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A Combinatorial Classification of Triangle Centers on the Line at Infinity

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Abstract

Many triangle centers on the line at infinity have barycentric coordinates that are polynomials. These are classified first by two types, called even and odd, and then further classified by bases with respect to which the polynomials are linear combinations. For each positive integer n, the polynomials in a basis are determined by the partitions of n into at most three parts.

1 Introduction

Suppose that ABC is a triangle with sidelengths $a = |BC|, b = |CA|, c = |AB|$. Every point P in the plane of ABC, extended to the line at infinity, has homogeneous barycentric coordinates [8]. We regard a, b, c as real variables, so that the barycentric coordinates (henceforth barycentrics) for P are regarded as functions of a, b, c. We are interested in cases in which the barycentrics of P are polynomials. Consider, for example, the circumcenter, for which barycentrics are $\cos A : \cos B : \cos C$. By the law of cosines, this point also has barycentrics

$$(b^2 + c^2 - a^2)/(2bc) : (c^2 + a^2 - b^2)/(2ca) : (a^2 + b^2 - c^2)/(2ab).$$

Multiplying through by $2abc$, we see that the circumcenter is given by polynomials:

$$p(a, b, c) : p(b, c, a) : p(c, a, b),$$

where $p(a, b, c) = a(b^2 + c^2 - a^2)$.

Generalizing, a function $p(a, b, c)$ is a *polynomial center function* if it satisfies these two conditions:

$$p(ta, tb, tc) = t^n p(a, b, c) \text{ for some nonnegative integer } n,$$
$$p(a, b, c) = p(a, c, b),$$

i.e., $p(a, b, c)$ is homogeneous of degree $n \geq 0$ and bisymmetric in b and c.

A *polynomial triangle center* (henceforth PTC) is a point in the plane of ABC whose barycentrics are

$$p(a, b, c) : p(b, c, a) : p(c, a, b)$$

for some polynomial center function. We shall frequently shorten the notation to $p(a, b, c) ::$.

Every PTC is given by many polynomial center functions; viz., for $X = p(a, b, c) ::$, we also have $X = s(a, b, c)p(a, b, c) ::$ for every function $s(a, b, c)$ that is symmetric in a, b, c. The letters IPTC will represent a PTC that lies on the line at infinity, L^∞, which consists of all points $x : y : z$ satisfying

$$x + y + z = 0. \tag{1}$$

The line L^∞ is important in triangle geometry for reasons such as these five:

(i) Each line L meets L^∞ in a point, $P(L)$, and *all* lines parallel to L meet L^∞ in $P(L)$. Specifically, every line L is given by an equation of the form

$$ux + vy + wz = 0, \tag{2}$$

from which we find $P(L) = v - w : w - y : u - v$. Moreover, the point $P(L)$ is an IPTC if and only if $u : v : w$ is a PTC. For example, the Euler line, given as in (2) by

$$(u, v, w) = \Big((b^2 + c^2 - a^2)(b^2 - c^2), (c^2 + a^2 - b^2)(c^2 - a^2), (a^2 + b^2 - c^2)(a^2 - b^2)\Big),$$

intersects L^∞ in the Euler infinity point, given by

$$2a^4 - (b^2 - c^2)^2 - a^2(b^2 + c^2) :: .$$

(ii) Conversely, each IPTC is the point on L^∞ common to a family of parallel lines that are "polynomial" in the sense that the coordinates u, v, w as in (2) are those of the PTC $u : v : w$.

(iii) Consequently, asymptotes of hyperbolas, axes of parabolas, and lines associated with cubics [1] intersect L^∞, and in many cases, these intersections are IPTCs, as in the case of the Kiepert hyperbola (but not the Jerabek hyperbola).

(iv) Isogonal conjugation maps the circumcircle, denoted by Γ, onto L^∞ (and L^∞ onto Γ). If $u : v : w$ is a PTC on Γ, then its isogonal conjugate, $a^2vw : b^2wu : c^2uv$, is an IPTC.

(v) Let $*$ denote barycentric product [8], defined for $P = p : q : r$ and $U = u : v : w$ by

$$P * Q = pu : qr : rw.$$

If L is a line then there exists a point P such that $P * L = L^\infty$, so that L is the barycentric quotient L^∞/P. Here, a point U is a PTC on L if and only if $P * L$ is an IPTC; see Section 6 for more about this.

The author [4] showed that the set of PTCs can be partitioned into two classes, designated as even and odd. Specifically, a triangle center X is an even PTC if it has a representation $p(a, b, c) ::$ where p is a polynomial center function such that $p(a, b, b) \neq 0$; that is, $p(a, b, c)$ is not a polynomial multiple of $b - c$. Otherwise, X is an odd PTC.

Examples of simple PTCs are $1 ::$ (centroid), $a ::$ (incenter), $a^2 ::$ (symmedian point), $a^2(b^2 + c^2) - (b^2 - c^2)^2 ::$ (center of the nine-point circle) and $b + c ::$, this last point being $X(37)$ in the Encyclopedia of Triangle Centers [2]. Among the simplest even IPTCs are $X(519) = 2a - b - c ::$, $X(524)) = 2a^2 - b^2 - c^2 ::$, and $X(536) = 2bc - ca - ab ::$; among the simplest odd IPTCs are $X(514) = b - c ::$, $X(513) = ab - ac ::$, and $X(523) = b^2 - c^2 ::$.

Every PTS, when written in trilinear coordinates [4], has a first coordinate that is a linear combination (with real coefficients) of polynomials of the form

$$a^h(b^i c^j + b^j c^i), \tag{3}$$

where $h \geq 0, i \geq 0, j \geq 0$. Now, a point with trilinear coordinates $p : q : r$ has barycentrics coordinates $ap : bq : cr$, so that in the preceding sentence, the word "trilinear" can be replaced by "barycentric" [5]; following is a proof. Suppose that $U = u(a, b, c) ::$ is a PTC. Trilinears for U are $(1/a)u(a, b, c) ::$, which is a linear combination of polynomials of the form $a^H(b^I c^J + b^J c^I)$. Switching to barycentrics, we have U as the same linear combination of polynomials of the form $a^h(b^i c^j + b^j c^i)$, where $h = H + 1, i = I$, and $j = J$.

As an example for representing an IPTC as a linear combination of polynomials as in (3), consider the IPTC $X(514)$:

$$\begin{aligned} b - c :: &= (b - c)^2(a - b)(a - c) :: \\ &= a^2(b^2 + c^2) - 2a^2bc - a(b^3 + c^3) \\ &+ a(b^2c + bc^2) + b^3c + bc^3 - 2b^2c^2 ::, \end{aligned}$$

which is a linear combination as asserted.

2 Even IPTCs

The set $\mathcal{S}_n$ of even IPTCs of fixed degree n can be generated by a set of polynomials (3) for which $h + i + j = n$, in the sense that a first coordinate for each point X is a linear combination of polynomials as in (3). If such a set of polynomials are linearly independent, we call it a basis for $\mathcal{S}_n$. The main purpose of this section is to construct a specific basis, denoted by $\mathcal{B}_n$, in a manner that depends on the partitions of n.

Regarding symmetric polynomials, we use the notation (i,j,k) for the sum $\Sigma a^i b^j c^k$, and abbreviate it if any exponent is 0, so that, as examples,

$$\begin{aligned}(6) &= a^6+b^6+c^6,\\ (4,2) &= a^4b^2+a^4c^2+b^4c^2+b^4a^2+c^4a^2+c^4b^2,\\ (3,2,1) &= a^3b^2c+a^3c^2b+b^3c^2a+b^3a^2c+c^3a^2b+c^3b^2a.\end{aligned}$$

For fixed n, the triples (i,j,k) of exponents exactly match the elements of the set $P(n)$ of partitions of n into at most 3 parts, as in the Online Encyclopedia of Integer Sequences ([7], A001399). We partition $P(n)$ into seven classes:

$$\mathcal{C}_1 = \{(n)\}, \tag{4}$$
$$\mathcal{C}_2 = \{(n-h,h) : h = 1,2,\ldots,\lfloor (n-1)/2\rfloor\}, \tag{5}$$
$$\mathcal{C}_3 = \{(h,h,n-2h) : h = \lfloor (n+3)/3\rfloor,\ldots,\lfloor (n-1)/2\rfloor\}, \tag{6}$$
$$\mathcal{C}_4 = \{(n-2k,k,k) : k = 1,2,\ldots,\lfloor (n-1)/3\rfloor\}, \tag{7}$$
$$\mathcal{C}_5 = \{i,j,k) : i>j>k, i+j+k=n\}, \tag{8}$$
$$\mathcal{C}_6 = \{(n/2,n/2)\} \text{ if } 2|n, \text{ and } \emptyset \text{ otherwise}, \tag{9}$$
$$\mathcal{C}_7 = \{(n/3,n/3,n/3)\} \text{ if } 3|n, \text{ and } \emptyset \text{ otherwise}. \tag{10}$$

Corresponding to the classes $\mathcal{C}_i$ are the following polynomials that are bisymmetric in b and c:

$$\overline{n} = a^n, \tag{11}$$
$$\overline{n-h,h} = a^{n-h}(b^h+c^h), h \text{ as in } (5), \tag{12}$$
$$\overline{h,h,n-2h} = a^h(b^{n-2h}+c^{n-2h}), h \text{ as in } (6), \tag{13}$$
$$\overline{n-2k,k,k} = a^{n-2k}(b^k+c^k), k \text{ as in } (7), \tag{14}$$
$$\overline{i,j,k} = a^i(b^jc^k+b^kc^j), i,j,k \text{ as in } (8), \tag{15}$$
$$\overline{h,h} = a^h(b^h+c^h) \text{ if } 2|n, h=n/2, \text{ as in } (9), \tag{16}$$
$$\overline{h,h,h} = a^h(b^h+c^h) \text{ if } 3|n, h=n/3, \text{ as in } (10). \tag{17}$$

In the polynomials (11)-(17), the exponent on a exceeds the exponents on b and c. We extend the bar-notation to polynomials in which the exponent on a is less than that on b and c, as in these examples:

$$\begin{aligned}\overline{0n} &= b^n+c^n,\\ \overline{h,n-h} &= a^h(b^{n-h}+c^{n-h}), h=1,2,\ldots,\lfloor (n-1)/2)\rfloor,\\ \overline{0,h,n-h} &= a^0(b^hc^{n-h}+b^{n-h}c^h), h=1,2,\ldots,\lfloor (n-1)/2)\rfloor.\end{aligned}$$

Corresponding to the seven extended classes of polynomials are the following classes of polynomials:

$$\begin{aligned}
\mathcal{S}_1 &= \{\overline{n}, \overline{0n}\}, && (18)\\
\mathcal{S}_{2,h} &= \{\overline{n-h,h}, \overline{h,n-h}, \overline{0,h,n-h}\}, h \text{ as in } (5), && (19)\\
\mathcal{S}_{3,h} &= \{\overline{h,h,n-2h}, \overline{n-2h,h,h}\}, h \text{ as in } (6), && (20)\\
\mathcal{S}_{4,h} &= \{\overline{n-2k,k,k}, \overline{k,n-2k,k}\}, k \text{ as in } (7), && (21)\\
\mathcal{S}_{5,i,j,k} &= \{\overline{i,j,k}, \overline{j,k,i}, \overline{k,i,j}\}, i,j,k \text{ as in } (8), && (22)\\
\mathcal{S}_6 &= \{\overline{h,h} \text{ if } 2|n, h = n/2\}, && (23)\\
\mathcal{S}_7 &= \{\overline{h,h,h} \text{ if } 3|n, h = n/3\}. && (24)
\end{aligned}$$

It follows from the definitions of these classes that several are empty for small n; specifically,

$$\begin{aligned}
\mathcal{S}_{2,h} &= \emptyset \text{ for } n \leq 2, \text{ since } n-h > h,\\
\mathcal{S}_{3,h} &= \emptyset \text{ for } n \leq 4, \text{ since } h > n-2h,\\
\mathcal{S}_{4,h} &= \emptyset \text{ for } n \leq 3, \text{ since } n-2k > k,\\
\mathcal{S}_{5,1,j,k} &= \emptyset \text{ for } n \leq 5.
\end{aligned}$$

For each $n \geq 1$, let $\mathcal{E}_n$ be the union of sets of polynomials listed in (18)-(24), for all $h, i, j,$ and k as indicated in (18)-(24).

Lemma 1. *Suppose that $n \geq 1$. Then the polynomials in $\mathcal{E}_n$ are linearly independent.*

Proof. This is a corollary to Theorem 2.1.1 in [6]. □

Theorem 2. *If $n \geq 1$, the following polynomials comprise a basis $\mathcal{B}_n$ for the set of even IPTCs.*

$$\begin{aligned}
&2a^n - b^n - c^n, && (25)\\
&a^{n-h}(b^h + c^h) - (b^h c^{n-h} + b^{n-h} c^h), h \textit{ as in } (19), && (26)\\
&a^h(b^{n-h} + c^{n-h}) - (b^h c^{n-h} + b^{n-h} c^h), h \textit{ as in } (19), && (27)\\
&2a^h(b^h c^{n-2h} + b^{n-2h} c^h) - 2a^{n-2h} b^h c^h, h \textit{ as in } (20), && (28)\\
&2a^{n-2k} b^k c^k - a^k(b^{n-2k} c^k + b^k c^{n-2k}), k \textit{ as in } (21), && (29)\\
&a^i(b^j c^k + b^k c^j) - a^k(b^i c^j + b^j c^i), i,j,k \textit{ as in } (22), && (30)\\
&a^j(b^k c^i + b^i c^k) - a^k(b^i c^j + b^j c^i), i,j,k \textit{ as in } (22), && (31)\\
&2b^{n/2} c^{n/2} - a^{n/2} b^{n/2} - a^{n/2} c^{n/2} \textit{(included in } \mathcal{B}_n \textit{ only if } n \textit{ is even).} && (32)
\end{aligned}$$

Proof. Suppose that $p = p(a,b,c)$ :: is an even IPTC, reduced in the sense that the only common factor of $p(a,b,c), p(b,c,a),$ and $p(c,a,b)$ is a constant, so that the polynomial $p(a,b,c)$ is a linear combination of polynomials in $\mathcal{E}_n$. Let m be the number of these polynomials, and index them as p_i for $i = 1, 2, \ldots, m$, where the indexing is in the order in which the polynomials are defined: $p_1 = \overline{n}$, $p_2 = \overline{0n}$, $p_4 = \overline{1, n-1}$, $p_5 = \overline{0, 1, n-1}$, $p_6 = \overline{n-2, 2}$ (assuming $n \geq 5$), and so on. For $i = 1, 2, \ldots, m$, let h_i be the coefficient of $p_i(a, b, c)$:

$$p(a, b, c) = \sum_{i=1}^{m} h_i p_i(a, b, c). \tag{33}$$

Equation (1), applied to $(x, y, z) = (p(a, b, c), p(b, c, a), p(c, a, b))$, yields

$$\sum_{i=1}^{m} h_i(p_i(a, b, c) + p_i(b, c, a) + p_i(c, a, b)) = 0. \tag{34}$$

Included as summands in (34) are sums

$$\sum h_{i_k}(p_{i_k}(a, b, c) + p_{i_k}(b, c, a) + p_{i_k}(c, a, b)) = 0. \tag{35}$$

for which the polynomials p_{i_k} range through each of the seven types in (18)-24). We consider each type individually.

Type 1: Corresponding to $\overline{n}$ and $\overline{0n}$, the contribution (35) to (34) is

$$h_1(a^n + b^n + c^n) + h_2(b^n + c^n + c^n + a^n + a^n + b^n) = h_1(n) + 2h_2(n).$$

By Lemma 1, $h_1 + 2h_2 = 0$, so that we can (and do) take $h_1 = 2$ and $h_2 = -1$ to construct the polynomial (25) as a member of $\mathcal{B}_n$.

Type 2: Corresponding to $\overline{n-h, n}, \overline{h, n-h}$, and $\overline{0, h, n-h}$, where

$$h \in \{1, 2, \ldots, \lfloor (n-1)/2 \rfloor\},$$

the contribution (35) to (34) has the form

$$\begin{aligned}
&k_1 a^{n-h}(b^h + c^h) + k_2 a^h(b^{n-h} + c^{n-h}) + k_3(b^h c^{n-h} + b^{n-h} c^h) \\
&+ k_1 b^{n-h}(c^h + a^h) + k_2 b^h(c^{n-h} + a^{n-h}) + k_3(c^h a^{n-h} + c^{n-h} a^h) \\
&+ k_1 c^{n-h}(a^h + b^h) + k_2 c^h(a^{n-h} + b^{n-h}) + k_3(a^h b^{n-h} + a^{n-h} b^h) \\
&= (k_1 + k_2 + k_3)(n-h, h).
\end{aligned}$$

By Lemma 1, $k_1 + k_2 + k_3 = 0$. Let

$$\begin{aligned}
f_1 &= a^{n-h}(b^h + c^h), \\
f_2 &= a^h(b^{n-h} + c^{n-h}, \\
f_3 &= b^h c^{n-h} + b^{n-h} c^h,
\end{aligned}$$

and note that in order to generate all linear combinations of $k_1 f_1 + k_2 f_2 + k3 f_3$, a necessary and sufficient condition is to use as a basis any two pairs (k_1, k_2) that are linearly independent, and then take $k_3 = -k_1 - k_2$. We choose the pairs $(1,0)$ and $(0,1)$ to obtain a basis $\{f_1 - f_3, f_2 - f_3\}$. Thus, in the construction of $\mathcal{B}_n$, we now include (26) and (27).

Type 3: Corresponding to $\overline{h,h,n-2h}$ and $\overline{n-2h,h,h}$, where

$$h \in \{1, 2, \ldots, \lceil (n-1)/3 \rceil\},$$

the contribution (35) to (34) has the form

$$\begin{aligned}
&k_1 a^h(b^h c^{n-2h} + b^{n-2h}c^h) + k_2 a^{n-2h} b^h c^h \\
&+ k_1 b^h(c^h a^{n-2h} + c^{n-2h}a^h) + k_2 b^{n-2h} c^h a^h \\
&+ k_1 c^h(a^h b^{n-2h} + a^{n-2h}b^h) + k_2 c^{n-2h} a^h b^h \\
&= (2k_1 + k_2)(n-2h, h, h).
\end{aligned}$$

By Lemma 1, $2k_1 + k_2 = 0$; we choose $(k_1, k_2) = (1, -2)$ and include in $\mathcal{B}_n$ the polynomial (28). *Type 4:* Corresponding to $\overline{n-2k,k,k}$ and $\overline{k,n-2k,k}$, where

$$k \in \{1, 2, \ldots, \lfloor (n+1)/3 \rfloor\},$$

we find that the method for Type 3 applies here also, so that we include (29) in $\mathcal{B}_n$.

Type 5: Corresponding to $\overline{i,j,k}$, $\overline{j,k,i}$, and $\overline{k,i,j}$, where $i \geq j \geq k$ and $i+j+k = n$, the contribution (35) to (34) has the form

$$\begin{aligned}
&k_1 a^i(b^j c^k + b^k c^j) + k_2 b^i(c^j a^k + c^k a^j) + k_3 c^i(a^j b^k + a^k b^j) \\
&+ k_1 b^i(c^j a^k + c^k a^j) + k_2 c^i(a^j b^k + a^k b^j) + k_3 a^i(b^j c^k + b^k c^j) \\
&+ k_1 c^i(a^j b^k + a^k b^j) + k_2 a^i(b^j c^k + b^k c^j) + k_3 b^i(c^j a^k + c^k a^j) \\
&= 3(k_1 + k_2 + k_3)(i, j, k).
\end{aligned}$$

By Lemma 1, $k_1 + k_2 + k_3 = 0$. As in the case for Type 2, we obtain, for each qualified (i, j, k),

$$\begin{aligned}
f_1 &= a^i(b^j c^k + b^k c^j) \\
f_2 &= a^j(b^k c^i + b^i c^k) \\
f_3 &= a^k(b^i c^j + b^j c^i)
\end{aligned}$$

and two more polynomials, $f_1 - f_3$ and $f_2 - f_3$, for inclusion in $\mathcal{B}_n$: (30) and (31).

Type 6: If $2|n$, the contribution (35) to (34) has the form

$$\begin{aligned}
&k_1 a^{n/2}(b^{n/2} + c^{n/2}) + k_2 b^{n/2} c^{n/2} \\
&+ k_1 b^{n/2}(c^{n/2} + a^{n/2}) + k_2 c^{n/2} a^{n/2} \\
&+ k_1 c^{n/2}(a^{n/2} + b^{n/2}) + k_2 a^{n/2} b^{n/2} \\
&= (2k_1 + k_2)(n/2, n/2),
\end{aligned}$$

so that the polynomial (32) is included in $\mathcal{B}_n$. (If n is odd, there is no contribution.)

Type 7: If $3|n$, the prospective contribution (35) to (34) has the form

$$k_1 a^{n/3} b^{n/3} c^{n/3} + k_1 b^{n/3} c^{n/3} a^{n/3} + k_1 c^{n/3} a^{n/3} b^{n/3} = 3k_1 (abc)^{n/3}.$$

By Lemma 1, $k_1 = 0$, so that for Type 7, there is no contribution to $\mathcal{B}_n$. □

We conclude this section with a list of the bases $\mathcal{B}_n$ for n up to 6.

Basis $\mathcal{B}_1$ for even IPTCs of degree 1:
$\{2a - b - c\}$.

Basis $\mathcal{B}_2$, for even IPTCs of degree 2:
$\{2a^2 - b^2 - c^2, 2bc - ca - ab\}$.

Basis $\mathcal{B}_3$, for even IPTCs of degree 3:
$\{2a^3 - b^3 - c^3, a^2 b + a^2 c - b^2 c - bc^2, ab^2 + ac^2 - b^2 c - bc^2\}$.

Basis $\mathcal{B}_4$, for even IPTCs of degree 4:

$$\begin{aligned}
&\{2a^4 - b^4 - c^4, \\
&\ a^3 b + a^3 c - b^3 c - bc^3, ab^3 + ac^3 - b^3 c - bc^3, \\
&\ 2b^2 c^2 - a^2 b^2 - a^2 c^2, 2a^2 bc - ab^2 c - abc^2\}.
\end{aligned}$$

Basis $\mathcal{B}_5$, for even IPTCs of degree 5:

$$\begin{aligned}
&\{2a^5 - b^5 - c^5, \\
&\ a^4 b + a^4 c - b^4 c - bc^4, ab^4 + ac^4 - b^4 c - bc^4, \\
&\ a^3 b^2 + a^3 c^2 - b^3 c^2 - b^2 c^3, a^2 b^3 + a^2 c^3 - b^3 c^2 - b^2 c^3, \\
&\ 2a^3 bc - ab^3 c - abc^3, 2ab^2 c^2 - a^2 b^2 c - a^2 bc^2\}.
\end{aligned}$$

Basis $\mathcal{B}_6$, for even IPTCs of degree 6:

$$\begin{aligned}
&\{2a^6 - b^6 - c^6, \\
&\ a^5 b + a^5 c - b^5 c - bc^5, ab^5 + ac^5 - b^5 c - bc^5, \\
&\ a^4 b^2 + a^4 c^2 - b^4 c^2 - b^2 c^4, a^2 b^4 + a^2 c^4 - b^4 c^2 - b^2 c^4, \\
&\ 2a^4 bc - ab^4 c - abc^4, \\
&\ 2b^3 c^3 - a^3 b^3 - a^3 c^3, \\
&\ a^3 b^2 c + a^3 bc^2 - ab^3 c^2 - ab^2 c^3, a^2 b^3 c + a^2 bc^3 - ab^3 c^2 - ab^2 c^3\}.
\end{aligned}$$

A different scheme for obtaining bases for even IPTCs is given in Section 4.

3 Odd IPTCs

Recall that an IPTC $p(a, b, c)$ :: is odd if

$$p(a,b,c) = (b-c)q(a,b,c) ::$$

for some PTC $q(a,b,c)$ :: that is not a polynomial multiple of $b-c$. Two odd IPTCs are $b-c$ :: and $a(b-c)$ ::, so that every IPTC has the form

$$(b-c)(s(a,b,c) + at(a,b,c)) :: \tag{36}$$

where $s(a,b,c)$ and $t(a,b,c)$ are polynomials symmetric in a,b,c. If the degree of (36) is $n \geq 2$, then clearly the degree of s is $n-1$ and the degree of t is $n-2$; for $n=1$, in (36), we take $t(a,b,c)$ to be the zero-polynomial and $s(a,b,c) \neq t(a,b,c)$, so that (36) represents $b-c$ :: , and $\{b-c\}$ is a basis for the odd IPTC of degree 1.

Let $P(n)$ be as in Section 2, so that the corresponding set $P^*(n)$ of polynomials is partitioned into the seven classes (18)-(24). A basis for each set of odd IPTCs that have fixed degree $n \geq 2$ is now generated from (36) by ranging $s(a,b,c)$ through the basis $P^*(n-1)$ for the symmetric polynomials of degree $n-1$ and ranging $t(a,b,c)$ through $P^*(n-2)$.

The resulting bases for odd IPTCs are quite different from those for even IPTCs, as illustrated by the following examples:

Basis $\mathcal{B}_2^*$ for odd IPTCs of degree 2:

$$\{(b-c)(a+b+c), (b-c)a)\}.$$

Referring to (36), here we have $s(a,b,c) = a+b+c$ and $t(a,b,c) = 1$, so that these IPTCs come from linear combinations of the form

$$h(b-c)(a+b+c) + k(b-c)a$$

and include, as examples,

$$\begin{aligned}
X(514) &= b-c ::, \text{ from } (h,k) = (1,0),\\
X(513) &= a(b-c) ::, \text{ from } (h,k) = (0,1),\\
X(523) &= b^2-c^2 ::, \text{ from } (h,k) = (1,-1),\\
X(4977) &= (b-c)(2a+b+c) ::, \text{ from } (h,k) = (1,1).
\end{aligned}$$

Basis $\mathcal{B}_3^*$ for odd IPTCs of degree 3:

$$\{(b-c)(a^2+b^2+c^2) + (b-c)(bc+ca+ab), a(b-c)(a+b+c) : h,k \text{ real}\}.$$

Again referrring to (36), here $s(a,b,c)$ is any linear combination of the polynomials corresponding to the partitions (2) and (1, 1) of 2, and $t(a,b,c) = a+b+c$.

Among the triangle centers generated from this basis are the following:

$$
\begin{aligned}
&X(824) = b^3 - c^3 ::, \\
&X(812) = (b-c)(a^2 - bc) ::, \\
&X(918) = (b-c)(b^2 + c^2 - ab - ac) ::, \\
&X(28840) = (b-c)(a^2 + 2ab + 2ac + bc) ::, \\
&X(30519) = (b-c)(2b^2 + 2c^2 + bc - ab - ac) ::, \\
&X(30520) = (b-c)(a^2 + 2b^2 + 2c^2 - ab - ac) :: .
\end{aligned}
$$

4 Even IPTCs: a second method

The method of Section 3 can be used to obtain bases for even IPTCs. Specifically, every IPTC has the form

$$s(a,b,c)(2a-b-c) + t(a,b,c)(2a^2 - b^2 - c^2), \tag{37}$$

where $s(a,b,c)$ and $t(a,b,c)$ are symmetric functions as in Section 3. Let $\mathcal{B}'_n$ be the basis for IPTCs of degree n as obtained from (37). Examples follow:

Basis $\mathcal{B}'_1$ for even IPTCs of degree 1:
$\{2a-b-c\}$.

Basis $\mathcal{B}'_2$, for even IPTCs of degree 2:
$\{(a+b+c)(2a-b-c), 2a^2-b^2-c^2\}$.

Basis $\mathcal{B}'_3$, for even IPTCs of degree 3:

$$
\begin{aligned}
&\{a^2+b^2+c^2)(2a-b-c), (bc+ca+ab)(2a-b-c), \\
&(a+b+c)(2a^2-b^2-c^2)\}.
\end{aligned}
$$

Basis $\mathcal{B}'_4$, for even IPTCs of degree 4:

$$
\begin{aligned}
&(a^3+b^3+c^3)(2a-b-c), \\
&(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)(2a-b-c), \\
&abc(2a-b-c), \\
&(a^2+b^2+c^2)(2a^2-b^2-c^2), \\
&(bc+ca+ab)(2a^2-b^2-c^2).
\end{aligned}
$$

Note that the polynomial $2a^2-b^2-c^2$ in (37) can be replaced by $2bc-ca-ab$ to produce yet another type of basis. In both types, the polynomials can consist of up to 18 terms, as in

$$(a^4b^5 + a^4c^5 + b^4c^5 + b^4a^5 + c^4a^5 + c^4b^5)(2a^2-b^2-c^2),$$

in contrast to the polynomials in the bases $\mathcal{B}_n$ in Section 2, which consist of only 3 or 4 terms.

5 Cardinalities of bases

We begin with cardinalities of the seven classes in Section 2.

$$|\mathcal{S}_1| = 1, \tag{38}$$
$$|\mathcal{S}_{2,h}| = \lfloor (n-1)/2) \rfloor, \tag{39}$$
$$|\mathcal{S}_{3,h}| = \lfloor (n-1)/2 \rfloor - \lfloor n/3) \rfloor, \tag{40}$$
$$|\mathcal{S}_{4,h}| = \lfloor (n-1)/3) \rfloor, \tag{41}$$
$$|\mathcal{S}_{5,i,j,k}| = \lfloor (n-3)^2/12 + 1/2 \rfloor, \tag{42}$$
$$|\mathcal{S}_6| = \lfloor n/2 \rfloor - \lfloor (n-1)/2 \rfloor \tag{43}$$
$$|\mathcal{S}_7| = \lfloor 2(n-1)/3) \rfloor - 2\lfloor (n-1)/3) \rfloor. \tag{44}$$

The cardinalities (38)-(44) can be easily verified, except perhaps for (42), for which a method of proof follows from the fact that the number of triples (i,j,k) satisfying $i < j < k$ and fixed $i = n-j-k$ is $\lfloor (n-3i-1)/2 \rfloor$; add these for $i = 1,2,\ldots,h$, where $h = \lfloor (n-1)/3 \rfloor$, to get

$$|\mathcal{S}_{5,i,j,k}| = (n/2)f_h - (f_1 + f_2 + \ldots + f_h),$$

where f_i, the fractional part of $(n-3i-1)/2$, is 0 or 1, according as $3i-1$ is even or odd. Then (42) follows after dealing with cases for n.

By Theorem 2,

$$|\mathcal{B}_n| = |\mathcal{S}_1| + 2|\mathcal{S}_{2,h}| + |\mathcal{S}_{3,h}| + |\mathcal{S}_{4,h}| + 2|\mathcal{S}_{5,i,j,k}| + |\mathcal{S}_6|. \tag{45}$$

Note, in accord with the end of the proof of 2, that $|\mathcal{S}_7|$ is not a term in the sum (45).

The sum of the cardinalities (38)-(44) is

$$\lfloor (n+3)^2/12 + 1/2 \rfloor,$$

corresponding to the sequence A001399, and, notably, a shift of the count in (42). The sequences (38)-(44) are indexed as A000012, A133872, A008615 (prefaced by $0,0$), A002264, A211540, A000035, and A079978 (prefaced by $0,0$), respectively. Using generating functions for these sequences, we find a simplification for the sum (45):

$$|\mathcal{B}_n| = \left\lfloor \frac{(n+1)(n+2)}{6} \right\rfloor,$$

given by

$$A001840 = (1,2,3,5,7,9,12,15,18,22,26,30,35,40,\ldots).$$

Similarly, one can show that

$$|\mathcal{B}_n^*| = |\mathcal{B}'_n| = |\mathcal{B}_n|.$$

6 From infinite to finite

Suppose that L is a "polynomial line", as in item (ii) of Section 1; that is, L is given by (2) where $U = u : v : w$ is a PTC (or, equivalently, L passes through at least two PTCs.) Suppose further that $X = x : y : z \in L$. Then $U * X \in L^{\infty}$, by (2). By the results in Sections 2 and 3, the first coordinate, ux, of $U * X$, is a linear combination of polynomials in a basis $\mathcal{B}_n$ or $\mathcal{B}_n^*$:

$$ux = \sum h_{i_k}(p_{i_k}(a,b,c) + p_{i_k}(b,c,a) + p_{i_k}(c,a,b)).$$

Consequently,

$$uvwx = \sum h_{i_k}(vwp_{i_k}(a,b,c) + vwp_{i_k}(b,c,a) + vwp_{i_k}(c,a,b)).$$

Since $X = uvwx : vwuy : wuvz$, the polynomials in $vw\mathcal{B}_n$ or $vw\mathcal{B}_n^*$ comprise a basis for the PTCs in L; however, polynomials in these bases may have include symmetric factors, which can be canceled.

As a first example, consider the Nagel line, which passes through the incenter and centroid and is given by

$$(b-c)x + (c-a)y + (a-b)z = 0,$$

so that $u = b - c$. Bases for PTCs on the Nagel line can be represented by $\mathcal{B}_n/p$ and $\mathcal{B}_n^*/p$, where $p : q : r = X(514) = b - c$ ::; e.g.,

$$\mathcal{B}_2^*/p = \mathcal{B}_2^*/(b-c) = \{a^2+b^2+c^2, bc+ca+ab, a(a+b+c)\},$$

relative to which, for example, the PTC $X(3661) = b^2 + c^2 + bc$ :: is given by the linear combination

$$1 \cdot (a^2+b^2+c^2) + 1 \cdot (bc+ca+ab) + (-1) \cdot a(a+b+c).$$

For a second example, we turn to the Euler line, for which the PTCs are given by $\mathcal{B}_n/p$ and $\mathcal{B}_n^*/p$, where

$$X(525) = p : q : r = (b^2-c^2)(b^2+c^2-a^2) :: .$$

In $\mathcal{B}_1/p$ and $\mathcal{B}_2^*/p$, for example, are the first coordinates of these PTCs on the Euler line:

$$X(27) = \frac{1}{(b+c)(b^2+c^2-a^2)} ::,$$
$$X(28) = \frac{a}{(b+c)(b^2+c^2-a^2)} :: .$$

As a third example, consider the antiorthic axis, given by

$$bcx + cay + abz = 0,$$

so that $u = bc$. Bases for the PTCs on this line are given by $a\mathcal{B}_n$ and $a\mathcal{B}_n^*$. The first coordinate of the PTC

$$X(672) = a^2(b^2+c^2-ab-ac) ::,$$

for instance, is obtained from $a\mathcal{B}_3$ as

$$a\Big(ab^2+ac^2-b^2c-bc^2-(a^2b+a^2c-b^2c-bc^2)\Big).$$

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