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Some New Restricted n-Color Composition Functions

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Abstract

An n-color composition is one in which a part of size m can come in m colors (denoted by subscripts). Let $\mathcal{C}(\nu)$ denote the set of n-color compositions of the positive integer ν. In this paper, we consider further modular restrictions on the subscripts of the parts within members of $\mathcal{C}(\nu)$. We first count members of $\mathcal{C}(\nu)$ in which all parts have subscripts of the form $\ell a+b$, where b and ℓ are fixed and $a \geq 0$ is arbitrary. Generating function and explicit formulas are found for general b and ℓ which extend earlier results when $\ell=2$ and $b \leq 3$. We study the case $\ell=b-1$ in further detail and find that the corresponding subset of $\mathcal{C}(\nu)$ is in bijection with various classes of compositions. Finally, we consider two related problems: one where the subscript restriction applies only to parts within a given modular class and another where the subscript of a part belongs to the same modular class mod ℓ as the part where ℓ is fixed.

1 Introduction

A *composition* of a positive integer ν is a sequence of positive integers $\sigma=\{\sigma_1,\sigma_2,\ldots,\sigma_r\}$ such that $\sigma_1+\sigma_2+\cdots+\sigma_r=\nu$. The summands σ_i are called the *parts* of σ and ν is the *weight* of σ. For example, the compositions of 4 are

$$\{4\},\ \{3,1\},\ \{1,3\},\ \{2,2\},\ \{2,1,1\},\ \{1,2,1\},\ \{1,1,2\},\ \{1,1,1,1\}.$$

Agarwal [1] introduced a generalization of the concept of a composition known as an *n-color composition* wherein a part of size $m \geq 1$ can come in one of m different colors. The colors of the part m are denoted by subscripts $m_1, m_2, \ldots, m_m$. For example, the n-color compositions of 4 are

$$\{4_1\},\ \{4_2\},\{4_3\},\{4_4\},\{3_1,1_1\},\{3_2,1_1\},\{3_3,1_1\},\{1_1,3_1\},\{1_1,3_2\},\{1_1,3_3\},\{2_1,2_1\},$$
$$\{2_1,2_2\},\{2_2,2_1\},\{2_2,2_2\},\{2_1,1_1,1_1\},\{2_2,1_1,1_1\},\{1_1,2_1,1_1\},\{1_1,2_2,1_1\},\{1_1,1_1,2_1\},$$
$$\{1_1,1_1,2_2\},\{1_1,1_1,1_1,1_1\}.$$

It is well-known that the total number of n-color compositions of ν is given by the Fibonacci number $F_{2\nu}$. Moreover, the number of n-color compositions of ν with exactly m parts is the binomial coefficient $\binom{\nu+m-1}{2m-1}$. For further results about n-color compositions, see, e.g., [1, 2, 4, 6, 7, 9, 10, 11, 13, 14, 15]. In this paper, we study some new restrictions on n-color compositions that generalize previous results given by Sachdeva and Agarwal [13].

The organization of this paper is as follows. In the next section, we count the members of $\mathcal{C}(\nu)$ in which the subscripts on all parts are of the form $\ell a+b$ for some $a \geq 0$, where $b, \ell \geq 1$ are fixed, providing generating function and explicit formulas. This extends recent work [13] in the case $\ell=2$. We consider further the case $\ell=b-1$, which yields several previously studied sequences from [16], and find bijections between various restricted classes of binary words and compositions and the corresponding subset of $\mathcal{C}(\nu)$. In the third section, we count members of $\mathcal{C}(\nu)$ in which only parts of the form $\ell a+b$ for some $a \geq 0$ satisfy a similar modular requirement with respect to their subscripts. An explicit formula for the generating function is found which extends prior results [13]. Finally, a comparable formula can be given which counts members of $\mathcal{C}(\nu)$ in which parts of the form $\ell a+b$ where $a \geq 0$ and $1 \leq b \leq \ell$ must have subscripts of the same form.

2 Generalized restricted n-color compositions

Given positive integers ℓ and b, let $\mathcal{C}_{\ell a+b}(\nu)$ denote the number of n-color compositions of ν into parts with subscripts of the form $\ell a+b$ for some integer $a \geq 0$. We also denote by $\mathcal{C}_{\ell a+b}(m,\nu)$ the number of n-color compositions of ν into m parts with subscripts of the form $\ell a+b$.

For example, $\mathcal{C}_{3a+1}(4)=9$, the compositions being

$$\{4_1\},\{4_4\},\{3_1,1_1\},\{1_1,3_1\},\{2_1,2_1\},\{2_1,1_1,1_1\},\{1_1,2_1,1_1\},\{1_1,1_1,2_1\},\{1_1,1_1,1_1,1_1\}.$$

Theorem 1. *Let $\mathcal{GC}_{\ell a+b}(m,x)$ and $\mathcal{GC}_{\ell a+b}(x)$ denote the generating functions for the sequences $\mathcal{C}_{\ell a+b}(m,\nu)$ and $\mathcal{C}_{\ell a+b}(\nu)$, respectively. Then we have*

$$\mathcal{GC}_{\ell a+b}(m,x)=\left(\frac{x^b}{(1-x)(1-x^\ell)}\right)^m,$$
$$\mathcal{GC}_{\ell a+b}(x)=\frac{x^b}{1-x-x^\ell+x^{\ell+1}-x^b}.$$

Proof. Let $\sigma=\sigma_1\cdots\sigma_m$ be a non-empty n-color composition having m parts where each subscript is of the form $\ell a+b$ for some $a \geq 0$. If $\sigma_j=i$ with $i \geq b$, then σ_j contributes to the generating function the term $w_i x^i$, where

$$w_i=\left\lfloor\frac{i-b+\ell}{\ell}\right\rfloor,$$

while if $i<b$, then it fails to contribute.

Note that the generating function of the sequence

$$\{w_i\}_{i\geq 0}=\left\{\underbrace{0,\ldots,0}_{b},\underbrace{1,\ldots,1}_{\ell},\underbrace{2,\ldots,2}_{\ell},\ldots\right\}$$

is given by

$$\frac{x^b}{(1-x)(1-x^\ell)}.$$

Therefore,

$$\mathcal{GC}_{\ell a+b}(m,x)=\left(\sum_{i\geq 0} w_i x^i\right)^m=\left(\frac{x^b}{(1-x)(1-x^\ell)}\right)^m.$$

Finally, summing the last expression over $m \geq 1$, we get

$$\mathcal{GC}_{\ell a+b}(x) = \frac{\frac{x^b}{(1-x)(1-x^\ell)}}{1 - \frac{x^b}{(1-x)(1-x^\ell)}} = \frac{x^b}{1 - x - x^\ell + x^{\ell+1} - x^b}.$$

□

We have the following combinatorial formula for the sequence $\mathcal{C}_{\ell a+b}(m, \nu)$.

Theorem 2. *The sequence $\mathcal{C}_{\ell a+b}(m, \nu)$ is given by the expression*

$$\mathcal{C}_{\ell a+b}(m, \nu) = \sum_{i=0}^{\lfloor \frac{\nu - bm}{\ell} \rfloor} \binom{m+i-1}{m-1} \binom{\nu - \ell i + m(1-b) - 1}{m-1}.$$

Moreover, $\mathcal{C}_{\ell a+b}(\nu) = \mathcal{C}_{\ell a+b}(\nu - 1) + \mathcal{C}_{\ell a+b}(\nu - \ell) - \mathcal{C}_{\ell a+b}(\nu - \ell - 1) + \mathcal{C}_{\ell a+b}(\nu - b)$ when $\nu > \max\{\ell + 1, b\}$.

Proof. By Theorem 1, we have

$$\begin{aligned}
\mathcal{GC}_{\ell a+b}(m, x) &= \left(\frac{x^b}{(1-x)(1-x^\ell)} \right)^m \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{m+i-1}{i} \binom{m+j-1}{j} x^{j+i\ell+bm}.
\end{aligned}$$

Taking $t = j + \ell i + bm$ gives

$$\mathcal{GC}_{\ell a+b}(m, x) = \sum_{i=0}^{\infty} \sum_{t=i\ell+bm}^{\infty} \binom{m+i-1}{m-1} \binom{t - \ell i + m(1-b) - 1}{m-1} x^t.$$

By comparing the ν-th coefficient of both sides of the last equation, we obtain the desired result. The recurrence relation follows from the generating function formula for $\mathcal{GC}_{\ell a+b}(x)$ given in Theorem 1. □

Remark 3. Setting $\ell = b = 1$ in Theorem 2, and using the binomial identity [5, Formula 5.26], recovers the fact that there are $\binom{\nu+m-1}{2m-1}$ n-color compositions of ν with exactly m parts and thus $F_{2\nu}$ altogether with no restriction as to the number of parts.

By setting $\ell = 2$ and $b = 1$, we have the following corollary (see Theorem 2.1 of [13]).

Corollary 4. *The generating functions for the number of n-color compositions of ν into m parts with odd subscripts and for the total number of n-color compositions of ν with odd subscripts are*

$$\begin{aligned}
\mathcal{GC}_{2a+1}(m, x) &= \left(\frac{x}{(1-x)(1-x^2)} \right)^m = \left(\frac{x}{(1+x)(1-x)^2} \right)^m, \\
\mathcal{GC}_{2a+1}(x) &= \frac{x}{1 - 2x - x^2 + x^3}.
\end{aligned}$$

Moreover,

$$\mathcal{C}_{2a+1}(m,\nu) = \sum_{i=0}^{\lfloor \frac{\nu-m}{2} \rfloor} \binom{m+i-1}{m-1}\binom{\nu-2i-1}{m-1}$$

and $\mathcal{C}_{2a+1}(\nu) = 2\mathcal{C}_{2a+1}(\nu-1) + \mathcal{C}_{2a+1}(\nu-2) - \mathcal{C}_{2a+1}(\nu-3)$ *for* $\nu > 3$*, with the initial values* $\mathcal{C}_{2a+1}(1) = 1, \mathcal{C}_{2a+1}(2) = 2, \mathcal{C}_{2a+1}(3) = 5.$

Letting $\ell = 2$ and $b = 2$ yields the following corollary (see Theorem 2.3 of [13]).

Corollary 5. *The generating functions for the number of n-color compositions of* ν *into* m *parts with even subscripts and for the total number of n-color compositions of* ν *with even subscripts are*

$$\mathcal{G}\mathcal{C}_{2a+2}(m,x) = \left(\frac{x^2}{(1-x)(1-x^2)}\right)^m = \left(\frac{x^2}{(1+x)(1-x)^2}\right)^m,$$

$$\mathcal{G}\mathcal{C}_{2a+2}(x) = \frac{x^2}{1-x-2x^2+x^3}.$$

Moreover,

$$\mathcal{C}_{2a+2}(m,\nu) = \sum_{i=0}^{\lfloor \frac{\nu-2m}{2} \rfloor} \binom{m+i-1}{m-1}\binom{\nu-2i-m-1}{m-1}$$

and $\mathcal{C}_{2a+2}(\nu) = \mathcal{C}_{2a+2}(\nu-1) + 2\mathcal{C}_{2a+2}(\nu-2) - \mathcal{C}_{2a+2}(\nu-3)$ *for* $\nu > 3$*, with the initial values* $\mathcal{C}_{2a+2}(1) = 0, \mathcal{C}_{2a+2}(2) = 1, \mathcal{C}_{2a+2}(3) = 1.$

Letting $\ell = 2$ and $b = 3$ yields the further corollary (see Theorem 2.2 of [13]).

Corollary 6. *The generating functions for the number of n-color compositions of* ν *into* m *parts with odd subscripts* > 1 *and for the total number of n-color compositions of* ν *with odd subscripts* > 1 *are*

$$\mathcal{G}\mathcal{C}_{2a+3}(m,x) = \left(\frac{x^3}{(1+x)(1-x)^2}\right)^m,$$

$$\mathcal{G}\mathcal{C}_{2a+3}(x) = \frac{x^3}{1-x-x^2}.$$

Moreover,

$$\mathcal{C}_{2a+3}(m,\nu) = \sum_{i=0}^{\lfloor \frac{\nu-3m}{2} \rfloor} \binom{m+i-1}{m-1}\binom{\nu-2i-2m-1}{m-1}$$

and $\mathcal{C}_{2a+3}(\nu) = \mathcal{C}_{2a+3}(\nu-1) + \mathcal{C}_{2a+3}(\nu-2)$ *for* $\nu > 3$*, with the initial values* $\mathcal{C}_{2a+3}(1) = 0, \mathcal{C}_{2a+3}(2) = 0, \mathcal{C}_{2a+3}(3) = 1.$

ℓ	b	Sequence $\mathcal{C}_{\ell a+b}(\nu)$	A-Sequence
3	1	1, 2, 4, 9, 19, 40, 85, 180, 381, 807, 1709, 3619, 7664, 16230, 34370	A052908
3	2	1, 1, 2, 4, 6, 11, 19, 32, 56, 96, 165, 285, 490, 844, 1454, 2503, 4311	A116732
3	3	1, 1, 1, 3, 4, 5, 10, 15, 21, 36, 56, 83, 134, 210, 320, 505, 791, 1221	A176848

Table 1: Some particular cases for $\ell = 3$.

When $\ell = 3$, we obtain some known sequences from the OEIS [16]. In Table 1, we give the first several non-zero values.

Note that the sequence A052908 does not have a combinatorial interpretation listed. For the sequence A116732, our combinatorial interpretation differs from the one given. Let $\mathcal{A}$ be the set of compositions with parts in $\{1, 2, 3\}$ such that the order of adjacent 1's and 3's is unimportant. Let $a(n)$ be the number of elements in $\mathcal{A}$ of weight n. For example, $a(6) = 19$, where the compositions are

$\{3, 3\}, \{3, 2, 1\}, \{3, 1, 2\}, \{2, 3, 1\}, \{1, 2, 3\}, \{3, 1, 1, 1\}, \{2, 2, 2\}, \{2, 2, 1, 1\}, \{2, 1, 2, 1\}, \{2, 1, 1, 2\}, \{1, 2, 2, 1\}, \{1, 2, 1, 2\}, \{1, 1, 2, 2\}, \{2, 1, 1, 1, 1\}, \{1, 2, 1, 1, 1\}, \{1, 1, 2, 1, 1\}, \{1, 1, 1, 2, 1\}, \{1, 1, 1, 1, 2\}, \{1, 1, 1, 1, 1, 1\}$.

Theorem 7. *For $n \geq 0$, $a(n) = \mathcal{C}_{3a+2}(n + 2)$.*

Proof. Let w be a composition in $\mathcal{A}$. Then w is either an integer partition (non-ordered composition) with parts in $\{1, 3\}$ or can be factorized as $p\,2\,w'$, where p is a partition with parts in $\{1, 3\}$ and $w' \in \mathcal{A}$. Thus, the generating function $A(x)$ of the sequence $a(n)$ satisfies the relation

$$A(x) = P_{1,3}(x) + P_{1,3}(x)x^2A(x),$$

where $P_{1,3}(x)$ counts integer partitions with parts in $\{1, 3\}$. Since

$$P_{1,3}(x) = \frac{1}{(1 - x)(1 - x^3)},$$

we have

$$A(x) = \frac{1}{1 - x - x^2 - x^3 + x^4}.$$

Finally, by Theorem 1,

$$\mathcal{GC}_{3a+2}(x) = x^2A(x),$$

which yields the desired result upon comparing n-th coefficients. $\square$

Let $b(n)$ be the number of compositions of n where each part of size j for $j \geq 1$ comes in $\lfloor j/3 \rfloor$ kinds (sequence A176848). For example, $b(7) = 4$, the enumerated compositions being $\{7_x\}, \{7_y\}, \{3_x, 4_x\}, \{4_x, 3_x\}$. It is clear from the definitions that $b(n) = \mathcal{C}_{3a+3}(n)$ for $n \geq 1$.

We now give a bijective proof of the prior theorem.

Combinatorial proof of Theorem 7.

Let $\mathcal{A}_n$ and $\mathcal{C}_n$ denote the set of compositions enumerated by $a(n)$ and $\mathcal{C}_{3a+2}(n)$, respectively. We will define a bijection between $\mathcal{A}_n$ and $\mathcal{C}_{n+2}$ for $n \geq 0$. Let us assume that 3 always precedes 1 whenever there is an adjacency of the two letters within a member of $\mathcal{A}_n$. Let $\lambda \in \mathcal{A}_n$. First assume λ contains no 2's. Then we may write $\lambda = 3^i1^j$, where $i, j \geq 0$ with $3i + j = n$. In this case, we map λ to the colored composition $\lambda' = (3i + j + 2)_{3i+2}$ of $n + 2$ containing a single part. So assume λ contains at least one 2, in which case we may write

$$\lambda = 3^{i_0}1^{j_0}2^{a_1}3^{i_1}1^{j_1}2^{a_2}3^{i_2}1^{j_2} \cdots 2^{a_r}3^{i_r}1^{j_r},$$

where all exponents are non-negative, $r \geq 1$, $a_1, \ldots, a_r \geq 1$, and $i_k + j_k \geq 1$ for $1 \leq k \leq r-1$. In this case, we let

$$\lambda' = (3i_0 + j_0 + 2)_{3i_0+2}, (2_2)^{a_1-1}, (3i_1 + j_1 + 2)_{3i_1+2}, \ldots, (2_2)^{a_r-1}, (3i_r + j_r + 2)_{3i_r+2},$$

where $(2_2)^t$ denotes a run of the part 2_2 of length t.

Note that λ' contains $r + 1$ parts and indeed belongs to $\mathcal{C}_{n+2}$. Also, while it is possible for the first or the last part of λ' to be 2_2, all parts of the form $(3i_k + j_k + 2)_{3i_k+2}$ where $1 \leq k \leq r - 1$ are greater than 2. Furthermore, since $j_k \geq 0$ for $0 \leq k \leq r$, arbitrary differences can occur between the part sizes and subscripts. Thus, the mapping $\lambda \mapsto \lambda'$ may be reversed and hence is a bijection between $\mathcal{A}_n$ and $\mathcal{C}_{n+2}$, as desired, upon decomposing members of $\mathcal{C}_{n+2}$ in the same way λ' was above. $\square$

2.1 The case $\ell = b - 1$

In this subsection, we provide additional combinatorial interpretations for the sequence $\mathcal{C}_{\ell a+\ell+1}(n)$, where $\ell \geq 1$. In Table 2, we give the first several non-zero values of these sequences for $2 \leq \ell \leq 6$.

ℓ	b	Sequence $\mathcal{C}_{\ell a+b}(\nu)$	A-Sequence
2	3	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597	A000045
3	4	1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595	A000930
4	5	1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19, 26, 36, 50, 69, 95, 131, 181, 250	A003269
5	6	1, 1, 1, 1, 1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 34, 45, 60, 80, 106, 140	A003520
6	7	1, 1, 1, 1, 1, 1, 2, 3, 4, 5, 6, 7, 9, 12, 16, 21, 27, 34, 43, 55, 71, 92	A005708

Table 2: Some particular cases of $\ell = b - 1$.

Let $F_\ell(n) := \mathcal{C}_{\ell a+\ell+1}(n)$. By Theorem 1, we have

$$F_\ell(x) := \sum_{n=0}^{\infty} F_\ell(n)x^n = \frac{x^{\ell+1}}{1 - x - x^\ell}.$$

Moreover, $F_\ell(n) = F_\ell(n-1) + F_\ell(n-\ell)$ for $n > \ell+1$, with the initial values $F_\ell(\ell+1) = 1$ and $F_\ell(n) = 0$ for $n \in [\ell] = \{1, 2, \ldots, \ell\}$. For $\ell = 2$, it is clear that the sequence $F_2(n)$ coincides with the Fibonacci numbers, i.e., $F_2(n) = F_{n-2}$ for $n \geq 2$. Moreover, $F_3(n)$ is seen to correspond to the Narayana sequence (cf. [12]).

Let $\mathcal{E}_\ell$ be the set of compositions into parts 1 and ℓ, where $\ell \geq 2$. Let $e_\ell(n)$ denote the number of elements in $\mathcal{E}_\ell$ of weight n. Chinn and Heubach [3] studied this family of compositions and, in particular, found

$$E_\ell(x) := \sum_{n=0}^{\infty} e_\ell(n) x^n = \frac{1}{1 - x - x^\ell}.$$

Then $x^{\ell+1} E_\ell(x) = F_\ell(x)$ and we have the following result.

Theorem 8. *For $n \geq 0$, $F_\ell(n+\ell+1) = e_\ell(n)$.*

Let $\mathcal{H}_\ell$ be the set of compositions into parts greater than or equal to ℓ. Let $h_\ell(n)$ be the number of elements in $\mathcal{H}_\ell$ of weight n. It is not difficult to show that (see, for example, [8, Theorem 3.13])

$$H_\ell(x) := \sum_{n=0}^{\infty} h_\ell(n) x^n = \frac{1}{1 - (x^\ell + x^{\ell+1} + \cdots)} = \frac{1-x}{1 - x - x^\ell}.$$

Therefore, we have the following relation.

Theorem 9. *For $n \geq 1$, $F_\ell(n+1) = h_\ell(n)$.*

Let $\mathcal{G}_\ell$ be the set of binary words such that between any two successive ones there are at least $\ell - 1$ zeros. Let $g_\ell(n)$ be the number of words in $\mathcal{G}_\ell$ of length n. Let w be a binary word in $\mathcal{G}_\ell$ of length $n > \ell$. Then w can be decomposed as $w = 0w_1$ or $w = 1\underbrace{0\cdots0}_{\ell-1}w_2$, where $w_1, w_2 \in \mathcal{G}_\ell$, which implies $g_\ell(n) = g_\ell(n-1) + g_\ell(n-\ell)$ for all $n > \ell$. Thus, this sequence satisfies the same recurrence relation as $F_\ell(n)$. Note that $g_\ell(n) = n+1$ if $n \in [\ell]$, which follows from the definitions. Since $F_\ell(n+\ell) = 1$ if $n \in [\ell]$, applying the recurrence for $F_\ell(n)$ implies $F_\ell(n+2\ell) = n+1$ for $n \in [\ell]$. Comparing the recurrences and initial values gives the following relation.

Theorem 10. *For $n \geq 0$, $F_\ell(n+2\ell) = g_\ell(n)$.*

We conclude this section by providing bijective proofs of the last three results.

Combinatorial proofs of Theorems 8 and 9.

Let $\mathcal{E}_\ell(n)$ denote the set of compositions of n with parts 1 and ℓ and $\mathcal{F}_\ell(n)$ the set of colored compositions enumerated by $F_\ell(n)$. We define a mapping $f : \mathcal{E}_\ell(n) \to \mathcal{F}_\ell(n+\ell+1)$ as follows. If $\lambda = 1^{n-b\ell}\ell^b$, where $0 \leq b \leq \lfloor n/\ell \rfloor$, then let $f(\lambda) = ((b+1)\ell + n - b\ell + 1)_{(b+1)\ell+1}$. Otherwise, we have

$$\lambda = 1^{a_0}\ell^{b_1}1^{a_1}\cdots\ell^{b_r}1^{a_r}\ell^{b_{r+1}},$$

where $r \geq 1$, $a_0 \geq 0$, $a_i, b_i \geq 1$ if $1 \leq i \leq r$ and $b_{r+1} \geq 0$. In this case, let

$$f(\lambda) = (b_1\ell + a_0 + 1)_{b_1\ell+1}, (b_2\ell + a_1)_{b_2\ell+1}, \ldots, (b_r\ell + a_{r-1})_{b_r\ell+1}, ((b_{r+1}+1)\ell + a_r)_{(b_{r+1}+1)\ell+1}.$$

Note that $f(\lambda)$ contains $r+1$ parts and indeed belongs to $\mathcal{F}_\ell(n+\ell+1)$ (a 1 not accounted for by λ occurs in the first part and there is an extra ℓ in the last part). Observe further that the last part of $f(\lambda)$ has subscript greater than or equal to $\ell+1$ depending on whether the last part of λ is ℓ or 1. Upon considering the number of parts in a member of $\mathcal{F}_\ell(n+\ell+1)$, the mapping f is seen to be reversible and hence yields the desired bijection.

To show Theorem 9, let $\mathcal{H}_\ell(n)$ denote the set of compositions of n having parts of size ℓ or more. We define $g : \mathcal{H}_\ell(n) \to \mathcal{F}_\ell(n+1)$ for $n \geq 1$ as follows. If $n \in [\ell-1]$, then both sets are empty, so assume $n \geq \ell$. Then we may express $\lambda \in \mathcal{H}_\ell(n)$ as

$$\lambda = x_1\ell^{a_1}x_2\ell^{a_2}\cdots x_r\ell^{a_r},$$

where $r \geq 1$, $x_1 \geq \ell$, $x_i \geq \ell+1$ if $i > 1$ and $a_i \geq 0$ for all i. Let

$$g(\lambda) = (a_1\ell + x_1 + 1)_{(a_1+1)\ell+1}, (a_2\ell + x_2)_{(a_2+1)\ell+1}, \ldots, (a_r\ell + x_r)_{(a_r+1)\ell+1}.$$

One may verify that the mapping g is a bijection, which completes the proof. $\square$

Combinatorial proof of Theorem 10.

Let $\mathcal{G}_\ell(n)$ denote the set of binary words enumerated by $g_\ell(n)$. We define a mapping $f : \mathcal{G}_\ell(n) \to \mathcal{F}_\ell(n+2\ell)$ in several steps as follows. Let $\lambda = \lambda_1\lambda_2\cdots\lambda_n \in \mathcal{G}_\ell(n)$ and first assume $n \in [\ell]$. In this case, let

$$f(\lambda) = \begin{cases} (n+2\ell)_{\ell+1}, & \text{if } \lambda = 0^n; \\ (n-s+\ell)_{\ell+1}, (s+\ell)_{\ell+1}, & \text{if } \lambda = 0^s10^{n-1-s}, \text{ where } 1 \leq s \leq n-1; \\ (n+2\ell)_{2\ell+1}, & \text{if } \lambda = 10^{n-1}. \end{cases}$$

Henceforth, assume $n > \ell$. We will also assume $\ell > 1$, as the adjustments necessary in the $\ell = 1$ case will be apparent. Note that $\lambda \in \mathcal{G}_\ell(n)$ may start with an initial (possibly empty) run of 0's with the remainder of λ being decomposed into sections of the form $u = 10^{\ell-1}$ (1 followed by $\ell-1$ 0's) and $v = 10^{m-1}$ where $m \geq \ell+1$ is arbitrary (to be specified). Furthermore, it is possible for λ to end in a section w of the form $w = 10^p$, where $0 \leq p \leq \ell-2$.

First assume λ contains no section of the form v above. Then either

$$\lambda = 0^{n-i\ell}u^i, \qquad 0 \leq i \leq \lfloor n/\ell \rfloor, \tag{1}$$

or

$$\lambda = 0^{n-p-1-i\ell}u^iw, \qquad 0 \leq p \leq \ell-2 \text{ and } 0 \leq i \leq \lfloor (n-p-1)/\ell \rfloor, \tag{2}$$

where $w = 10^p$. We define f in this case by considering whether or not n is divisible by ℓ. If ℓ divides n, then let $f(\lambda) = (n+2\ell)_{(i+1)\ell+1}$, if λ is of the form (1), and let

$$f(\lambda) = (\ell+p+1)_{\ell+1}, ((i+1)\ell + n - p - 1 - i\ell)_{(i+1)\ell+1},$$

if of form (2). If ℓ does not divide n, then we define $f(\lambda)$ the same way as before provided λ is not of the form (2) with $n-p-1=i\ell$. Note that $n-p-1=i\ell$ corresponds to exactly one λ in (2) since $0 \le p \le \ell-2$. We set $f(\lambda)=(n+2\ell)_{q\ell+1}$ in this case where $q=\lfloor n/\ell \rfloor+2$ (note that $q\ell+1 \le n+2\ell$ if and only if ℓ does not divide n). Observe that in either case f maps the members of $\mathcal{G}_\ell(n)$ not containing a v section in a one-to-one manner to the subset of $\mathcal{F}_\ell(n+2\ell)$ whose members either have one part or have two parts where the first part is less than 2ℓ.

Assume henceforth that λ contains at least one section of the form v above. Then we may write

$$\lambda = 0^j u^{i_1} v_1 \cdots u^{i_r} v_r u^{i_{r+1}}, \tag{3}$$

where $r \ge 1$, $j, i_1, \ldots, i_{r+1} \ge 0$, and $v_i = 10^{m_i-1}$ with $m_i \ge \ell+1$ for $1 \le i \le r$, or

$$\lambda = 0^j u^{i_1} v_1 \cdots u^{i_r} v_r u^{i_{r+1}} w, \tag{4}$$

with all the same restrictions as before and $w=10^p$ for some $0 \le p \le \ell-2$. If λ is of the form (3), then let

$$f(\lambda) = ((i_1+2)\ell+j)_{(i_1+1)\ell+1}, (i_2\ell+m_1)_{(i_2+1)\ell+1}, \ldots, (i_{r+1}\ell+m_r)_{(i_{r+1}+1)\ell+1}.$$

Observe that $r \ge 1$ implies $f(\lambda)$ contains at least two parts in this case and $m_i \ge \ell+1$ for all i implies the size of the part always exceeds the size of the subscript (with the first part of size at least 2ℓ).

Now suppose λ is of form (4). To define f, we consider cases on j. If $j \ge 1$ in (4), then let

$$f(\lambda) = (\ell+p+1)_{\ell+1}, ((i_1+1)\ell+j)_{(i_1+1)\ell+1}, (i_2\ell+m_1)_{(i_2+1)\ell+1}, \ldots, (i_{r+1}\ell+m_r)_{(i_{r+1}+1)\ell+1}.$$

Note $f(\lambda)$ here must contain at least three parts and therefore this covers the remaining cases where the first part is less than 2ℓ. If $j=0$ in (4), then let

$$f(\lambda) = ((i_1+2)\ell+p+1)_{(i_1+2)\ell+1}, (i_2\ell+m_1)_{(i_2+1)\ell+1}, \ldots, (i_{r+1}\ell+m_r)_{(i_{r+1}+1)\ell+1}.$$

Notice that this covers the remaining $\rho \in \mathcal{F}_\ell(n+2\ell)$ in which the first part of ρ is at least 2ℓ with ρ containing at least two parts. The inverse of f can then be constructed (we leave the details to the reader) in a composite manner in much the same way as f was above upon considering the number of parts and whether or not the first part is at least 2ℓ. $\square$

3 Subscript restrictions only on certain parts

Given integers $\ell, \ell', b, b' \ge 1$, let $\mathcal{D}_{\ell a+b}^{\ell' a'+b'}(\nu)$ denote the number of n-color compositions of ν such that the parts of the form $\ell a+b$ for some $a \ge 0$ have only subscripts of the form $\ell' a'+b'$ for some $a' \ge 0$. Additionally, we denote by $\mathcal{D}_{\ell a+b}^{\ell' a'+b'}(m,\nu)$ the number of such n-color compositions of ν that have exactly m parts.

For example, $\mathcal{D}_{4a+3}^{3a'+1}(3)=6$, the compositions being

$$\{3_1\}, \{2_1, 1_1\}, \{2_2, 1_1\}, \{1_1, 2_1\}, \{1_1, 2_2\}, \{1_1, 1_1, 1_1\}.$$

Theorem 11. *Let $\mathcal{GD}_{\ell a+b}^{\ell' a'+b'}(m,x)$ and $\mathcal{GD}_{\ell a+b}^{\ell' a'+b'}(x)$ denote the generating functions for the sequences $\mathcal{D}_{\ell a+b}^{\ell' a'+b'}(m,\nu)$ and $\mathcal{D}_{\ell a+b}^{\ell' a'+b'}(\nu)$, respectively. Then we have*

$$\mathcal{GD}_{\ell a+b}^{\ell' a'+b'}(m,x) = \left(x^b H(x^\ell) + P(x) + \sum_{i=1,\ i\not\equiv b \pmod{\ell}}^{\ell} \frac{i+(\ell-i)x^\ell}{(1-x^\ell)^2} x^i \right)^m,$$

$$\mathcal{GD}_{\ell a+b}^{\ell' a'+b'}(x) = \frac{x^b H(x^\ell) + P(x) + \sum_{i=1,\ i\not\equiv b \pmod{\ell}}^{\ell} \frac{i+(\ell-i)x^\ell}{(1-x^\ell)^2} x^i}{1-\left(x^b H(x^\ell) + P(x) + \sum_{i=1,\ i\not\equiv b \pmod{\ell}}^{\ell} \frac{i+(\ell-i)x^\ell}{(1-x^\ell)^2} x^i \right)},$$

where $H(x)$ is the generating function of the sequence

$$h_n = \begin{cases} \lfloor \frac{\ell n+b-b'}{\ell'} \rfloor + 1, & \text{if } \ell n + b \geq b' \text{ and } n \geq 0; \\ 0, & \text{otherwise,} \end{cases}$$

and $P(x)$ is the polynomial given by

$$P(x) = \sum_{\substack{i\equiv b \pmod{\ell} \\ 0\leq i<b}} i x^i.$$

Proof. Summing the first expression over $m \geq 1$ gives the second, so we need only prove the first. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ be a non-empty n-color composition having m parts such that parts of the form $\ell a + b$ where $a \geq 0$ have only subscripts of the form $\ell' a' + b'$ where $a' \geq 0$. First assume $\sigma_j \equiv b \pmod{\ell}$ and suppose $\sigma_j = r = \ell a + b$. If $a \geq 0$ and $r \geq b'$, then σ_j contributes to the generating function a $w_a x^r$ term, where

$$w_a = \left\lfloor \frac{\ell a + b - b'}{\ell'} \right\rfloor + 1.$$

If $a \geq 0$ and $r < b'$, then there are no possible such parts for otherwise the index would exceed the part (note that this case can occur only if $b < b'$).

If $a < 0$, then $\sigma_j = r < b$ and there is a contribution to the generating function of rx^r per the definitions, and combining all such r yields the polynomial $P(x)$ defined above. If $\sigma_j \not\equiv b \pmod{\ell}$, then there is again a contribution of rx^r. Thus, for each $i \in [\ell]$ such that $i \not\equiv b \pmod{\ell}$, we have a total contribution of

$$\begin{aligned} &ix^i + (\ell+i)x^{\ell+i} + (2\ell+i)x^{2\ell+i} + \cdots \\ &= ix^i(1 + x^\ell + x^{2\ell} + \cdots) + \ell x^i(x^\ell + 2x^{2\ell} + 3x^{3\ell} + \cdots) \\ &= \frac{ix^i}{1-x^\ell} + \ell x^i \left[\frac{y}{(1-y)^2} \right]_{y=x^\ell} = \frac{ix^i}{1-x^\ell} + \frac{\ell x^{i+\ell}}{(1-x^\ell)^2}, \end{aligned}$$

which gives the final part of the formula for $\mathcal{GD}_{\ell a+b}^{\ell' a'+b'}(m,x)$ above. □

For example, the generating function for the sequence $\mathcal{D}^{4a'+3}_{3a+7}(m,\nu)$ is given by

$$\mathcal{GD}^{4a'+3}_{3a+7}(m,x) = \left(x^7H(x^3) + x + 4x^4 + \frac{(2+x^3)}{(1-x^3)^2}x^2 + \frac{3}{(1-x^3)^2}x^3\right)^m,$$

where $H(x) = \frac{2+x^2+x^3-x^4}{(1-x)^2(1+x+x^2+x^3)}$. Note that $H(x)$ is the generating function for the sequence

$$\{2,2,3,4,5,5,6,7,8,8,9,10,11,11,12,13,14,14,15,16,17,17,18,19,20,20,21,22,\ldots\}.$$

Moreover,

$$\begin{aligned}
&\mathcal{GD}^{4a'+3}_{3a+7}(x)\\
&= \frac{-3x^{19}+2x^{16}-x^{14}-3x^{12}-3x^{11}-3x^9-3x^8+2x^7-3x^6-3x^5-3x^4-3x^3-2x^2-x}{3x^{19}-2x^{16}-x^{15}+x^{14}+4x^{12}+3x^{11}+3x^9+3x^8-2x^7+3x^6+3x^5+3x^4+4x^3+2x^2+x-1}\\
&= x+3x^2+8x^3+21x^4+55x^5+144x^6+372x^7+977x^8+2549x^9+6647x^{10}+\cdots.
\end{aligned}$$

For example, $\mathcal{D}^{4a'+3}_{3a+7}(7) = 372$, as all n-color compositions of $n=7$ are counted except

$$\{7_1\},\{7_2\},\{7_4\},\{7_5\},\{7_6\}.$$

Remark 12. Taking all of the relevant parameters to be one in Theorem 11 gives

$$\mathcal{GD}^{a'+1}_{a+1}(m,x) = \frac{x^m}{(1-x)^{2m}}, \qquad m \geq 1,$$

and

$$\mathcal{GD}^{a'+1}_{a+1}(x) = \frac{x}{1-3x+x^2},$$

which are the generating functions for the number with m parts and the total number of n-color compositions of ν for $\nu \geq 1$, respectively.

By setting $\ell = 2 = \ell'$ in Theorem 11, we have the following corollaries.

Corollary 13 (Theorem 2.4 of [13])**.** *The generating functions for the number of n-color compositions of ν into m parts such that the odd parts have only even subscripts and for the total number of n-color compositions of ν such that the odd parts have only even subscripts are*

$$\mathcal{GD}^{2a'+2}_{2a+1}(m,x) = \left(\frac{2x^2+x^3}{(1-x^2)^2}\right)^m,$$
$$\mathcal{GD}^{2a'+2}_{2a+1}(x) = \frac{2x^2+x^3}{1-4x^2-x^3+x^4}.$$

Corollary 14 (Theorem 2.5 of [13])**.** *The generating functions for the number of n-color compositions of ν into m parts such that the odd parts have only odd subscripts and for the* *total number of n-color compositions of ν such that the odd parts have only odd subscripts are*

$$\mathcal{GD}_{2a+1}^{2a'+1}(m,x) = \left(\frac{x+2x^2}{(1-x^2)^2}\right)^m,$$
$$\mathcal{GD}_{2a+1}^{2a'+1}(x) = \frac{x+2x^2}{1-x-4x^2+x^4}.$$

Corollary 15 (Theorem 2.6 of [13]). *The generating functions for the number of n-color compositions of ν into m parts such that the even parts have only even (odd) subscripts and for the total number of n-color compositions of ν such that the even parts have only even (odd) subscripts are*

$$\mathcal{GD}_{2a+2}^{2a'+2}(m,x) = \mathcal{GD}_{2a+2}^{2a'+1}(m,x) = \left(\frac{x+x^2+x^3}{(1-x^2)^2}\right)^m,$$
$$\mathcal{GD}_{2a+2}^{2a'+2}(x) = \mathcal{GD}_{2a+2}^{2a'+1}(x) = \frac{x+x^2+x^3}{1-x-3x^2-x^3+x^4}.$$

4 A further related restriction

Given $\ell \geq 1$, let $\mathcal{T}_\ell(\nu)$ denote the number of n-color compositions of ν such that any part of the form $\ell a + b$ for some $a \geq 0$ and $1 \leq b \leq \ell$ has a subscript of the same form. Additionally, we denote by $\mathcal{T}_\ell(m,\nu)$ the number of such n-color compositions of ν that have m parts.

For example, $\mathcal{T}_4(5) = 17$, the compositions being

$$\{5_1\},\{5_5\},\{4_4,1_1\},\{1_1,4_4\},\{3_3,2_2\},\{2_2,3_3\},\{3_3,1_1,1_1\},\{1_1,3_3,1_1\},\{1_1,1_1,3_3\},$$
$$\{2_2,2_2,1_1\},\{2_2,1_1,2_2\},\{1_1,2_2,2_2\},\{2_2,1_1,1_1,1_1\},\{1_1,2_2,1_1,1_1\},\{1_1,1_1,2_2,1_1\},$$
$$\{1_1,1_1,1_1,2_2\},\{1_1,1_1,1_1,1_1,1_1\}.$$

Similar to the proof of Theorems 1 and 2 above, we have the following result.

Theorem 16. *Let $\mathcal{GT}_\ell(m,x)$ and $\mathcal{GT}_\ell(x)$ denote the generating functions for the sequences $\mathcal{T}_\ell(m,\nu)$ and $\mathcal{T}_\ell(\nu)$, respectively. Then we have*

$$\mathcal{GT}_\ell(m,x) = \left(\frac{x}{(1-x)(1-x^\ell)}\right)^m,$$
$$\mathcal{GT}_\ell(x) = \frac{x}{1-2x-x^\ell+x^{\ell+1}}.$$

Moreover, the sequence $\mathcal{T}_\ell(m,\nu)$ for $1 \leq m \leq \nu$ is given explicitly by

$$\mathcal{T}_\ell(m,\nu) = \sum_{i=0}^{\left\lfloor \frac{\nu-m}{\ell} \right\rfloor} \binom{m+i-1}{m-1}\binom{\nu-\ell i-1}{m-1},$$

with $\mathcal{T}_\ell(\nu) = 2\mathcal{T}_\ell(\nu-1) + \mathcal{T}_\ell(\nu-\ell) - \mathcal{T}_\ell(\nu-\ell-1)$ for $\nu > \ell+1$.

Note that the sequences $\mathcal{T}_\ell(\nu)$ and $\mathcal{C}_{\ell a+1}(\nu)$ are the same which can be shown using the definitions.

We now describe a statistic on n-color compositions which accounts for the expression given for $\mathcal{T}_\ell(m,\nu)$ above. More precisely, let $\mathcal{S}_\ell(m,\nu)$ denote the set of n-color compositions enumerated by $\mathcal{T}_\ell(m,\nu)$ and we determine a statistic σ on $\mathcal{S}_\ell(m,\nu)$ such that

$$|\{\pi \in \mathcal{S}_\ell(m,\nu) : \sigma(\pi) = i\}| = \binom{m+i-1}{m-1}\binom{\nu-\ell i-1}{m-1}.$$

Given a part α_β of $\pi \in \mathcal{S}_\ell(m,\nu)$, let $\sigma(\alpha_\beta) = \lfloor(\beta-1)/\ell\rfloor$. Define $\sigma(\pi)$ to be the sum of the σ-values of its individual parts. For example, if $\ell = 3$ and $\pi = 5_2, 7_7, 8_5, 12_9, 3_3 \in \mathcal{S}_3(5,35)$, then $\sigma(\pi) = 0+2+1+2+0 = 5$. Note that if β corresponds to the i-th smallest possible subscript on a part of π of size α, then α_β contributes $i-1$ towards the $\sigma(\pi)$ statistic value. If $\ell = 1$, then it is seen that $\sigma(\pi)$ is simply the sum of the subscripts of all the parts minus the number of parts of π. Define

$$t_{\nu,m}^{(\ell)}(q) = \sum_{\pi \in \mathcal{S}_\ell(m,\nu)} q^{\sigma(\pi)}, \qquad \nu \geq m \geq 1,$$

where q is an indeterminate. We have the following explicit formula for $t_{\nu,m}^{(\ell)}(q)$.

Theorem 17. *If $\nu \geq m \geq 1$ and $\ell \geq 1$, then*

$$t_{\nu,m}^{(\ell)}(q) = \sum_{i=0}^{\lfloor \frac{\nu-m}{\ell} \rfloor} \binom{m+i-1}{m-1}\binom{\nu-\ell i-1}{m-1} q^i. \tag{5}$$

Proof. Let σ' be the statistic defined on $\pi \in \mathcal{S}_\ell(m,\nu)$ as follows. Given a part α_β of π, let $\sigma'(\alpha_\beta) = \frac{\alpha-\beta}{\ell}$ and define $\sigma'(\pi)$ to be the sum of the σ' values of its parts. For example, if $\pi \in \mathcal{S}_3(5,35)$ is as before, then $\sigma'(\pi) = 3$. We first show that σ and σ' are identically distributed on $\mathcal{S}_\ell(m,\nu)$. To do so, we change the subscripts on each part of $\pi \in \mathcal{S}_\ell(m,\nu)$ as follows. Let r_s be a part of π. First assume r is not divisible by ℓ. Then $r = \ell a + b$ where $a \geq 0$ and $1 \leq b \leq \ell - 1$ and $s = \ell a' + b$ for some $0 \leq a' \leq a$. In this case, we replace r_s with r_t, where $t = \ell(a-a') + b$. If r is divisible be ℓ, then $r = \ell a$ and $s = \ell a'$ for some $1 \leq a' \leq a$, in which case we replace the part r_s with r_t, where $t = \ell(a-a'+1)$. Let π' denote the resulting member of $\mathcal{S}_\ell(m,\nu)$. One may verify that the mapping $\pi \mapsto \pi'$ is a bijection with $\sigma(\pi) = \sigma'(\pi')$ for all π.

We now count members $\pi \in \mathcal{S}_\ell(m,\nu)$ such that $\sigma'(\pi) = i$ where $0 \leq i \leq \lfloor(\nu-m)/\ell\rfloor$. We denote these π by

$$\pi = (a_1 + \ell b_1)_{a_1}, \ldots, (a_m + \ell b_m)_{a_m},$$

where $a_j \geq 1$ and $b_j \geq 0$ for all j. Then $b_1 + \cdots + b_m = i$ implies there are $\binom{m+i-1}{m-1}$ possibilities for the b_j. Thus, $a_1 + \cdots + a_m = \nu - \ell i$ so that there are $\binom{\nu-\ell i-1}{m-1}$ possibilities for the a_j. Since the a_j and b_j may be chosen independently of one another, it follows that there are $\binom{m+i-1}{m-1}\binom{\nu-\ell i-1}{m-1}$ such π, which completes the proof of (5). □

Let $t_\nu^{(\ell)}(q,u)=\sum_{m=1}^{\nu} t_{\nu,m}^{(\ell)}(q)u^m$ for $\nu\geq 1$ and define the generating function

$$T^{(\ell)}(x;q,u)=\sum_{\nu\geq 1} t_\nu^{(\ell)}(q,u)x^\nu.$$

Using (5) and interchanging summation yields the following result.

Corollary 18. *We have*

$$T^{(\ell)}(x;q,u)=\frac{xu}{1-x(1+u)-x^\ell q+x^{\ell+1}q} \tag{6}$$

and thus

$$t_\nu^{(\ell)}(q,u)=(1+u)t_{\nu-1}^{(\ell)}(q,u)+qt_{\nu-\ell}^{(\ell)}(q,u)-qt_{\nu-\ell-1}^{(\ell)}(q,u), \qquad \nu>\ell+1. \tag{7}$$

Formulas (6) and (7) reduce, respectively, to the generating function and recurrence formulas for $\mathcal{T}_\ell(\nu)$ in Theorem 16 when $q=u=1$. Note that the $\ell=u=1$ case of recurrence (7) was previously considered in [9]. A combinatorial proof may be given for (7) by considering whether or not the last part is 1_1, and if not, whether or not the last part is equal to its subscript. Finally, taking $\ell=2$ in the preceding yields a polynomial generalization of the problem of counting n-color compositions of a given size in which each part and its respective subscript always have the same parity.

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