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Arithmetic Subderivatives: Discontinuity and Continuity

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Abstract

We first prove that any arithmetic subderivative of a rational number defines a function that is everywhere discontinuous in a very strong sense. Second, we show that although the restriction of this function to the set of integers is continuous (in the relative topology), it is not Lipschitz continuous. Third, we see that its restriction to a suitable infinite set is Lipschitz continuous. This follows from the solutions of certain arithmetic differential equations.

1 Introduction

Let $0 \neq x \in \mathbb{Q}$. There exists a unique sequence $(\nu_p(x))_{p\in\mathbb{P}}$ of integers (with only finitely many nonzero terms) such that

$$x = (\operatorname{sgn} x) \prod_{p\in\mathbb{P}} p^{\nu_p(x)}. \tag{1}$$

Here $\mathbb{P}$ stands for the set of primes, and $\operatorname{sgn} x = x/|x|$. Define $\operatorname{sgn} 0 = 0$ and $\nu_p(0) = 0$ for all $p \in \mathbb{P}$; then Eq. (1) also holds for $x = 0$.

Let $\emptyset \neq S \subseteq \mathbb{Q}$. The *arithmetic subderivative* [5] of $x \in \mathbb{Q}$ with respect to S is

$$x'_S = D_S(x) = x \sum_{p\in S} \frac{\nu_p(x)}{p}.$$

The *arithmetic partial derivative* [4, 3] of x with respect to $p \in \mathbb{P}$ is $x'_p = D_p(x) = D_{\{p\}}(x)$. The *arithmetic derivative* [6, 2, 8] of x is $x' = D(x) = D_{\mathbb{P}}(x)$. Arithmetic differential equations and arithmetic partial differential equations have been studied [8, 4, 3, 7].

We have $D_S(p) = 1$ for all $p \in S$. If x is near to p and suitably chosen, then $D_S(x)$ can be far from one. So it is natural to expect that D_S is discontinuous at any $a \in \mathbb{P}$. However, it is not equally apparent that D_S is discontinuous at an arbitrary $a \in \mathbb{Q}$.

We say that a real-valued function f, defined on an open real (or rational) interval I, is *superdiscontinuous* at $a \in I$ if it attains values of an arbitrarily large absolute value arbitrarily near to a and on its both sides. More precisely, given any $\delta, M > 0$, there are $x, y \in I$ such that

$$a - \delta < x < a < y < a + \delta \quad \text{and} \quad |f(x)|, |f(y)| > M.$$

If x (respectively, y) exists, then f is *superdiscontinuous from the left* (*right*) at a. We prove in Section 2 that D_S is superdiscontinuous at any $a \in \mathbb{Q}$. We need two lemmas.

Lemma 1. *Let $p, q \in \mathbb{P}$, $p \neq q$. The set $\{\pm p^m q^n \mid m, n \in \mathbb{Z}\}$ is dense in $\mathbb{Q}$.*

Proof. See the proof of [3, Lemma 30]. □

Lemma 2. *Let f be a real-valued function, defined on an open real (or rational) interval I, and let $a \in I$. If f is superdiscontinuous at each point of $I \setminus \{a\}$, then it is superdiscontinuous also at a.*

Proof. Let $\delta, M > 0$. It is no restriction to assume that $a \pm \delta \in I$. Since f is superdiscontinuous (from the right) at $a - \delta$, there is $x \in I$ satisfying $a - \delta < x < a$ and $|f(x)| > M$. Similarly, since f is superdiscontinuous (from the left) at $a + \delta$, there is $y \in I$ satisfying $a < y < a + \delta$ and $|f(y)| > M$. □

Let f and I be as above. The function f is *Lipschitz continuous* (in I) if there exists $L > 0$ satisfying

$$|f(x) - f(y)| \leq L|x - y| \tag{2}$$

for all $x, y \in I$. Since a Lipschitz continuous function is continuous in the ordinary sense, the results in Section 2 imply that D_S is Lipschitz discontinuous. But the restriction of D_S to $\mathbb{Z}$, denoted by $D_S|_{\mathbb{Z}}$, is continuous (in the relative topology); so, is it Lipschitz continuous? We show in Section 3 that it is not. This raises a further question: Is there an infinite set $A \subset \mathbb{Q}$ such that $D_S|_A$ is Lipschitz continuous? Applying the previously known solutions of the differential equations $x' = 0$ and $x'_p = ax$, we give a positive answer. This, in turn, motivates us to consider the more general differential equation $x'_S = ax$. We solve it in Section 4. Finally, we complete our paper with concluding remarks in Section 5.

2 Superdiscontinuity of D_S

We state in Theorem 3 an additional assumption, in order to make the proof easier.

Theorem 3. *Let $\emptyset \neq S \subseteq \mathbb{P}$. If there are distinct $p, q \in \mathbb{P} \setminus S$, then D_S is superdiscontinuous at any $a \in \mathbb{Q}$.*

Proof.
Case 1: $a > 0$. Let $\delta, M > 0$, $s \in S$, and $k \in \mathbb{Z}_+$. By Lemma 1, there are $m, n \in \mathbb{Z}$ such that

$$\frac{a}{s^k} < p^m q^n < \frac{a}{s^k} + \frac{\delta}{s^k}.$$

Then $x = s^k p^m q^n$ satisfies

$$a < x < a + \delta.$$

The subderivative

$$D_S(x) = ks^{k-1} p^m q^n > ks^{k-1} \frac{a}{s^k} = \frac{ka}{s} > M$$

if

$$k > \frac{sM}{a}.$$

Consequently, superdiscontinuity from the right follows.

To prove superdiscontinuity from the left, we use the above notation but let $\delta < a$ (which is no restriction). Now, by Lemma 1, there are $m, n \in \mathbb{Z}$ such that

$$\frac{a}{s^k} - \frac{\delta}{s^k} < p^m q^n < \frac{a}{s^k}.$$

Hence,

$$a - \delta < x < a,$$

and

$$D_S(x) = ks^{k-1}p^m q^n > ks^{k-1}\frac{a-\delta}{s^k} = \frac{k(a-\delta)}{s} > M$$

if

$$k > \frac{sM}{a-\delta}.$$

This verifies superdiscontinuity from the left.

Case 2: $a < 0$. Apply the fact that $D_S(a) = -D_S(-a)$.

Case 3: $a = 0$. Apply Lemma 2. ☐

What about trying the same idea for $S = \mathbb{P} \setminus \{p\}$, $p \in \mathbb{P}$? It is enough to assume that $a > 0$. Let $q \in S$ and $T = S \setminus \{q\}$; then $p, q \notin T$. Let δ, M, s, and k be as above. Again, there are m and n such that $x = s^k p^m q^n$ satisfies

$$a < x < a + \delta.$$

If

$$k > \frac{sM}{a},$$

then

$$D_S(x) = D_T(x) + \frac{n}{q}s^k p^m q^n > M + ns^k p^m q^{n-1}.$$

Consequently, if $n \geq 0$ (i.e., $\nu_q(x) \geq 0$ for some $q \in S$), then $D_S(x) > M$, and superdiscontinuity from the right follows. Unfortunately, a problem arises if $n < 0$ (i.e., $\nu_q(x) < 0$ for all $q \in S$). Then $D_S(x) < 0$, and we get nothing reasonable. It seems that our method cannot be revised to work in this case. Instead of a single x, we apparently should consider a suitable sequence (x_i). We do this in the proof of the next theorem, where S is arbitrary. Theorem 3 could now be omitted, but we find its proof interesting on its own and therefore keep it.

Theorem 4. *Let $\emptyset \neq S \subseteq \mathbb{P}$. Then D_S is superdiscontinuous at any $a \in \mathbb{Q}$.*

Proof. We can assume that $a \neq 0$; then superdiscontinuity at $a = 0$ follows from Lemma 2. Let $p, q \in \mathbb{P}$, $p \neq q$, and denote $m = \nu_p(a)$, $n = \nu_q(a)$. Then

$$a = p^m q^n r,$$

where $r \neq 0$ has $\nu_p(r) = \nu_q(r) = 0$. We show that there is a sequence (x_i) of rational numbers with

$$x_i \to a, \quad x_1, x_2, \ldots > a, \tag{3}$$

and

$$|D_S(x_i)| \to \infty. \tag{4}$$

By a simple modification of this proof, it can be seen that there also exists a sequence (x_i) satisfying

$$x_i \to a, \quad x_1, x_2, \ldots < a$$

and (4).

By Lemma 1, there are integer sequences (m_i) and (n_i) such that the sequence

$$(x_i),\, x_i = p^{m_i} q^{n_i}$$

satisfies (3). To prove (4), we must know their limiting behaviour.

Case 1: Both (m_i) and (n_i) are bounded from above. Then they are bounded also from below, since otherwise $x_i \to 0$ or (x_i) diverges, contradicting (3). Therefore, (x_i) has only finitely many different terms, implying that its limit a is one of them. This again contradicts (3). Consequently, this case is impossible.

Case 2: The sequence (m_i) is not bounded from above. If, instead, (n_i) is not bounded from above, we can proceed similarly. Now, (m_i) has a subsequence (m_{i_k}) satisfying $m_{i_k} \to \infty$. Since $p^{m_{i_k}} \to \infty$ but $x_{i_k} \to a \neq 0$, necessarily $q^{n_{i_k}} \to 0$, i.e., $n_{i_k} \to -\infty$. Including also the case when (n_i) is not bounded from above, we therefore have

$$m_{i_k} \to \pm\infty \quad \text{and} \quad n_{i_k} \to \mp\infty. \tag{5}$$

In order to keep the notation simple, we let (x_i) denote the subsequence (x_{i_k}). Then Eq. (5) reads

$$m_i \to \pm\infty \quad \text{and} \quad n_i \to \mp\infty. \tag{6}$$

By the convergence, the sequence (x_i) is bounded also from above. Let $u \in \mathbb{Q}$ satisfy

$$x_1, x_2, \ldots < u.$$

We can assume in Eq. (6) that

$$m_i \to -\infty \quad \text{and} \quad n_i \to \infty. \tag{7}$$

If the signs are opposite, then a simple modification applies. There is $i_0 \in \mathbb{Z}_+$ such that $n_i > 0$ for all $i \geq i_0$. Denote

$$c_i = \frac{m_i - m}{n_i}, \quad \text{where} \quad i \geq i_0.$$

Then

$$\begin{aligned}
a < x_i < u \iff p^m q^n r < p^{m_i} q^{n_i} < u \iff& \\
q^{n-n_i} r < p^{m_i - m} < p^{-m} q^{-n_i} u \iff& \\
(n - n_i)\log_p q + \log_p r < m_i - m < -m - n_i \log_p q + \log_p u \iff& \\
(n - n_i + \log_q r)\log_p q < m_i - m < -m + (-n_i + \log_q u)\log_p q& \\
\iff \left(\frac{n + \log_q r}{n_i} - 1\right)\log_p q < c_i < -\frac{m}{n_i} + \left(\frac{\log_q u}{n_i} - 1\right)\log_p q.&
\end{aligned} \tag{8}$$

By Eq. (7),

$$\Big(\frac{n+\log_q r}{n_i}-1\Big)\log_p q \to -\log_p q, \quad -\frac{m}{n_i}+\Big(\frac{\log_q u}{n_i}-1\Big)\log_p q \to -\log_p q.$$

Hence, by Eq. (8),

$$c_i \to -\log_p q. \tag{9}$$

The function D_S has not yet had any role. Let us now focus on it.

If $p, q \in S$ (i.e., there are at least two elements in S), then

$$\begin{aligned} D_S(x_i) = x_i\Big(\frac{m_i}{p}+\frac{n_i}{q}\Big) = x_i\Big(\frac{m+c_i n_i}{p}+\frac{n_i}{q}\Big) = \\ x_i\frac{m}{p}+x_i\Big(\frac{c_i}{p}+\frac{1}{q}\Big)n_i =: A_i + B_i. \end{aligned} \tag{10}$$

By (3),

$$A_i \to a\frac{m}{p}. \tag{11}$$

Further, (3) and (9) imply that

$$x_i\Big(\frac{c_i}{p}+\frac{1}{q}\Big) \to a\Big(\frac{1}{q}-\frac{\log_p q}{p}\Big) = \frac{a}{p}\Big(\frac{p}{q}-\log_p q\Big) \neq 0,$$

because $a \neq 0$ and $p^p \neq q^q$. Consequently, by Eq. (7),

$$|B_i| \to \infty. \tag{12}$$

Finally, Eqs. (12), (11), and (10) imply (4).

The case of $S = \{p\}$ (i.e., there is only one element in S) remains. Then

$$D_S(x_i) = x_i\frac{m_i}{p}.$$

Since $x_i \to a(\neq 0)$ and $m_i \to \pm\infty$, the result (4) again follows. □

3 Lipschitz discontinuity of $D_S|_{\mathbb{Z}}$

We recall Dirichlet's theorem on arithmetic progressions.

Theorem 5. *Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}_+$. If $\gcd(a, b) = 1$, then there exist infinitely many primes of the form $a + kb$, where $k \in \mathbb{Z}_+$.*

Proof. See the proof of [1, Theorem 7.9]. □

Since all integer points are isolated (i.e., for all $a \in \mathbb{Z}$, there is $r > 0$ such that $N(a,r) \cap \mathbb{Z} = \{a\}$, where $N(a,r) = \{x \in \mathbb{Q} \mid |x-a| < r\}$), the restriction $D_S|_{\mathbb{Z}}$ is continuous (in the relative topology). However, we prove that even $D_S|_{\mathbb{Z}_+}$ is Lipschitz discontinuous.

Theorem 6. *Let $\emptyset \neq S \subseteq \mathbb{P}$. The restriction $D_S|_{\mathbb{Z}_+}$ is Lipschitz discontinuous.*

Proof. Let $L > 0$. We show that there exist $x, y \in \mathbb{Z}_+$ such that

$$x - y = 1 \quad \text{and} \quad D_S(x) - D_S(y) > L. \tag{13}$$

Let $p \in S$. Choose $n \in \mathbb{Z}_+$ with

$$np^{n-1} > L + 1. \tag{14}$$

By Theorem 5, there is $k \in \mathbb{Z}_+$ such that

$$y = kp^n - 1 \in \mathbb{P}.$$

Let $x = kp^n$. Then

$$x - y = 1, \quad D_S(y) \in \{0, 1\}, \tag{15}$$

and

$$D_S(x) = kD_S(p^n) + p^n D_S(k) \geq kD_S(p^n) = knp^{n-1} \geq np^{n-1}. \tag{16}$$

Now, Eqs. (15), (16), and (14) imply Eq. (13). □

On the other hand, we give two examples showing that if $S = \mathbb{P}$ or $S = \{p\}$, $p \in \mathbb{P}$, then there is an infinite set $A \subset \mathbb{Q}$ such that $D_S|_A$ is Lipschitz continuous.

Example 7. Let $x \in \mathbb{Q}$. Then [8, Theorem 15]

$$x' = 0 \tag{17}$$

if (not "only if", contrary to this theorem)

$$x = 0 \quad \text{or} \quad x = \pm \prod_{p \in \mathbb{P}} p^{\xi_p(x)p}, \tag{18}$$

where $(\xi_p(x))_{p \in \mathbb{P}}$ is an integer sequence with a finite number of nonzero terms such that

$$\sum_{p \in \mathbb{P}} \xi_p(x) = 0.$$

We present a correct "only if" part in Theorem 9. Anyway, the set

$$A = \{x \in \mathbb{Q} \mid x \,\text{satisfies}\, (18)\}$$

is clearly infinite. Since the restriction $D|_A$ is identically zero by Eq. (17), it is Lipschitz continuous.

Example 8. Let $p \in \mathbb{P}$ and $a \in \mathbb{Q}$ satisfy

$$ap \in \mathbb{Z}, \tag{19}$$

and let $x \in \mathbb{Q}$. Then [3, Theorem 3]

$$x'_p = ax \tag{20}$$

if and only if

$$x = cp^{ap}, \quad \text{where} \quad \nu_p(c) = 0. \tag{21}$$

The set

$$A = \{x \in \mathbb{Q} \mid x \,\text{satisfies}\,(21)\}$$

is clearly infinite. By Eq. (20), we have $D_p|_A(x) = ax$. It is Lipschitz continuous with $L = |a|$.

4 The differential equation $x'_S = ax$

We extend Examples 7 and 8.

Theorem 9. *Let $\emptyset \neq S \subseteq \mathbb{P}$ and $a, x \in \mathbb{Q}$. Then*

$$x'_S = ax \tag{22}$$

if and only if

$$x = c \prod_{p \in S} p^{\xi_p(x)p}. \tag{23}$$

Here

$$\nu_p(c) = 0 \quad \textit{for all} \quad p \in S, \tag{24}$$

and ($c = 0$ or) $(\xi_p(x))_{p \in S}$ is a sequence of rational numbers with a finite number of nonzero terms (this is needed if S is infinite) satisfying

$$\xi_p(x)p \in \mathbb{Z} \quad \textit{for all} \quad p \in S \tag{25}$$

and

$$\sum_{p \in S} \xi_p(x) = a. \tag{26}$$

Proof. If x is as in Eq. (23), then, by Eq. (26), we get

$$x'_S = x \sum_{p \in S} \frac{\xi_p(x)p}{p} = x \sum_{p \in S} \xi_p(x) = xa,$$

verifying Eq. (22).

Conversely, suppose that x satisfies Eq. (22). We show that it satisfies Eq. (23). The case of $x = 0$ is trivial. If the claim holds for x, then it also holds for $-x$. Therefore, it is enough to consider

$$x = \prod_{p \in \mathbb{P}} p^{\nu_p(x)}.$$

Denoting $\nu_p = \nu_p(x)$ for short, we can write

$$x = c \prod_{p \in S} p^{\nu_p} = c \prod_{p \in S} p^{\xi_p p},$$

where c satisfies Eq. (24) and

$$\xi_p = \frac{\nu_p}{p}, \quad p \in S.$$

Now,

$$\sum_{p \in S} \xi_p = \sum_{p \in S} \frac{\nu_p}{p} = \frac{x'_S}{x} = \frac{ax}{x} = a$$

by Eq. (22), i.e., Eq. (26) holds. Also Eq. (25) is obviously satisfied. Thus Eq. (23) follows. $\square$

The set

$$A = \{x \in \mathbb{Q} \mid x \,\text{satisfies}\, (23)\}$$

is clearly infinite. By Eq. (22), the restriction $D_S|_A(x) = ax$. It is Lipschitz continuous with $L = |a|$.

If $S = \mathbb{P}$ and $a = 0$, then Eq. (23) reads

$$x = c \prod_{p \in \mathbb{P}} p^{\xi_p(x) p}.$$

Here

$$\sum_{p \in \mathbb{P}} \xi_p(x) = 0$$

and $\nu_p(c) = 0$ for all $p \in \mathbb{P}$, i.e., $c \in \{0, \pm 1\}$. Thus we encounter Eq. (18), but it is enough that $\xi_p(x)p \in \mathbb{Z}$ for all $p \in \mathbb{P}$, not necessarily $\xi_p(x) \in \mathbb{Z}$.

If $S = \{p\}$, then

$$a = \sum_{q \in \{p\}} \xi_q(x) = \xi_p(x).$$

Hence, by Eq. (23),

$$x = c \prod_{q \in \{p\}} p^{\xi_q(x) q} = c p^{\xi_p(x) p} = c p^{ap}, \quad \text{where} \quad \nu_p(c) = 0,$$

repeating Eq. (21). The condition (25) reduces to (19).

5 Concluding remarks

We proved that D_S is superdiscontinuous at any $a \in \mathbb{Q}$. We also proved that its restriction to a suitable infinite set A is Lipschitz continuous. This happens if, for example, the set A consists of the solutions of $x' = 0$ or those of

$$x'_p = ax, \tag{27}$$

where $ap \in \mathbb{Z}$. These equations have been discussed in the literature, while the more general equation,

$$x'_S = ax, \tag{28}$$

apparently has not. Anyway, it also provides us a suitable A if there is a sequence $(\xi_p(x))_{p\in S}$ satisfying Eqs. (25) and (26).

According to Example 8, $ap \in \mathbb{Z}$ is a sufficient condition for Eq. (27) to have nontrivial solutions. In fact, it is also necessary [3, Theorem 3]. But what about Eq. (28)? It has nontrivial solutions if and only if there are $p_1, \ldots, p_k \in S$ such that $ap_1 \cdots p_k \in \mathbb{Z}$. We will present the proof in a forthcoming paper.

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