{z_j,c_j} \circ \cdots \circ \tau_{z_2,c_2}(1)$ is defined and equal to $(z_1 C_{1,j}, \ldots, z_j C_{j,j})$.

For $j = 2$, since $z_2 \in \langle 1 \rangle$, $\tau_{z_2,c_2}(1) = (c_2, z_2) = (z_1 C_{1,2}, z_2 C_{2,2})$.

Now, for induction assume $\tau_{z_j,c_j} \circ \cdots \circ \tau_{z_2,c_2}(1)$ is defined and equal to $(z_1 C_{1,j}, \ldots, z_j C_{j,j})$. Since G is telescopic, $c_{j+1} g_{j+1} \in \langle G_j \rangle$, so $z_{j+1} C_{j,k} \in \langle z_1 C_{1,k}, \ldots, z_j C_{j,k} \rangle$. It follows that $z_{j+1} \in \langle z_1 C_{1,j}, \ldots, z_j C_{j,j} \rangle$, and therefore $z_{j+1} \in \langle \tau_{z_j,c_j} \circ \cdots \circ \tau_{z_2,c_2}(1) \rangle$. Then

$$\begin{aligned}
(\tau_{z_{j+1},c_{j+1}} \circ \cdots \circ \tau_{z_2,c_2})(1) &= \tau_{z_{j+1},c_{j+1}}(z_1 C_{1,j}, \ldots, z_j C_{j,j}) \\
&= (z_1 C_{1,j} c_{j+1}, \ldots, z_j C_{j,j} c_{j+1}, z_{j+1}) \\
&= (z_1 C_{1,j+1}, \ldots, z_j C_{j,j+1}, z_{j+1} C_{j+1,j+1}).
\end{aligned}$$

In particular, for $j = k$, we have

$$(\tau_{z_k,c_k} \circ \tau_{z_{k-1},c_{k-1}} \circ \cdots \circ \tau_{z_2,c_2})(1) = (z_1 C_{1,k}, \ldots, z_k C_{k,k}) = G.$$

Now, suppose $\gcd(G) = d \geq 1$ and let $H = G/d$, so $H = (h_1, \ldots, h_k)$ with $h_i = g_i/d$ and $c(H) = c(G)$. Let $y_i = z_i/d$. By Proposition 17, since $g_i = z_i C_{i,k}$ with $z_i \in d\mathbb{N}_0$ for $i = 1, \ldots, k$, we have $h_i = y_i C_{i,k}$ with $y_i \in \mathbb{N}_0$. By above,

$$(\tau_{y_k,c_k} \circ \tau_{y_{k-1},c_{k-1}} \circ \cdots \circ \tau_{y_2,c_2})(1) = (y_1 C_{1,k}, \ldots, y_k C_{k,k}) = H.$$

Since $\tau_{dg,m}(G) = \tau_{g,m}(dG)$ (by Lemma 27), we multiply by d and get

$$(\tau_{dy_k,c_k} \circ \tau_{dy_{k-1},c_{k-1}} \circ \cdots \circ \tau_{dy_2,c_2})(d) = (dy_1 C_{1,k}, \ldots, dy_k C_{k,k}) = dH.$$

Finally, since $z_i = dy_i$ and $dH = G$,

$$(\tau_{z_k,c_k} \circ \tau_{z_{k-1},c_{k-1}} \circ \cdots \circ \tau_{z_2,c_2})(d) = (z_1 C_{1,k}, \ldots, z_k C_{k,k}) = G.$$

□

We now have our result. Given any two telescopic sequences G and H with $\gcd(G) = \gcd(H)$, we can perform a sequence of ρ_n and $\tau_{g,m}$ operations on G to produce H. We send G to $\gcd(G) = \gcd(H)$ and then send $\gcd(H)$ to H.

Proposition 36. *Let G and H be telescopic sequences with $\gcd(G) = \gcd(H)$. Then there exist functions $\phi_{G,H}$ and $\phi_{H,G}$, each a finite composition of functions ρ_{n_i} and τ_{g_j,m_j} (for parameters n_i and g_j, m_j) such that $\phi_{G,H}(G) = H$ and $\phi_{H,G}(H) = G$.*

Proof. By Lemmas 34 and 35, $T_H(R_G(G)) = T_H(\gcd(G)) = T_H(\gcd(H))$. Since H is telescopic, $T_H(\gcd(H)) = H$. Thus, $\phi_{G,H} = T_H \circ R_G$ is composition of ρ_{n_i} and τ_{g_j,m_j} functions such that $\phi_{G,H}(G) = H$.

Reversing the roles of G and H, since G is telescopic, we have the same result for $\phi_{H,G}$. □

We conclude this section with an example.

Example 37. Let $G = (4, 6, 9)$ and $H = (30, 18, 20, 33)$. Then $\gcd(G) = \gcd(H) = 1$, $c(G) = (2, 2)$, and $c(H) = (5, 3, 2)$. Both sequences are telescopic. Then $R_G = \rho_2 \circ \rho_3$, and

$$R_G(G) = (\rho_2 \circ \rho_3)(4, 6, 9) = \rho_2(2, 3) = 1.$$

For T_H, we first need the z_i values for H. Since $z_i = h_i/C_{i,4}$ for $i = 1, 2, 3, 4$, we get $z_1 = 1$, $z_2 = 3$, $z_3 = 10$, and $z_4 = 33$. Therefore $T_H = \tau_{33,2} \circ \tau_{10,3} \circ \tau_{3,5}$, and

$$T_H(1) = (\tau_{33,2} \circ \tau_{10,3} \circ \tau_{3,5})(1) = (\tau_{33,2} \circ \tau_{10,3})(5, 3) = \tau_{33,2}(15, 9, 10) = (30, 18, 20, 33).$$

Thus, for $\phi_{G,H} = T_H \circ R_G = \tau_{33,2} \circ \tau_{10,3} \circ \tau_{3,5} \circ \rho_2 \circ \rho_3$, we have $\phi_{G,H}(G) = H$.

We follow the same process for the reverse direction. Here, $T_G = \tau_{9,2} \circ \tau_{3,2}$ and $R_H = \rho_2 \circ \rho_3 \circ \rho_4$. Therefore $\phi_{H,G} = T_G \circ R_H = \tau_{9,2} \circ \tau_{3,2} \circ \rho_2 \circ \rho_3 \circ \rho_4$ and $\phi_{H,G}(H) = G$.

5 Reduction of a telescopic sequence to a minimal telescopic sequence

In this section, we present a method to take a telescopic sequence G and produce a minimal telescopic sequence G' such that $\langle G' \rangle = \langle G \rangle$. The main result is Theorem 52. In the context of free numerical semigroups, this shows that every free numerical semigroup is generated by a telescopic sequence that is minimal. As we will see, the main idea is in Proposition 43, which says there are only two ways in which a telescopic sequence can fail to be minimal.

Before we begin, we introduce one more sequence operation and two helpful lemmas. For $G \in \mathbb{N}_0^k$ and $n = 1, \ldots, k$, let

$$\pi_n(G) = G_{n-1} \times G_{n,k}. \tag{15}$$

In other words, for $G = (g_1, \ldots, g_k)$,

$$\pi_n(G) = (g_1, \ldots, \widehat{g_n}, \ldots, g_k) = (g_1, \ldots, g_{n-1}, g_{n+1}, \ldots, g_k) \in \mathbb{N}_0^{k-1}. \tag{16}$$

The nth term is removed from G and the indices of all subsequent terms are decreased by one.

With this notation, we have that a sequence $G \in \mathbb{N}_0^k$ is minimal if and only if $\langle \pi_n(G) \rangle \neq \langle G \rangle$ for all $n = 1, \ldots, k$, which occurs if and only if $g_n \notin \langle \pi_n(G) \rangle$ for all $n = 1, \ldots, k$.

As in previous sections, we will continue to assume that $g_1 > 0$.

Lemma 38. *For any sequence $G \in \mathbb{N}_0^k$, let $S = \langle G \rangle$. Then $\gcd(G) = \gcd(S)$.*

Proof. Let $d_G = \gcd(G)$ and $d_S = \gcd(S)$. Since $G \subset S$, $d_S \mid d_G$. For any $s \in S$, $s = \sum_{i=1}^k a_i g_i$ for some non-negative integers a_i. Since $d_G \mid g_i$ for all i, we have $d_G \mid s$ for all $s \in S$. Since d_G is a common divisor for all elements of S, $d_G \mid d_S$. Thus $d_G = d_S$. □

Lemma 39. *Suppose $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ is any sequence (not necessarily telescopic). If $g_n \in \langle \pi_n(G) \rangle$ for some n, then $\gcd(\pi_n(G)) = \gcd(G)$.*

Proof. Since $\langle \pi_n(G) \rangle = \langle G \rangle$, we use Lemma 38 twice with $\pi_n(G)$ and G to get

$$\gcd(\pi_n(G)) = \gcd(\langle \pi_n(G) \rangle) = \gcd(\langle G \rangle) = \gcd(G).$$

□

5.1 Determination of the two cases for non-minimality of a telescopic sequence

As we will see, there are two ways for which a telescopic sequence can fail to be minimal. In Lemmas 40 and 41, we will show that if one entry of a telescopic sequence can be written as a non-negative linear combination of the remaining elements, then we have a divisibility condition on the coefficient of the term with highest index.

Lemma 40. *Let $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ be telescopic with $c(G) = (c_2, \ldots, c_k)$. Suppose $g_n \in \langle \pi_n(G) \rangle$ for some $n \in \{1, \ldots, k\}$. That is, suppose*

$$g_n = \sum_{i=1,\, i \neq n}^{k} a_i g_i \tag{17}$$

for non-negative integers a_i. Then $c_k \mid a_k$.

Proof. Since $g_n \in \langle \pi_n(G) \rangle$, by Lemma 39, $\gcd(\pi_n(G)) = \gcd(G)$. We consider two cases: $n = k$ and $n < k$.

Assume $n = k$. Since $G_{k-1} = (g_1, \ldots, g_{k-1}) = \pi_k(G)$, we have $\gcd(G_{k-1}) = \gcd(G_k)$, so $c_k = \gcd(G_{k-1})/\gcd(G_k) = 1$, and consequently $c_k \mid a_k$.

Now assume $n < k$. Since $\gcd(G_k) = \gcd(G)$, then $c_k = \gcd(G_{k-1})/\gcd(G)$. Therefore, in order to have that $c_k \mid a_k$, it is enough to show that $\gcd(G_{k-1}) \mid (a_k \gcd(G))$. From Equation (17) we solve for $a_k g_k$ to get

$$a_k g_k = g_n - \sum_{i=1,\, i \neq n}^{k-1} a_i g_i,$$

so $a_k g_k$ is a linear combination of $g_1, \ldots, g_{k-1}$. Since $\gcd(G_{k-1}) \mid g_1, \ldots, g_{k-1}$, we have $\gcd(G_{k-1}) \mid (a_k g_k)$. Then, since $\gcd\left(\gcd(G_{k-1}), g_k\right) = \gcd(G)$, we conclude that $\gcd(G_{k-1}) \mid (a_k \gcd(G))$. $\square$

Lemma 41. *Let $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ be a telescopic sequence with $c(G) = (c_2, \ldots, c_k)$. Suppose there exist n and m with $1 \leq n < m \leq k$ such that $g_n \in \langle \pi_n(G_m) \rangle$. That is, suppose*

$$g_n = \sum_{i=1,\, i \neq n}^{m} a_i g_i \tag{18}$$

for non-negative integers a_i. Then $c_m \mid a_m$.

Proof. Suppose $g_n \in \langle \pi_n(G_m) \rangle$ for some $m > n$. Let $H = G_m$, a sequence of length m. By Lemma 16, H is telescopic with $c(H) = (c_2, \ldots, c_m)$. Since $g_n \in \langle \pi_n(H) \rangle$, we apply Lemma 40 to H to conclude that $c_m \mid a_m$. $\square$

We need one more lemma before we can state our main result on the form of a non-minimal telescopic sequence.

Lemma 42. *For $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ telescopic with $c(G) = (c_2, \ldots, c_k)$, suppose $g_n \in \langle \pi_n(G_m) \rangle$ for some m and n with $1 \leq n \leq m \leq k$. Then $g_n \in \langle \pi_n(G_{m-1}) \rangle$ or $g_n = c_m g_m$.*

Proof. First, suppose $n = m$. Since $\pi_n(G_n) = (g_1, \ldots, g_{n-1}) = \pi_n(G_{n-1})$, if $g_n \in \langle \pi_n(G_m) \rangle$ then $g_n \in \langle \pi_n(G_{m-1}) \rangle$, as desired. (Additionally, we have that $c_m = \gcd(G_m)/\gcd(G_{m-1}) = 1$ in this case. Since $g_n = g_m$, we also have $g_n = c_m g_m$.)

Now suppose $n < m$. Suppose $g_n \in \langle \pi_n(G_m) \rangle$ and $g_n \notin \langle \pi_n(G_{m-1}) \rangle$. We wish to show that $g_n = c_m g_m$.

Since $g_n \in \langle \pi_n(G_m) \rangle$, there exist non-negative integers a_i such that

$$g_n = \sum_{i=1,\, i \neq n}^{m} a_i g_i. \tag{19}$$

Since $g_n \notin \langle \pi_n(G_{m-1}) \rangle$, we must have $a_m > 0$. By Lemma 41, $c_m \mid a_m$, so we let $q = a_m/c_m \in \mathbb{N}$. For G telescopic, $c_m g_m \in \langle G_{m-1} \rangle$, so there are $b_1, \ldots, b_{m-1} \in \mathbb{N}_0$ such that

$$c_m g_m = b_1 g_1 + \cdots + b_{m-1} g_{m-1}. \tag{20}$$

Thus $a_m g_m = q c_m g_m = q \sum_{i=1}^{m-1} b_i g_i$, which we plug into Equation (19) to obtain

$$g_n = q b_n g_n + \sum_{i=1,\, i \neq n}^{m-1} (q b_i + a_i) g_i. \tag{21}$$

Since $g_n \notin \langle \pi_n(G_{m-1}) \rangle$, we must have $q b_n \neq 0$. Since $q b_n \in \mathbb{N}$, this implies $q b_n = 1$, so $(q b_i + a_i) g_i = 0$ for $i = 1, \ldots, m-1$, with $i \neq n$. Therefore $q = b_n = 1$ and $a_i g_i = b_i g_i = 0$ for $i = 1, \ldots, m-1$, with $i \neq n$. Since $q = 1$, $a_m = c_m$. Finally, we plug these values into either Equation (19) or Equation (20) to conclude that $g_n = c_m g_m$. $\square$

We now have our result.

Proposition 43. *For $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ a telescopic sequence, suppose $g_n \in \langle \pi_n(G) \rangle$. Then $g_n \in \langle G_{n-1} \rangle$ or $g_n = c_m g_m$ for some $m > n$.*

Proof. This follows immediately from Lemma 42. $\square$

Therefore, if a telescopic sequence G is not minimal, there are two cases: Case 1, where $g_n \in \langle G_{n-1} \rangle$; and Case 2, where $g_n = c_m g_m$ for some $m > n$.

Proposition 43 gives us a straightforward criterion to determine whether a given telescopic sequence is minimal.

Corollary 44. *Suppose $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ is telescopic with $c(G) = (c_2, \ldots, c_k)$. Let $H = (h_2, \ldots, h_k) = (c_2 g_2, \ldots, c_k g_k)$. Then G is minimal if and only if $g_i \neq h_j$ for all i, j.*

Proof. ($\Longrightarrow$) Suppose G is minimal. Then $g_j \notin \langle G_{j-1} \rangle$ for all $j \geq 2$. Since G is telescopic, $c_j g_j \in \langle G_{j-1} \rangle$ for all $j \geq 2$. Thus $g_j \neq c_j g_j$ for all $j \geq 2$.

Also, since G is minimal, $g_i \notin \langle g_j \rangle$ for all $i \neq j$. In particular, we have $g_i \neq c_j g_j$ for all $i \neq j$. Combined with the previous paragraph, we have $g_i \neq h_j$ for all i, j.

($\Longleftarrow$) Suppose G is not minimal, so $g_i \in \langle \pi_i(G) \rangle$ for some i. By Proposition 43, we have either $g_i \in \langle \pi_i(G_{i-1}) \rangle$ or $g_i = c_j g_j$ for some $j > i$. In the first case, $\gcd(g_1, \ldots, g_i) = \gcd(g_1, \ldots, g_{i-1})$, so $c_i = 1$. Therefore $g_i = c_i g_i = h_i$. In the second case, $g_i = h_j$. $\square$

The following example illustrates the two cases of Proposition 43.

Example 45. Let $G = (660, 550, 352, 902, 50, 201)$ a telescopic sequence with $c(G) = (6, 5, 1, 11, 2)$. We have $g_2 \in \langle \pi_2(G) \rangle$ and $g_4 \in \langle \pi_4(G) \rangle$. Observe that $g_2 = c_5 g_5$, which is Case 2. And $g_4 = g_2 + g_3$, which is Case 1.

For each case, we will see how to remove g_n without losing the telescopic property of the generating sequence. We will illustrate our methods with the above example.

5.1.1 Case 1

For the case where $g_n \in \langle G_{n-1} \rangle$, we use a result (Proposition 31) about ρ_n from Section 4.

Corollary 46. *Suppose $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ is telescopic with $g_1 > 0$ and that $g_n \in \langle G_{n-1} \rangle$ for some n. Then $n > 1$, $c_n = 1$, and $\pi_n(G)$ is a telescopic sequence with $\langle \pi_n(G) \rangle = \langle G \rangle$.*

Proof. Suppose $n = 1$. Then $G_{n-1} = G_0 = ()$, the empty sequence, so $\langle G_0 \rangle = \{0\}$. Thus $g_1 = 0$. However, since we have assumed that $g_1 > 0$, this cannot occur. Therefore $n > 1$.

Now, let $c(G) = (c_2, \ldots, c_k)$. If $g_n \in \langle G_{n-1} \rangle$, then $\langle G_{n-1} \rangle = \langle G_n \rangle$ so

$$c_n = \gcd(G_{n-1}) / \gcd(G_n) = 1.$$

By Proposition 31, the sequence $\rho_n(G)$ is telescopic. Since $c_n = 1$,

$$\rho_n(G) = (g_1/c_n, \ldots, g_{n-1}/c_n, g_{n+1}, \ldots, g_k) = (g_1, \ldots, g_{n-1}, g_{n+1}, \ldots, g_k) = \pi_n(G),$$

so $\pi_n(G)$ is telescopic. Also, since $\langle G_{n-1} \rangle = \langle G_n \rangle$, $\langle G_{n-1} \times G_{n,k} \rangle = \langle G_n \times G_{n,k} \rangle$, so $\langle \pi_n(G) \rangle = \langle G \rangle$, as desired. □

In this case, we can therefore remove g_n to produce a shorter telescopic sequence that generates the same submonoid.

Example 47. For the telescopic sequence $G = (660, 550, 352, 902, 50, 201)$ with $c(G) = (6, 5, 1, 11, 2)$ from Example 45, $g_4 = 902 = 550 + 352 = g_2 + g_3 \in \langle G_3 \rangle$. To remove g_4 from G, we compute $\pi_4(G) = (660, 550, 352, 50, 201)$, a telescopic sequence with $c(\pi_4(G)) = (6, 5, 11, 2)$ and $\langle \pi_4(G) \rangle = \langle G \rangle$.

5.1.2 Case 2

Now we consider the case where $g_n = c_m g_m$ for $m > n$. Instead of just removing g_n (like we did in Case 1), we will first swap the positions of g_n and g_m before removing g_n. As the following lemma states, this is equivalent to a permutation from S_{k-1} acting on $\pi_n(G)$.

Lemma 48. *For $G \in \mathbb{N}_0^k$, suppose $1 \le n < m \le k$ and consider the transposition $(n\ m) \in S_k$. Then $\pi_m((n\ m)(G)) = \sigma(\pi_n(G))$ for $\sigma \in S_{k-1}$ defined by*

$$\sigma(i) = \begin{cases} i, & \text{if } i < n \text{ or } i \ge m; \\ n, & \text{if } i = m - 1; \\ i + 1, & \text{if } n \le i < m - 1. \end{cases} \tag{22}$$

We now have a lemma with two results about greatest common divisors that will be useful for the main result of the $g_n = c_m g_m$ case. Since c_1 is not defined, we will assume $n > 1$ for the following lemma and proposition. We will then address the possibility of $n = 1$ in Theorem 52 with the help of the permutation $(1\ 2)$.

Lemma 49. *For $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ a telescopic sequence with $c(G) = (c_2, \ldots, c_k)$ and $\gcd(G) = d$, if $g_n = c_m g_m$ for some m, n with $1 < n < m \leq k$, then $g_n = z_n C_{n,k}$ with $z_n \in \langle d \rangle$ and $\gcd(z_n/d, c_n) = 1$. In addition we have $\gcd(z_n/d, c_m) = 1$ and $\gcd(c_j, c_m) = 1$ for all $n < j < m$.*

Proof. By Proposition 17, $g_n = z_n C_{n,k}$ with $\gcd(z_n/d, c_n) = 1$ and $g_m = z_m C_{m,k}$ with $\gcd(z_m/d, c_m) = 1$. Since $g_n = c_m g_m$, we have $z_n C_{n,k} = c_m z_m C_{m,k}$, so $z_n C_{n,m-1} = z_m$. Therefore $(z_n/d) C_{n,m-1} = (z_m/d)$.

For the first greatest common divisor statement, $(z_n/d) \mid (z_m/d)$, so $\gcd(z_n/d, c_m) \mid \gcd(z_m/d, c_m)$, which is 1. Thus $\gcd(z_n/d, c_m) = 1$.

For the second greatest common divisor statement, for $n < j < m$ since $c_j \mid C_{n,m-1}$, $c_j \mid (z_m/d)$. Then $\gcd(c_j, c_m) \mid \gcd(z_m/d, c_m)$, which is 1. Thus $\gcd(c_j, c_m) = 1$. □

We now have our main result for this case.

Proposition 50. *For $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ telescopic with $c(G) = (c_2, \ldots, c_k)$, suppose $g_n = c_m g_m$ for some m, n with $1 < n < m \leq k$. For the transposition $(n\ m) \in S_k$, the sequence $\pi_m((n\ m)(G))$ is telescopic and $\langle \pi_m((n\ m)(G)) \rangle = \langle G \rangle$.*

Proof. For $H = \pi_m((n\ m)(G))$, by Lemma 48, H is a permutation of $\pi_n(G)$, and since $g_n \in \langle \pi_n(G) \rangle$, $\langle H \rangle = \langle \pi_n(G) \rangle = \langle G \rangle$.

We have $H = (h_1, \ldots, h_{k-1}) \in \mathbb{N}_0^{k-1}$ where

$$h_i = \begin{cases} g_i, & \text{for } i = 1, \ldots, m-1 \text{ with } i \neq n; \\ g_m, & \text{for } i = n; \\ g_{i+1}, & \text{for } i = m, \ldots, k-1. \end{cases}$$

Let $c(H) = (e_2, \ldots, e_{k-1})$. To show H is telescopic, we must show $e_j h_j \in \langle H_{j-1} \rangle$ for $2 \leq j \leq k-1$. We will consider five subintervals of indices: $2 \leq j < n$; $j = n$; $n < j < m$; $j = m$; and $m < j \leq k-1$.

Since $H_{n-1} = G_{n-1}$ and G is telescopic, by Lemma 16 H_{n-1} is telescopic, which implies that $e_j h_j \in \langle H_{j-1} \rangle$ for $2 \leq j \leq n-1$.

Suppose $j = n$. Let $d = \gcd(G)$. By Corollary 18, $g_n = z_n C_{n,k}$ for some $z_n \in d\mathbb{N}_0$ with $\gcd(z_n/d, c_n) = 1$. Then

$$\begin{aligned} \gcd(H_n) &= \gcd(h_1, \ldots, h_n) \\ &= \gcd(g_1, \ldots, g_{n-1}, g_m) \\ &= \gcd(\gcd(G_{n-1}), g_n/c_m) \\ &= \gcd(C_{n-1,k} d, z_n C_{n,k}/c_m) \\ &= \frac{C_{n,k} d}{c_m} \gcd(c_m c_n, z_n/d). \end{aligned}$$

Note that $C_{n,k}/c_m \in \mathbb{N}$ since $m > n$. Since $\gcd(z_n/d, c_n) = 1$ and $\gcd(z_n/d, c_m) = 1$ (the latter from Lemma 49), $\gcd(H_n) = C_{n,k} d/c_m$. Since $\gcd(H_{n-1}) = \gcd(G_{n-1})$, we have

$e_n = \gcd(H_{n-1})/\gcd(H_n) = \gcd(G_{n-1})/(C_{n,k}d/c_m) = c_m c_n$, so $e_n h_n = c_m c_n g_n/c_m = c_n g_n$, which is in $\langle G_{n-1}\rangle$ since G is telescopic. And since $H_{n-1} = G_{n-1}$, we have $e_n h_n \in \langle H_{n-1}\rangle$.

From the $j = n$ case, we saw $\gcd(H_n) = C_{n,k}d/c_m$. By Corollary 18 again, $g_{n+1} = z_{n+1}C_{n+1,k}$ for some $z_{n+1} \in d\mathbb{N}$ with $\gcd(z_{n+1}/d, c_{n+1}) = 1$. Then,

$$\begin{aligned}
\gcd(H_{n+1}) &= \gcd(g_1, \ldots, g_{n-1}, g_m, g_{n+1}) \\
&= \gcd(C_{n,k}d/c_m, g_{n+1}) \\
&= \gcd(C_{n,k}d/c_m, z_{n+1}C_{n+1,k}) \\
&= \frac{C_{n+1,k}d}{c_m}\gcd(c_{n+1}, c_m z_{n+1}/d) \\
&= \frac{C_{n+1,k}d}{c_m},
\end{aligned}$$

since $\gcd(c_{n+1}, z_{n+1}/d) = 1$ and $\gcd(c_{n+1}, c_m) = 1$ (the latter from Lemma 49).

Continuing in this way, we find that $\gcd(H_i) = C_{i,k}d/c_m$ for $n \le i < m$. Therefore, for $n < j < m$, $e_j = \gcd(H_{j-1})/\gcd(H_j) = c_j$. Since G is telescopic, $c_j g_j \in \langle G_{j-1}\rangle$. With $g_n = c_m g_m$ and $h_n = g_m$, we have $g_n \in \langle h_n\rangle$, so $\langle G_{j-1}\rangle \subset \langle H_{j-1}\rangle$. Thus $e_j h_j = c_j g_j \in \langle G_{j-1}\rangle \subseteq \langle H_{j-1}\rangle$ for $n < j < m$.

Suppose $j = m$. We first note that $\langle G_m\rangle = \langle H_{m-1}\rangle$. Then, since H_m is a permutation of $\pi_n(G_{m+1})$, we have $\langle H_m\rangle = \langle \pi_n(G_{m+1})\rangle = \langle G_{m+1}\rangle$, the latter equality holding because $g_n \in \langle g_m\rangle \subset \langle \pi_n(G_{m+1})\rangle$. Thus $\gcd(H_m) = \gcd(G_{m+1}) = C_{m+1,k}d$, from which it follows that $e_m = \gcd(H_{m-1})/\gcd(H_m) = (C_{m-1,k}d/c_m)/(C_{m+1,k}d) = c_{m+1}$. Since G is telescopic, $c_{m+1}g_{m+1} \in \langle G_m\rangle$. Then $e_m h_m = c_{m+1}g_{m+1} \in \langle G_m\rangle = \langle H_{m-1}\rangle$.

Finally, for $m \le i \le k-1$, H_i is a permutation of $\pi_n(G_{i+1})$, so $\langle H_i\rangle = \langle G_{i+1}\rangle$ and $\gcd(H_i) = \gcd(G_{i+1}) = C_{i+1,k}d$. Then for $m < j \le k-1$, $e_j = \gcd(H_{j-1})/\gcd(H_j) = \gcd(G_j)/\gcd(G_{j+1}) = c_{j+1}$. Since G is telescopic, $c_{j+1}g_{j+1} \in \langle G_j\rangle$. Thus $e_j h_j = c_{j+1}g_{j+1} \in \langle G_j\rangle = \langle H_{j-1}\rangle$ for $m < j \le k-1$. □

As a result, we can therefore remove g_n and reorder the remaining terms to produce a shorter telescopic sequence that generates the same submonoid.

Example 51. For the telescopic sequence $G = (660, 550, 352, 902, 50, 201)$ with $c(G) = (6, 5, 1, 11, 2)$ from Example 45, $g_2 = 550 = 11 \cdot 50 = c_5 g_5$. To remove g_2 from G, we compute

$$\pi_5((2\ 5)(G)) = \pi_5(660, 50, 352, 902, 550, 201) = (660, 50, 352, 902, 201).$$

Then $c(\pi_5((2\ 5)(G))) = (66, 5, 1, 2)$, $\pi_5((2\ 5)(G))$ is telescopic, and $\langle \pi_5((2\ 5)(G))\rangle = \langle G\rangle$.

5.2 Answers to Questions 2 and 3

If a telescopic sequence G is not minimal, then there is some n such that $g_n \in \langle \pi_n(G)\rangle$. In Cases 1 and 2, we saw how to remove g_n and produce a shorter telescopic sequence that generates the same submonoid. We can now answer Question 2.

Theorem 52. *Let $G \in \mathbb{N}_0^k$ be telescopic. Then there exists a minimal telescopic sequence G' such that $\langle G' \rangle = \langle G \rangle$.*

Proof. If G is minimal, we are done. Otherwise, for $G = (g_1, \ldots, g_k)$, there is some n for which $g_n \in \langle \pi_n(G) \rangle$. By Proposition 43, either $g_n \in \langle G_{n-1} \rangle$ (Case 1) or, for $c(G) = (c_2, \ldots, c_k)$, $g_n = c_m g_m$ for some $m > n$ (Case 2).

For Case 1, if $g_n \in \langle G_{n-1} \rangle$, then by Corollary 46, $\pi_n(G)$ is a telescopic sequence of length $k-1$ such that $\langle \pi_n(G) \rangle = \langle G \rangle$.

For Case 2, if $g_n = c_m g_m$ for some $m > n$, we consider two cases of n. If $n > 1$, then by Proposition 50, $\pi_m((n\ m)(G))$ is a telescopic sequence of length $k-1$ such that $\langle \pi_m((n\ m)(G)) \rangle = \langle G \rangle$.

If $n = 1$ and $g_2 > 0$, we transpose the first two terms and use the previous paragraph to remove the (new) second term. Let $H = (1\ 2)(G)$. We have $H = (h_1, \ldots, h_k)$, a telescopic sequence (by Proposition 14) with $h_1 > 0$, $h_2 \in \langle \pi_2(H) \rangle$, and $\langle H \rangle = \langle G \rangle$. By Proposition 50, $\pi_m((2\ m)(H))$ is a telescopic sequence of length $k-1$ such that $\langle \pi_m((2\ m)(H)) \rangle = \langle H \rangle = \langle G \rangle$.

Now, to construct the minimal telescopic sequence, starting with G we first iteratively remove all of the Case 1 terms (in any order). Observe that this removes any zeros. Then, from the resulting sequence, we can iteratively remove all of the Case 2 terms (with $n > 1$ and $n = 1$). Since $G \in \mathbb{N}_0^k$ and we remove one term with each step, this process terminates in fewer than k steps, producing a telescopic sequence G' that is necessarily minimal and satisfies $\langle G' \rangle = \langle G \rangle$. $\square$

Example 53. Once again using the telescopic sequence $G = (660, 550, 352, 902, 50, 201)$ with $c(G) = (6, 5, 1, 11, 2)$ from Example 45, we have $g_n \in \langle \pi_n(G) \rangle$ for $n = 2$ and $n = 4$. In Example 47, we removed g_4 from G to get the telescopic sequence $H = \pi_4(G) = (660, 550, 352, 50, 201)$ with $c(H) = (6, 5, 11, 2)$. In Example 51, we removed g_2 from G to get the telescopic sequence $\bar{H} = \pi_5((2\ 5)(G)) = (660, 50, 352, 902, 201)$ with $c(\bar{H}) = (66, 5, 1, 2)$. Of course, neither of these resulting sequences is minimal.

In $H = (h_1, \ldots, h_5)$, $h_2 = 11h_4$, which is Case 2. We therefore compute $\pi_4((2\ 4)(H)) = \pi_4(660, 50, 352, 550, 201) = (660, 50, 352, 201)$, a telescopic sequence that generates the same submonoid as H. Observe that this sequence is also minimal. (On its own, this paragraph addresses the motivating example in Example 1.)

In $\bar{H} = (\bar{h}_1, \ldots, \bar{h}_5)$, $\bar{h}_4 = 11\bar{h}_2 + 1\bar{h}_3$, which is Case 1. We therefore compute $\pi_4(\bar{H}) = (660, 50, 352, 201)$, which is the same sequence as in the previous paragraph.

The result is that, starting with the telescopic sequence $G = (660, 550, 352, 902, 50, 201)$, after two steps we produce the sequence $G' = (660, 50, 352, 201)$ for which $c(G') = (66, 5, 2)$. Then G' is telescopic and minimal, and $\langle G' \rangle = \langle G \rangle$.

Since a free numerical semigroup is one that is generated by a (not necessarily minimal) telescopic sequence G with $\gcd(G) = 1$, we get the following.

Corollary 54. *Suppose S is a free numerical semigroup. Then S is generated by a telescopic sequence that is minimal.*

We conclude this section by answering Question 3.

Corollary 55. *Let G and H be sequences with $\langle G \rangle = \langle H \rangle$. Then a permutation of G is telescopic if and only if a permutation of H is telescopic.*

Proof. Suppose G is telescopic and that $\langle G \rangle = \langle H \rangle$. By Theorem 52 there is a minimal telescopic sequence G' such that $\langle G' \rangle = \langle G \rangle$, so $\langle G' \rangle = \langle H \rangle$ as well. Since G' is minimal, each term in G' must appear in H. Consider the permutation

$$\sigma(H) = G' \times H',$$

where H' consists of the remaining terms of H not included in G'. Each term in H' is in $\langle G' \rangle$, so $c(\sigma(H)) = c(G') \times (1, \ldots, 1)$. Since G' is telescopic, the telescopic conditions hold for the G' portion of $\sigma(H)$. And since 1 times each term in H' is in $\langle G' \rangle$, the telescopic conditions hold for the H' portion of $\sigma(H)$. Therefore, $\sigma(H)$ is a telescopic sequence.

Switching the roles of G and H, we find the reverse implication holds as well. □

6 Construction of minimal telescopic sequences with desired properties

We now look at necessary and sufficient conditions for a constructed telescopic sequence to be minimal. Suppose $G = (g_1, \ldots, g_k) \in \mathbb{N}_0^k$ and $c(G) = (c_2, \ldots, c_k)$. If G is telescopic and not minimal, then $g_n \in \langle \pi_n(G) \rangle$ for some n, so by Proposition 43, either

1. $g_n \in \langle G_{n-1} \rangle$; or
2. there is some $m > n$ such that $g_m \mid g_n$.

The first case occurs precisely when $c_n = 1$, which we can avoid by requiring $c_j > 1$ for all $j = 2, \ldots, k$. The second case occurs precisely when $z_m C_{m,k} \mid z_n C_{n,k}$, which we can avoid by requiring $z_j \nmid z_i C_{i,j}$ for all i, j with $1 \leq i < j \leq k$.

Corollary 56. *Suppose $G = (g_1, \ldots, g_k)$ is a telescopic sequence (with notation as in Corollary 18). Then G is minimal if and only if we additionally have $c_j > 1$ for $j = 2, \ldots, k$ and $z_j \nmid z_i C_{i,j}$ for all $1 \leq i < j \leq k$.*

Example 57. Suppose we want a free numerical semigroup $S = \langle G \rangle$ where G is minimal and telescopic with

$$c(G) = (c_2, c_3, c_4, c_5) = (2, 3, 4, 5).$$

To generate a numerical semigroup, we need $\gcd(G) = 1$, so we have $z_1 = \gcd(G) = 1$. Since $c_j > 1$ for all j, for minimality we need only satisfy the non-divisibility conditions for $z_2, \ldots, z_5$. We need $z_j \in \mathbb{N}$ where

- $z_2 \in \langle z_1 \rangle$, $\gcd(z_2, 2) = 1$, and $z_2 \nmid 2z_1$;

- $z_3 \in \langle 2z_1, z_2 \rangle$, $\gcd(z_3, 3) = 1$, and $z_3 \nmid 6z_1, 3z_2$;
- $z_4 \in \langle 6z_1, 3z_2, z_3 \rangle$, $\gcd(z_4, 4) = 1$, and $z_4 \nmid 24z_1, 12z_2, 4z_3$;
- $z_5 \in \langle 24z_1, 12z_2, 4z_3, z_4 \rangle$, $\gcd(z_5, 5) = 1$, and $z_5 \nmid 120z_1, 60z_2, 20z_3, 5z_4$.

Any such $z_2, \ldots, z_5$ satisfying the above will produce the desired result, and conversely every such numerical semigroup S can be constructed in this way.

For a concrete example, we can take $z_2 = 3$, $z_3 = 5$, $z_4 = 11$, and $z_5 = 22$. For $g_i = z_i C_{i,5}$, we get $G = (120, 180, 100, 55, 22)$. We check that $\gcd(G) = 1$, G is minimal, and G is telescopic with $c(G) = (2, 3, 4, 5)$, as desired.

Corollary 58. *Suppose $G = (g_1, \ldots, g_k) \in \mathbb{N}^k$ is a telescopic and non-decreasing sequence with $c(G) = (c_2, \ldots, c_k)$ where $c_j > 1$ for all j. Then G is minimal.*

In particular, if $\gcd(G) = 1$, then for the numerical semigroup $S = \langle G \rangle$, we have $e(S) = |G|$.

Proof. If $g_j = g_{j-1}$ for some $j > 1$, then $c_j = 1$. Since we assume $c_j > 1$ for all j, we must have $g_j \neq g_{j+1}$, so G is a strictly increasing sequence. In other words, $g_n < g_m$ whenever $n < m$.

Since G is telescopic and $c_j > 1$ for all j, G is minimal if and only if $g_m \nmid g_n$ for all $1 \leq n < m \leq k$. Since $g_m > g_n > 0$, we have $g_m \nmid g_n$. Therefore G is minimal. $\square$

7 Acknowledgments

The author wishes to thank the anonymous referee for substantially helping to improve the presentation of this paper in many ways. Of particular note is the suggestion to write Lemma 16, which simplified a few subsequent proofs.

References

[1] R. Apéry, Sur les branches superlinéaires des courbes algébriques, *C. R. Acad. Sci. Paris* **222** (1946), 1198–1200.

[2] T. Ayano, Sigma functions for telescopic curves, *Osaka J. Math.* **51** (2014), 459–480.

[3] J. Bertin and P. Carbonne, Semi-groupes d'entiers et application aux branches, *J. Algebra* **49** (1977), 81–95.

[4] A. Brauer, On a problem of partitions, *Amer. J. Math.* **64** (1942), 299–312.

[5] A. Brauer and J. E. Shockley, On a problem of Frobenius, *J. Reine Angew. Math.* **211** (1962), 215–220.

[6] F. Curtis, On formulas for the Frobenius number of a numerical semigroup, *Mathematica Scandinavica* **67** (1990), 190–192.

[7] R. Fröberg, C. Gottlieb, and R. Häggkvist, On numerical semigroups, *Semigroup Forum* **35** (1987), 63–84.

[8] T. A. Gassert and C. M. Shor, On Sylvester sums of compound sequence semigroup complements, *J. Number Theory* **180** (2017), 45–72.

[9] T. A. Gassert and C. M. Shor, Characterizations of numerical semigroup complements via Apéry sets, *Semigroup Forum* (2018).

[10] J. Herzog, Generators and relations of abelian semigroups and semigroup rings, *Manuscripta Math.* **3** (1970), 175–193.

[11] C. Kiers, C. O'Neill, and V. Ponomarenko, Numerical semigroups on compound sequences, *Comm. Algebra* **44** (2016), 3842–3852.

[12] C. Kirfel and R. Pellikaan, The minimum distance of codes in an array coming from telescopic semigroups, *IEEE Trans. Inform. Theory* **41** (1995), 1720–1732.

[13] E. Leher, *Applications of the minimal transversal method in numerical semigroups*, PhD thesis, Tel Aviv University, 2007.

[14] J. L. Ramírez Alfonsín, Complexity of the Frobenius problem, *Combinatorica* **16** (1996), 143–147.

[15] J. L. Ramírez Alfonsín, *The Diophantine Frobenius problem*, Vol. 30 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, Oxford, 2005.

[16] A. M. Robles-Pérez and J. C. Rosales, The Frobenius number for sequences of triangular and tetrahedral numbers, *J. Number Theory* **186** (2018), 473–492.

[17] Ö. J. Rödseth, On a linear Diophantine problem of Frobenius, *J. Reine Angew. Math.* **301** (1978), 171–178.

[18] J. C. Rosales and P. A. García-Sánchez, On free affine semigroups, *Semigroup Forum* **58** (1999), 367–385.

[19] J. C. Rosales and P. A. García-Sánchez, *Numerical Semigroups*, Vol. 20 of *Developments in Mathematics*, Springer-Verlag New York, 2009.

[20] E. S. Selmer, On the linear diophantine problem of Frobenius, *J. Reine Angew. Math.* **293/294** (1977), 1–17.

[21] H. J. H. Tuenter, The Frobenius problem, sums of powers of integers, and recurrences for the Bernoulli numbers, *J. Number Theory* **117** (2006), 376–386.

[22] K. Watanabe, Some examples of one dimensional Gorenstein domains, *Nagoya Math. J.* **49** (1973), 101–109.

2010 *Mathematics Subject Classification*: Primary 20M14; Secondary 11B75.
Keywords: telescopic sequence, free numerical semigroup, minimal generating sequence, additive submonoid.

Received June 27 2018; revised versions received January 2 2019; January 6 2019. Published in *Journal of Integer Sequences*, February 23 2019.

Return to Journal of Integer Sequences home page.