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On Second-Order Linear Sequences of Composite Numbers

Dan Ismailescu
Department of Mathematics
Hofstra University
Hempstead, NY 11549
USA
dan.p.ismailescu@hofstra.edu

Adrienne Ko
Ethical Culture Fieldston School
Bronx, NY 10471
USA
19ako@ecfs.org

Celine Lee
Chinese International School
Hong Kong SAR
China
celinl@student.cis.edu.hk

Jae Yong Park
The Lawrenceville School
Lawrenceville, NJ 08648
USA
jpark19@lawrenceville.org

Abstract

We present a new proof of the following result of Somer:

Let $(a, b) \in \mathbb{Z}^2$ and let $(x_n)_{n\geq 0}$ be the sequence defined by some initial values x_0 and x_1 and the second-order linear recurrence

$$x_{n+1} = ax_n + bx_{n-1}$$

for $n \geq 1$. Suppose that $b \neq 0$ and $(a, b) \neq (2, -1), (-2, -1)$. Then there exist two relatively prime positive integers x_0, x_1 such that $|x_n|$ is a composite integer for all $n \in \mathbb{N}$.

The above theorem extends a result of Graham, who solved the problem when $(a, b) = (1, 1)$.

1 Introduction

We give a new proof of the following result of Somer:

Theorem 1. *[12, 3] Let $(a, b) \in \mathbb{Z}^2$ and let $(x_n)_{n\geq 0}$ be the sequence defined by some initial values x_0 and x_1 and the second-order linear recurrence*

$$x_{n+1} = ax_n + bx_{n-1} \tag{1}$$

for $n \geq 1$. Suppose that $b \neq 0$ and $(a, b) \neq (2, -1), (-2, -1)$. Then there exist two relatively prime positive integers x_0, x_1 such that $|x_n|$ is a composite integer for all $n \in \mathbb{N}$.

Throughout the paper we will use the following convention: a nonnegative integer n is said to be *composite* if $n \neq 0, 1$, and n is not a prime number.

Graham [5] considered the problem above in the particular case $(a, b) = (1, 1)$. He found two relatively prime positive integers x_0 and x_1 such that the sequence $x_{n+1} = x_n + x_{n-1}$, consists of composite numbers only. Graham's starting pair is

$$(x_0, x_1) = (331635635998274737472200656430763, 1510028911088401971189590305498785).$$

Graham's technique was successively refined by Knuth [6], Wilf [16], and Nicol [10], who all found smaller pairs (x_0, x_1). The current record is due to Vsemirnov [15]:

$$(x_0, x_1) = (106276436867, 35256392432). \tag{2}$$

The above results are based on the fact that the Fibonacci sequence is a *divisibility sequence*, that is, $F_n \mid F_m$ whenever $n \mid m$, and on finding a finite *covering system of congruences* $r_i \pmod{m_i}$, $1 \leq i \leq t$, such that there exist distinct primes $p_1, p_2, \ldots p_t$, so that $p_i \mid F_{m_i}$ for all $i = 1, 2, \ldots, t$.

Somer's proof of Theorem 1 is relatively short since it relies on several results of Bilu, Hanrot and Voutier [1], Choi [2], and Parnami and Shorey [11].

A few years later, unaware of Somer's article, Dubickas, Novikas and Šiurys [3] published a new solution that, although somewhat lengthier, is essentially self-contained.

In this paper, we present another free-standing proof of this theorem that, while comparable to the one in [3], differs from it in several important ways.

We summarize our plan for proving Theorem 1 as follows.

In Section 2 we prove three easy lemmata, which will be useful later on. We believe that Lemma 2 is of independent interest. In Section 3 we study two simple cases: (i) $a = 0$ and (ii) $a^2 + 4b = 0$. In this section we also show that the condition $(a, b) \neq (\pm 2, -1)$ is necessary. Section 4 deals with the case $|b| \geq 2$. Following [3], we choose $x_1 \equiv 0 \pmod{|b|}$, since then (1) implies that $x_n \equiv 0 \pmod{|b|}$ for all $n \geq 2$. The main difficulty is to show that x_0 and x_1 can be chosen such that $x_n \neq -b, 0, b$ for every $n \geq 0$. In this case, Dubickas et al. present a mainly existential proof which relies on a series of six lemmata. In contrast, our proof is constructive, as we provide explicit expressions for x_0 and x_1 as polynomials in a and b.

In Section 5 we consider the case $|b| = 1$. Dubickas, Novikas and Šiurys prove that except for finitely many values of $|a|$, one can take x_0 and x_1 so that each x_n is divisible by one of five distinct appropriately chosen prime numbers. We prove that, with the exception of finitely many values of $|a|$, four primes suffice. The proof in the case $|b| = 1$ relies on the divisibility properties of the *Lucas sequence of the first kind* $(u_n)_{n\geq 0}$, defined as

$$u_0 = 0, u_1 = 1 \quad \text{and} \quad u_{n+1} = au_n + bu_{n-1}, \text{ for } n \geq 1. \tag{3}$$

Finally, in Section 6 we prove that if $|a| \geq 3$ and $b = -1$, then u_n is composite for all $n \geq 3$. The interesting fact is that in this case it seems likely that there is no finite set of prime numbers $p_1, p_2, \ldots p_t$ such that each u_n is divisible by some p_i, $i = 1, 2, \ldots t$.

2 Three useful lemmata

Lemma 2. *Consider the sequence $(x_n)_{n\geq 0}$ given by (1). Then*

$$x_{n+1}^2 - ax_nx_{n+1} - bx_n^2 = (-b)^n(x_1^2 - ax_0x_1 - bx_0^2). \tag{4}$$

Proof.

$$\begin{bmatrix} x_{n+2} & x_{n+1} \\ x_{n+1} & x_n \end{bmatrix} = \begin{bmatrix} ax_{n+1} + bx_n & ax_n + bx_{n-1} \\ x_{n+1} & x_n \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_{n+1} & x_n \\ x_n & x_{n-1} \end{bmatrix} =$$

$$= \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_n & x_{n-1} \\ x_{n-1} & x_{n-2} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^2 \cdot \begin{bmatrix} x_n & x_{n-1} \\ x_{n-1} & x_{n-2} \end{bmatrix} = \cdots = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} x_2 & x_1 \\ x_1 & x_0 \end{bmatrix}.$$

Taking determinants on both sides we obtain

$$\det\begin{bmatrix} x_{n+2} & x_{n+1} \\ x_{n+1} & x_n \end{bmatrix} = \det\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^n \cdot \det\begin{bmatrix} x_2 & x_1 \\ x_1 & x_0 \end{bmatrix} \implies x_{n+2}\cdot x_n - x_{n+1}^2 = (-b)^n \cdot (x_2 x_0 - x_1^2)$$

$\implies (ax_{n+1}+bx_n)x_n - x_{n+1}^2 = (-b)^n \cdot ((ax_1+bx_0)x_0 - x_1^2)$, which after expanding becomes $x_{n+1}^2 - ax_nx_{n+1} - bx_n^2 = (-b)^n(x_1^2 - ax_0x_1 - bx_0^2)$, as claimed.

□

Lemma 3. *Consider the sequence $(x_n)_{n\geq 0}$ given by (1). Suppose that $1 \leq |x_0| < |x_1|$ and $|a| > |b| \geq 1$. Then the sequence $(|x_n|)_{n\geq 0}$ is strictly increasing.*

Proof. We use induction on n. By hypothesis, the statement is true for $n = 0$.

Suppose that $|x_{n-1}| < |x_n|$ for some $n \geq 1$. We intend to prove that $|x_n| < |x_{n+1}|$. Indeed

$$\begin{aligned} |x_{n+1}| = |ax_n + bx_{n-1}| \geq |ax_n| - |bx_{n-1}| = |a||x_n| - |b||x_{n-1}| > \\ > |a||x_n| - |b||x_n| = (|a| - |b|)|x_n|, \text{ by the induction hypothesis.} \end{aligned}$$

Using $|a| - |b| \geq 1$, we obtain $|x_{n+1}| > |x_n|$, which completes the induction. □

Lemma 4. *Let n_1, n_2 and n_3 be three positive integers such that no prime number p divides all of them. Then there exists an integer $k \geq 2$ such that n_1 and $n_2 + kn_3$ are relatively prime.*

Proof. Let $d := \gcd(n_2, n_3)$. Note that $\gcd(n_1, d) = 1$, otherwise d divides n_1, n_2 and n_3.

By Dirichlet's theorem on arithmetic progressions there exists a k such that $n_2/d + kn_3/d$ is a prime number greater than n_1. Then $d\,(n_2/d + k\cdot n_3/d) = n_2 + k\,n_3$ is both greater and relatively prime to n_1. □

3 Two simple special cases: (i) $a = 0$ and (ii) $a^2 + 4b = 0$

Case (i): Since $a = 0$, it can be easily proved that $x_{2n} = b^n x_0$ and $x_{2n+1} = b^n x_1$. It suffices to take $x_0 = 4$ and $x_1 = 9$ to obtain that x_n is composite for all $n \geq 0$.

Case (ii): If $a^2 + 4b = 0$, then a must be even; $a = 2c$ and therefore $b = -a^2/4 = -c^2$. We divide the proof into two cases: $|b| \geq 2$ and $b = -1$.

If $|b| \geq 2$, we have $|c| \geq 2$. Let us now take $x_0 = 4c^2 - 1$ and $x_1 = 2c^3$. Then x_0 and x_1 are relative prime positive composite integers. One immediately obtains $x_2 = c^2$; thus x_2 is also composite. Also, it can be easily proved that $x_n = c^n\left((n-1) - (2n-4)c^2\right)$ for all $n \geq 3$.

Note that for any $n \geq 3$ one cannot have $x_n = 0$; otherwise, $4 \leq c^2 = (n-1)/(2n-4) \leq 1$, which is impossible. It follows that $|x_n|$ is composite for all $n \geq 3$.

If $b = -1$, then $|a| = 2$. If $a = 2$, then a simple induction shows that

$$x_{n+1} = (n+1)x_1 - nx_0 = x_1 + n(x_1 - x_0) \quad \text{for all } n \geq 0,$$

that is, $(x_n)_{n\geq 0}$ is an arithmetic sequence whose first term and common difference are relatively prime. By Dirichlet's theorem on primes in arithmetic progressions, it follows that $|x_n|$ is a prime number for infinitely many values of n. If $a = -2$, then one can show that $x_{n+1} = (-1)^n(x_1 + n(x_0 + x_1))$. In this case, $(x_n)_{n\geq 0}$ is the union of two arithmetic sequences, both of which have the first term and the common difference relatively prime. Again, for any choice of x_0 and x_1 relatively prime, $|x_n|$ is a prime for infinitely many n. This proves the necessity of the condition $(a, b) \neq (\pm 2, -1)$.

4 The case $|b| \geq 2$

Based on the results from the previous section, from now on we can assume that

$$|a| \geq 1 \quad \text{and} \quad a^2 + 4b \neq 0. \tag{5}$$

We divide the proof into three separate cases:

Case I: $|a| > |b|$,
Case II: $1 \leq |a| \leq |b|$ and $|b|$ is composite, and
Case III: $1 \leq |a| \leq |b|$ and $|b|$ is a prime.

Case I: $|a| > |b|$. To complete the proof of Theorem 1, in this case we take $x_0 = b^4 - 1, x_1 = b^4$. Clearly, x_0 and x_1 are both positive, relatively prime and composite. Moreover, $x_0 < x_1$. Then $x_n \equiv 0 \pmod{b}$ for all $n \geq 1$, and by using Lemma 3 it follows that $(|x_n|)_{n\geq 0}$ is strictly increasing. Hence, we have $|x_n| \geq x_0 > |b|$ for all $n \geq 0$, and therefore $|x_n|$ is composite for all $n \geq 0$.

Case II: $1 \leq |a| \leq |b|$ and $|b|$ is composite. Choose $x_0 = 4b^4 - 1$, $x_1 = 2b^2$. Clearly, $\gcd(x_0, x_1) = 1$, and x_0, x_1 are positive composite integers. It also follows that $x_n \equiv 0 \pmod{b}$ for all $n \geq 1$, and since $|b|$ is composite, it follows that $|x_n|$ is composite, unless $x_n = 0$ for some n. We will prove that this cannot happen.

To get a contradiction, suppose that $x_{n+1} = 0$ for some $n \geq 2$. Then, by Lemma 2, we have $x_n^2 = (-b)^{n-1}(x_1^2 - ax_0x_1 - bx_0^2)$.

Since $x_0 = 4b^4 - 1$ and $x_1 = 2b^2$, we have

$$x_1^2 - ax_0x_1 - bx_0^2 = (-b)(16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1), \quad \text{which implies that}$$

$$x_n^2 = (-b)^n(16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1). \tag{6}$$

If n is even, this implies that $16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1$ is a perfect square. If n is odd, this implies that $(-b)(16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1)$ is a perfect square. We will prove that none of these are possible if $a^2 + 4b \neq 0$.

Note first that since $|a| \le |b|$, we have

$$(4b^4 + ab - 2)^2 < 16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1 < (4b^4 + ab)^2. \tag{7}$$

Indeed, these inequalities are equivalent to the following ones:

$$8b^4 - 2 - (4b^3 + (ab-1)^2) > 0, \quad \text{and} \quad 8b^4 + 4b^3 - 2 + (ab+1)^2 > 0. \tag{8}$$

We have

$$|a| \le |b| \implies |ab| \le b^2 \implies |ab-1| \le |ab| + 1 \le b^2 + 1 \implies (ab-1)^2 \le b^4 + 2b^2 + 1. \tag{9}$$

From this we obtain

$$|4b^3 + (ab-1)^2| \le 4|b|^3 + (ab-1)^2 \le b^4 + 4|b|^3 + 2b^2 + 1. \tag{10}$$

To prove the first inequality in (8), note that

$$\begin{aligned} &8b^4 - 2 - (4b^3 + (ab-1)^2) \ge 8b^4 - 2 - |4b^3 + (ab-1)^2| \ge \\ &\ge 8b^4 - 2 - (b^4 + 4|b|^3 + 2b^2 + 1) = 7b^4 - 4|b|^3 - 2b^2 - 3 = \\ &= 7|b|^4 - 4|b|^3 - 2|b|^2 - 3 = 4|b|^3(|b|-1) + 2|b|^2(|b|^2-1) + (|b|^4 - 3) \ge \\ &\ge 4|b|^3 + 2|b|^2 + |b|^4 - 3 \ge 4 \cdot 2^3 + 2 \cdot 2^2 + 2^4 - 3 > 0. \end{aligned}$$

The second inequality in (8) is much easier to prove:

$$8b^4 + 4b^3 - 2 + (ab+1)^2 \ge 8b^4 + 4b^3 - 2 \ge 8|b|^4 - 4|b|^3 - 2 \ge 8 \cdot 2^4 - 4 \cdot 2^3 - 2 > 0.$$

Hence, the inequalities (7) hold.

If the middle term in (7) were to be a perfect square, the only option remaining is

$$16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1 = (4b^4 + ab - 1)^2, \text{ which implies } b^2(a^2 + 4b) = 0.$$

However, this is impossible if $a^2 + 4b \ne 0$. Hence, if one assumes that $|a| \le |b|$ and $a^2 + 4b \ne 0$, then $16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1$ cannot be a perfect square.

Suppose next that $(-b)(16b^8 + 8ab^5 - 8b^4 - 4b^3 - 2ab + 1)$ is a perfect square. Since the two factors are mutually prime, it follows that both have to be perfect squares. In particular, the second one has to be a perfect square, and we have already proved that it cannot be. It follows that the sequence (x_n) does not contain any terms equal to 0. This completes the proof of Theorem 1 in case II.

Case III: $1 \le |a| \le |b|$ *and* $|b|$ *is a prime.*

We divide the proof of this case into three subcases: $|a| = 1$, $|a| = |b|$ and $2 \le |a| < |b|$.

Subcase IIIa: $|a| = 1$, $|b|$ *is a prime.*

If $a = 1$ and $b > 0$, then take $x_0 = (2b^2-1)^2, x_1 = b(b^2-1)$. Clearly, x_0 and x_1 are positive, relatively prime, composite integers. It follows immediately that $x_2 = b^3(4b^2 - 3)$ is also composite. Finally, an easy induction shows that $x_n \equiv 0 \pmod{b^2}$ and that $\frac{x_n}{b^2} \equiv -1 \pmod{b}$ for all $n \geq 3$. Hence, $|x_n|$ is necessarily composite for all $n \geq 3$.

The other situations can be dealt with similarly.

If $a = -1$ and $b < 0$, take $x_0 = (2b^2 - 1)^2, x_1 = -b(b^2 - 1)$. Then $x_2 = b^3(4b^2 - 3)$, and for $n \geq 3$ one can show that $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv (-1)^{n+1} \pmod{b}$. Again, all $|x_n|$ are composite.

If $a = 1$ and $b < 0$, we take $x_0 = (2b^2 - 1)^2, x_1 = -b(b^2 + 1)$. Then $x_2 = b^3(4b^2 - 5)$, and for all $n \geq 3$ one can prove that $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv -1 \pmod{b}$. Hence, all $|x_n|$ are composite.

Finally, if $a = -1$ and $b > 0$, select $x_0 = (2b^2 - 1)^2, x_1 = b(b^2 + 1)$. Then $x_2 = b^3(4b^2 - 5)$, and for all $n \geq 3$ we have that $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv (-1)^{n+1} \pmod{b}$. All $|x_n|$ are composite.

This completes the proof of Theorem 1 in subcase IIIa.

Subcase IIIb: $|a| = |b|$, $|b|$ *is a prime.*

If $a = b$, take $x_0 = 4b^4 - 1, x_1 = 2b^2$. Then $x_2 = b(4b^4 + 2b^2 - 1)$ is composite, and $x_n \equiv 0 \pmod{b^2}$ for all $n \geq 3$. It remains to show that $x_n \neq 0$ for all n.

As in Case II, $x_{n+1} = 0$ implies that either $16b^8 + 8b^6 - 8b^4 - 4b^3 - 2b^2 + 1$ or $(-b)(16b^8 + 8b^6 - 8b^4 - 4b^3 - 2b^2 + 1)$ is a perfect square. One can use the same technique as in Case II to prove that this cannot happen if $a^2 + 4b = b^2 + 4b \neq 0$.

If $a = -b$, take $x_0 = 4b^4 - 1, x_1 = 2b^2$ (same choice). Then $x_2 = b(4b^4 - 2b^2 - 1)$ is composite, and $x_n \equiv 0 \pmod{b^2}$ for all $n \geq 3$. It remains to show that $x_n \neq 0$ for all n.

As in Case II, $x_{n+1} = 0$ implies that either $16b^8 - 8b^6 - 8b^4 - 4b^3 + 2b^2 + 1$ or $(-b)(16b^8 - 8b^6 - 8b^4 - 4b^3 + 2b^2 + 1)$ is a perfect square. The same approach as in Case II shows that this is impossible if $a^2 + 4b = b^2 + 4b \neq 0$.

Subcase IIIc: $2 \leq |a| < |b|$, $|b|$ *is a prime.*

It follows that $\gcd(a, b) = 1$.

If $a > 0, b > 0$, take $x_0 = a^3, x_1 = b(b^2 - a^2)$. Then $x_2 = ab^3$. For $n \geq 3$, $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv -a^{n-1} \pmod{b}$. Since $\gcd(a, b) = 1$, $x_n \neq 0$.

If $a < 0, b < 0$, take $x_0 = -a^3, x_1 = -b(b^2 - a^2)$. Then $x_2 = -ab^3$. For $n \geq 3$, $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv a^{n-1} \pmod{b}$. Hence, $x_n \neq 0$ for all n.

If $a < 0, b > 0$, take $x_0 = -a^3, x_1 = b(b^2 + a^2)$. Then $x_2 = ab^3$, and for $n \geq 3$, $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv a^{n-1} \pmod{b}$. So, $x_n \neq 0$ for all n.

Lastly, if $a > 0, b < 0$, take $x_0 = a^3, x_1 = -b(b^2 + a^2)$. Then $x_2 = -ab^3$. For $n \geq 3$, $x_n \equiv 0 \pmod{b^2}$ and $\frac{x_n}{b^2} \equiv -a^{n-1} \pmod{b}$. Since $\gcd(a, b) = 1$, we have $x_n \neq 0$ for all n.

Hence, in each case one can choose x_0 and x_1 to be positive, composite and relatively prime so that x_2 is composite, and for all $n \geq 3$ we have that $x_n \equiv 0 \pmod{b^2}$ and $x_n \neq 0$. It follows that $|x_n|$ is composite for all $n \geq 0$, and this completes the proof of Theorem 1 in the case $|b| \geq 2$.

5 The case $|b| = 1$

The main idea behind the proof of Theorem 1 in this particular case can be summarized as follows: we want to find a finite set of primes $p_1, p_2, \ldots, p_t$ such that for every $n \geq 0$ the number $|x_n|$ is divisible by at least one of these primes.

We start the analysis with the simple case in which $|a|$ has at least two distinct prime factors.

Lemma 5. *Let $|b| = 1$, and suppose that $|a|$ has at least two distinct prime factors: p_1 and p_2, with $p_1 < p_2$. Then the sequence given by (1) and the initial terms $x_0 = p_1^2$, $x_1 = p_2^2$ satisfies the conditions in Theorem 1.*

Proof. Clearly, x_0 and x_1 are positive relatively prime composite integers. An easy induction shows that $x_n \equiv 0 \pmod{p_1}$ if n is even and $x_n \equiv 0 \pmod{p_2}$ if n is odd. Since $|x_0| < |x_1|$ and $1 = |b| < |a|$, the hypotheses of Lemma 3 are satisfied; hence $(|x_n|)_{n\geq 0}$ is strictly increasing. It follows that x_n is composite for all $n \geq 0$, and therefore Theorem 1 is verified in this case. □

Remark 6. In all what follows we will assume that $|a| = p_1^s$ for some prime p_1 and some nonnegative integer $s \geq 0$. Note that this allows the possibility of $|a| = 1$.

Next, we introduce a special sequence $(u_n)_{n\geq 0}$ given by

$$u_0 = 0, u_1 = 1, \quad u_{n+1} = au_n + bu_{n-1} \quad \text{for all } n \geq 1. \tag{11}$$

This sequence is called the *Lucas sequence of the first kind*.

One can prove that $(u_n)_{n\geq 0}$ is a *divisibility sequence*; that is, u_m divides u_n whenever m divides n. Indeed, suppose that $a^2 + 4b \neq 0$, which implies that the roots α and β of the characteristic equation are distinct, $\alpha \neq \beta$. Assume that $n = mk$. Then

$$\begin{aligned} u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^{mk} - \beta^{mk}}{\alpha - \beta} &= \frac{\alpha^m - \beta^m}{\alpha - \beta} \cdot \left(\alpha^{m(k-1)} + \alpha^{m(k-2)}\beta^m + \cdots + \beta^{m(k-1)}\right) \\ &= u_m \left(\alpha^{m(k-1)} + \alpha^{m(k-2)}\beta^m + \cdots + \beta^{m(k-1)}\right). \end{aligned}$$

The second factor of the last term in the equality above is a symmetric function in α and β, and therefore it can be written as a polynomial function of $\alpha + \beta = a$ and $\alpha\beta = -b$. It follows that u_n/u_m is an integer, as claimed.

Of particular interest in the sequel are the values of u_4 and u_6:

$$u_4 = a(a^2 + 2b) \quad \text{and} \quad u_6 = a(a^2 + b)(a^2 + 3b). \tag{12}$$

The next two results give information regarding the prime factors of u_4 and u_6 when $|b| = 1$.

Lemma 7. *Suppose that $b = -1$, $|a| \geq 4$ and $|a| = p_1^s$ for some prime p_1 and some $s \geq 1$. Then $u_6 = a(a^2 - 1)(a^2 - 3)$ has at least three additional distinct prime factors, p_2, p_3 and p_4, all three different from p_1.*

Proof. Note that a^2-3 is not divisible by 4 or 9. Since $a^2-3 \geq 13$, it follows that a^2-3 has an odd prime factor p_4 different from 3, and since $\gcd(a, a^2-3) \in \{1,3\}$ we have that $p_4 \neq p_1$. Clearly, neither p_1 nor p_4 are factors of a^2-1. Note that a^2-1 has at least two distinct prime factors. Indeed, if one assumes the opposite, then both $|a-1|$ and $|a+1|$ are powers of some prime. But this cannot happen, since $|a| \geq 4$. It follows that a^2-1 has at least two distinct prime factors, p_2 and p_3, both different from p_1 and p_4. $\square$

To illustrate Lemma 7, let us take $(a,b) = (-9,-1)$. Then $a^2-3 = 78 = 2 \cdot 3 \cdot 13$ while $a^2-1 = 80 = 2^4 \cdot 5$. Hence, one has $p_1 = 3$, $p_4 = 13$, $p_2 = 2$, and $p_3 = 5$.

Lemma 8. *Suppose that $b = 1$, $|a| \geq 6$ and $|a| = p_1^s$ for some prime p_1 and some $s \geq 1$.*

If $p_1 \neq 3$, then $u_4 = a(a^2+2)$ has at least two additional distinct prime factors, p_2 and p_3, both different from p_1.

If $|a| = 3^s$ for some $s \geq 2$, then $u_6 = a(a^2+1)(a^2+3)$ has at least three additional distinct prime factors, p_2, p_3 and p_4, all three different from p_1.

Proof. Suppose first that $p_1 = 2$; that is, $|a| = 2^s$ for some $s \geq 3$. Note that $a^2+2 \equiv 0 \pmod 3$, hence one can choose $p_2 = 3$. Note that $a^2 = 2(2^{2s-1}+1)$ must have a prime factor different from 2 and 3. Indeed, if one assumes the opposite, then $2^{2s-1}+1 = 3^t$ for some $t \geq 2$. However, under the assumption that $2s-1 \geq 2$ and $t \geq 2$, the Catalan-Mihăilescu theorem implies that the only solution of this equation is $s = 2, t = 2$. But this implies that $|a| = 2^2 = 4$, a contradiction.

Consider next the case $|a| = p_1^s$ with $p_1 > 3$. In particular, a^2+2 is odd and therefore $\gcd(a, a^2+2) = 1$. Moreover, since $a^2+2 \equiv 0 \pmod 3$ one can safely take $p_2 = 3$. We claim that a^2+2 has at least one other prime factor, p_3, different from p_1 and p_2. Indeed, if one assumes otherwise, then $a^2+2 = p_1^{2s}+2 = 3^t$ for some $t \geq 2$. However, it was shown by Ljunggren [7] that the more general equation $x^2+2 = y^n$, $n \geq 2$ has the unique solution $x = 5, y = 3, n = 3$. This would give $|a| = 5$, which is excluded from our analysis.

Finally, suppose that $p_1 = 3$ and therefore $|a| = 3^s$ for some $s \geq 2$. Then both a^2+1 and a^2+3 are even, so one can take $p_2 = 2$. Note that $a^2+1 = 9^s+1$; hence $a^2+1 \equiv 10 \pmod{72}$ and $a^2+3 \equiv 12 \pmod{72}$. Hence, there exists a positive integer c such that $a^2+1 = 2(36c+5)$ and $a^2+3 = 12(6c+1)$. Let p_3 be a prime factor of $36c+5$ and let p_4 be a prime factor of $6c+1$. It follows immediately that $p_1 = 3, p_2 = 2, p_3$, and p_4 are all distinct. $\square$

We present a couple of particular instances covered by the above lemma.

Suppose first that $a = 8$ and $b = 1$. Then $u_2 = a = 2^3$ while $u_4 = 2^4 \cdot 3 \cdot 11$. Thus, in this case one can take $p_1 = 2, p_2 = 3, p_3 = 11$.

Suppose next that $a = -49$ and $b = 1$. Then $u_2 = a = -7^2$, while $u_4 = -7^2 \cdot 3^3 \cdot 89$. It follows that we can choose $p_1 = 7, p_2 = 3, p_3 = 89$.

Finally, let us assume that $a = 9$ and $b = 1$. Then $u_2 = a = 3^2$, while $a^2+1 = 2 \cdot 41$, $a^2+3 = 2^2 \cdot 3 \cdot 7$. Hence, in this case we have $p_1 = 3, p_2 = 2, p_3 = 41, p_4 = 7$.

The next lemma uses the concept of covering system introduced by Erdős in [4].

Definition 9. A collection of residue classes $r_i \pmod{m_i}$, $0 \le r_i < m_i$, where $1 \le i \le t$ is said to be a *covering system* if every integer n satisfies at least one equality $n \equiv r_i \pmod{m_i}$.

In the proof of the theorem when $|b| = 1$, except for finitely many values of a we will use one of the following two covering systems

$$\{0 \pmod 2, \quad 1 \pmod 6, \quad 3 \pmod 6, \quad 5 \pmod 6\} \quad \text{or}$$
$$\{0 \pmod 2, \quad 1 \pmod 4, \quad 3 \pmod 4\}.$$

Lemma 10. *Let a, b be two integers such that $|a| \ge 2$ and $|b| = 1$. Let $(u_n)_{n\ge 0}$ be the Lucas sequence defined in (11), and suppose that there exists a finite collection of triples (p_i, m_i, r_i), $1 \le i \le t$ with the following properties:*

(i) All primes p_i are distinct.

(ii) The residue classes $r_i \pmod{m_i}$ form a covering system.

(iii) $p_i \mid u_{m_i}$ for all $1 \le i \le t$.

Then there exist two relatively prime positive integers x_0 and x_1 such that each term of the sequence $(x_n)_{n\ge 0}$ defined in (1) is composite.

Proof. Let $P = p_1 p_2 \dots p_t$.

By the Chinese remainder theorem, there exist $y, z \in \{0, 1, \dots P-1\}$ satisfying

$$\begin{aligned} y &\equiv u_{m_i - r_i} \pmod{p_i}, \\ z &\equiv u_{m_i - r_i + 1} \pmod{p_i}, \end{aligned} \tag{13}$$

for $i = 1, 2, \dots, t$. Note that there is no prime which divides y, z, and P simultaneously. Indeed, if such a prime p_j were to exist, then there would be two consecutive terms u_n and u_{n+1} both divisible by p_j. Since $u_{n+1} = au_n + bu_{n-1}$ and $|b| = 1$, then $p_j \mid u_{n-1}$. By induction, it follows that $p_j \mid u_1$ which is impossible since $u_1 = 1$.

Let $x_0 \equiv y \pmod P$ and $x_1 \equiv z \pmod P$.

Then we have $x_0 \equiv u_{m_i - r_i} \pmod{p_i}$ and $x_1 \equiv u_{m_i - r_i + 1} \pmod{p_i}$ for all $i = 1, 2, \dots, t$. By induction on n, we obtain $x_{n+1} \equiv u_{m_i - r_i + n} \pmod{p_i}$ for every $n \ge 0$ and every $1 \le i \le t$.

Since the residue classes $r_i \pmod{m_i}$ form a covering system, each nonnegative integer n belongs to one of these classes, say $n = r_i + k\, m_i$ for some $k \ge 0$ and some $i \in \{1, 2, \dots, t\}$. This implies that

$$x_{n+1} \equiv u_{m_i - r_i + n} \pmod{p_i} \equiv u_{m_i(k+1)} \pmod{p_i} \equiv 0 \pmod{p_i}, \tag{14}$$

since $p_i \mid u_{m_i}$ and $u_{m_i} \mid u_{m_i(k+1)}$. Hence, every term of the sequence $(x_n)_n$ is divisible by some prime p_i. It remains to choose x_0 and x_1, relatively prime positive integers $x_0 \equiv y \pmod P$ and $x_1 \equiv z \pmod P$ such that $|x_n| \ge P$ for every $n \in \mathbb{N}$.

In order to achieve this, take $x_0 = y+P$ and $x_1 = z+kP$ where $k \geq 2$ and $\gcd(x_0, x_1) = 1$. Using Lemma 4 with $n_1 = y + P$, $n_2 = z$ and $n_3 = P$ shows that such a choice is always possible. Recall that we proved earlier that $\gcd(y, z, P) = 1$. Such a choice implies that $0 < P \leq x_0 < x_1$, and since $|a| > |b| = 1$, Lemma 3 implies that $(|x_n|)_n$ is a strictly increasing sequence. It follows that $|x_n| \geq P$ for all $n \geq 0$, and therefore each such x_n is composite. □

We can now prove the theorem if $|b| = 1$.

Suppose first that $b = -1$ and $|a| = p_1^s \geq 4$. Then, by Lemma 7, there are four distinct primes p_1, p_2, p_3, p_4 dividing u_2, u_6, u_6, u_6, respectively. The theorem follows after using Lemma 10 for the triples $(p_1, 2, 0), (p_2, 6, 1), (p_3, 6, 3), (p_4, 6, 5)$.

As a numerical illustration, suppose that $a = -9, b = -1$. Then, as described in the paragraph following the proof of Lemma 7, we have $p_1 = 3, p_2 = 2, p_3 = 5$ and $p_4 = 13$.

The system (13) becomes

$$\begin{array}{llll} y \equiv u_2 & \pmod{3} & \qquad z \equiv u_3 & \pmod{3} \\ y \equiv u_5 & \pmod{2} & \qquad z \equiv u_6 & \pmod{2} \\ y \equiv u_3 & \pmod{5} & \qquad z \equiv u_4 & \pmod{5} \\ y \equiv u_1 & \pmod{13} & \qquad z \equiv u_2 & \pmod{13} \end{array}$$

and its solution is $P = p_1p_2p_3p_4 = 390, y = 105, z = 134$. Since $y < z$ and $\gcd(y, z) = 1$, one can safely take $x_0 = y = 105$ and $x_1 = z = 134$. Then the sequence $(|x_n|)_{n\geq 0}$ is strictly increasing, and for every $n \geq 0$ we have that $x_{2n} \equiv 0 \pmod{3}, x_{6n+1} \equiv 0 \pmod{2}, x_{6n+3} \equiv 0 \pmod{5}$ and $x_{6n+5} \equiv 0 \pmod{13}$. It follows that all terms of the sequence are composite.

Next suppose that $b = 1$ and $|a| = p_1^s \geq 6$ for some prime p_1. If $p_1 \neq 3$, then by Lemma 8 there exist three distinct primes p_1, p_2, p_3, dividing u_2, u_4, u_4, respectively.

The theorem follows after using Lemma 10 for the triples $(p_1, 2, 0), (p_2, 4, 1), (p_3, 4, 3)$. As in the case $b = -1$, we present the details in a couple of particular cases.

Suppose first that $a = 8, b = 1$. Then, as described in the paragraph following the proof of Lemma 8, we have $p_1 = 2, p_2 = 3, p_3 = 11$. The system (13) becomes

$$\begin{array}{llll} y \equiv u_0 & \pmod{2} & \qquad z \equiv u_1 & \pmod{2} \\ y \equiv u_3 & \pmod{3} & \qquad z \equiv u_4 & \pmod{3} \\ y \equiv u_1 & \pmod{11} & \qquad z \equiv u_2 & \pmod{11} \end{array}$$

and its solution is $P = p_1p_2p_3 = 66, y = 56, z = 63$. Note that in this case $\gcd(y, z) = 7 > 1$. Still, one can safely take $x_0 = y = 56$ and $x_1 = z + P = 129$, and now $\gcd(x_0, x_1) = 1$ as desired. Then, since $0 < x_0 < x_1$, the sequence $(|x_n|)_{n\geq 0}$ is strictly increasing and for every $n \geq 0$ we have that $x_{2n} \equiv 0 \pmod{2}, x_{4n+1} \equiv 0 \pmod{3}$, and $x_{4n+3} \equiv 0 \pmod{11}$. It follows that all terms of the sequence are composite.

Next assume that $a = -49, b = 1$. Then $p_1 = 7, p_2 = 3, p_3 = 89$, and the system (13) becomes

$$\begin{aligned} y &\equiv u_0 \pmod 7 & z &\equiv u_1 \pmod 7 \\ y &\equiv u_3 \pmod 3 & z &\equiv u_4 \pmod 3 \\ y &\equiv u_1 \pmod{89} & z &\equiv u_2 \pmod{89}. \end{aligned}$$

The solution is $P = p_1p_2p_3 = 1869, y = 980, z = 1464$. Notice that in this case $\gcd(y, z) = 4 > 1$. Still, one can safely take $x_0 = y = 980$ and $x_1 = z+P = 3333$ and now $\gcd(x_0, x_1) = 1$ as desired. Moreover, since $0 < x_0 < x_1$, the sequence $(|x_n|)_{n\geq 0}$ is strictly increasing, and for every $n \geq 0$ we have that $x_{2n} \equiv 0 \pmod 7), x_{4n+1} \equiv 0 \pmod 3$, and $x_{4n+3} \equiv 0 \pmod{89}$. It follows that all terms of the sequence are composite.

For the case $b = 1$ and $|a| = 3^s$ with $s \geq 2$, we use the second part of Lemma 8 to conclude that there are four distinct primes p_1, p_2, p_3, p_4, dividing u_2, u_6, u_6, u_6, respectively. The theorem follows after using Lemma 10 for the triples $(p_1, 2, 0), (p_2, 6, 1), (p_3, 6, 3), (p_4, 6, 5)$.

We show the full details if $a = 9, b = 1$. Then, as mentioned in the paragraph following the proof of Lemma 8, we have $p_1 = 3, p_2 = 2, p_3 = 41, p_4 = 7$. The system (13) becomes

$$\begin{aligned} y &\equiv u_2 \pmod 3 & z &\equiv u_3 \pmod 3 \\ y &\equiv u_5 \pmod 2 & z &\equiv u_6 \pmod 2 \\ y &\equiv u_3 \pmod{41} & z &\equiv u_4 \pmod{41} \\ y &\equiv u_1 \pmod 7 & z &\equiv u_2 \pmod 7. \end{aligned}$$

Solving, we obtain $P = p_1p_2p_3p_4 = 1722, y = 1107, z = 1444$. In this case we can simply choose $x_0 = y = 1107$ and $x_1 = z = 1144$.

Then $\gcd(x_0, x_1) = 1$, and since $0 < x_0 < x_1$, the sequence $(|x_n|)_{n\geq 0}$ is strictly increasing. Moreover, for every $n \geq 0$ we have that $x_{2n} \equiv 0 \pmod 3, x_{6n+1} \equiv 0 \pmod 2, x_{6n+3} \equiv 0 \pmod{41}$, and $x_{6n+5} \equiv 0 \pmod 7$. It follows that all terms of the sequence are composite as desired.

At this point we have proved the main theorem when $|b| = 1$ for all but finitely many values of a. We still have to study what happens when $b = -1$ and $|a| \leq 3$ as well as the cases when $b = 1$ and $|a| \leq 5$. Recall that the cases $a = 0$ and $(a, b) = (\pm 2, -1)$ were already handled in section 2.

For most of these cases, we will still use Lemma 10; the only difference is that the set of triples $\{p_i, m_i, r_i\}_{i=1}^{i=t}$ is occasionally going to be slightly more numerous.

We summarize our findings in the table below. We invite the reader to verify that the collections $\{p_i, m_i, r_i\}_{i=1}^{i=t}$ do indeed satisfy the three conditions in Lemma 10. Note that in each case we have $0 < x_0 < x_1$, and since $|a| > |b| = 1$, Lemma 3 implies that $(|x_n|)_{n\geq 0}$ is strictly increasing.

It remains to see what happens when $|a| = |b| = 1$, as in these cases Lemma 10 does not apply.

a	b	$\{(p_i, m_i, r_i)\}$	x_0	x_1
5	1	$(5,2,0),(2,6,1),(7,6,3),(13,6,5)$	495	1136
-5	1	$(5,2,0),(2,6,1),(7,6,3),(13,6,5)$	495	866
4	1	$(2,2,0),(3,4,1),(7,8,3),(23,8,7)$	116	165
-4	1	$(2,2,0),(3,4,1),(7,8,3),(23,8,7)$	116	801
3	1	$(3,2,0),(11,4,1),(7,8,3),(17,8,7)$	1803	3454
-3	1	$(3,2,0),(11,4,1),(7,8,3),(17,8,7)$	1803	3091
2	1	$(2,2,0),(5,3,0),(3,4,1),(7,6,5),(11,12,7)$	260	807
-2	1	$(2,2,0),(5,3,0),(3,4,1),(7,6,5),(11,12,7)$	260	1503
3	-1	$(3,2,0),(2,3,0),(7,4,3),(47,8,5),(23,12,5),(1103,24,1)$	7373556	2006357
-3	-1	$(3,2,0),(2,3,0),(7,4,3),(47,8,5),(23,12,5),(1103,24,1)$	7373556	14686445

Table 1: Covering triples for the cases $b = 1, a = \pm 2, \pm 3, \pm 4, \pm 5$ and $b = -1, a = \pm 3$

If $a = -1, b = -1$, then it can be easily verified that the sequence given by the recurrence $x_{n+1} = -x_n - x_{n-1}$ has period 3. Hence, if one chooses $x_0 = 8$ and $x_1 = 27$, then $x_2 = -35$, and due to the periodic behavior all terms of the sequence are composite.

Similarly, if $a = 1, b = -1$, then the sequence given by the recurrence $x_{n+1} = x_n - x_{n-1}$ has period 6. Again, if one chooses $x_0 = 8$ and $x_1 = 35$, then the first few terms of the sequence are $8, 35, 27, -8, -35, -37, 8, 35, 27, \ldots$; that is, x_n is always composite.

If $a = b = 1$, then Vsemirnov's pair $v_0 = 106276436867$, $v_1 = 35256392432$ shows that all the numbers

$$v_n = v_{n-1} + v_{n-2} = v_1 F_n + v_0 F_{n-1} \tag{15}$$

are composite. Here, F_n is the nth Fibonacci number, where $F_{-1} = 1, F_0 = 0, F_1 = 1$.

For the case $a = -1, b = 1$, we follow the solution in [3].

It can be easily checked that the general term of the sequence $x_{n+1} = -x_n + x_{n-1}$ can be written as

$$x_n = (-1)^{n+1} x_1 F_n + (-1)^n x_0 F_{n-1}, \quad n \geq 0. \tag{16}$$

We choose $x_0 = v_0 - v_1 = 71020044435$ and $x_1 = v_0 = 106276436867$. It is easy to check that x_0 and x_1 are relatively prime composite integers. Moreover, from (15) and (16) we obtain that

$$\begin{aligned} x_n &= (-1)^{n+1} v_0 F_n + (-1)^n (v_0 - v_1) F_{n-1} + = (-1)^{n+1} v_1 F_{n-1} + (-1)^{n+1} v_0 (F_n - F_{n-1}) = \\ &= (-1)^{n+1} v_1 F_{n-1} + (-1)^{n+1} v_0 F_{n-2} = (-1)^{n+1} \left(v_1 F_{n-1} + v_0 F_{n-2} \right) = (-1)^{n-1} v_{n-1}. \end{aligned}$$

Hence, $|x_n| = v_{n-1}$ is composite for all $n \geq 0$. The proof of the theorem is now complete.

6 A surprising result

In this section we prove the following:

Theorem 11. *Consider the integers a, b such that $|a| \geq 3$ and $b = -1$. Let $u_0 = 0$, $u_1 = 1$ and $u_{n+1} = au_n + bu_{n-1} = au_n - u_{n-1}$ be the Lucas sequence of the first kind associated with a and -1. Then $|u_n|$ is composite for all $n \geq 3$.*

Proof. One has $u_3 = a^2 - 1$, and $u_4 = a^3 - 2a$, which are obviously composite. Since $|a| > |b| = 1$, Lemma 3 implies that the sequence $(|u_n|)_{n\geq 0}$ is strictly increasing. Suppose for the sake of contradiction that there exists an $n \geq 2$ such that $|u_{n+1}| = p$, where p is some prime number. Since $|u_{n+1}| \geq |u_3| = a^2 - 1$ it follows that necessarily $p > |a|$.

Now using Lemma 2 for the sequence $(u_n)_{n\geq 0}$, equality (4) becomes

$$u_{n+1}^2 - au_nu_{n+1} + u_n^2 = u_1^2 - au_1u_0 + u_0^2,$$

and since $u_0 = 0$, $u_1 = 1$, and $|u_{n+1}| = p$, we obtain that

$$u_n^2 \pm apu_n + p^2 - 1 = 0. \tag{17}$$

Regard the above equation as a quadratic in u_n. Since $u_n \in \mathbb{Z}$, it is necessary that the discriminant is a perfect square, that is, there exist a nonnegative integer c such that

$$a^2p^2 - 4(p^2 - 1) = c^2 \quad \text{from which} \quad (a^2 - 4)p^2 = c^2 - 4 = (c - 2)(c + 2). \tag{18}$$

Since $|a| \geq 3$, one can assume that $c \geq 3$. Since p is a prime and p^2 divides $(c-2)(c+2)$, we have two possibilities. If p divides both $c - 2$ and $c + 2$ then p divides 4, which means that $p = 2$. However, this is impossible since $p > |a| \geq 3$. Otherwise, p^2 divides either $c - 2$ or $c + 2$. In either case we obtain that $c + 2 \geq p^2$. Using this inequality in (18), it follows that

$$(a^2 - 4)p^2 = (c - 2)(c + 2) \geq p^2(p^2 - 4) \implies |a| \geq p, \quad \text{a contradiction.}$$

This completes the proof. ∎

In particular, the above theorem holds if $a = 4$ and $b = -1$, thus answering a question of Vos Post (see A001353 in [14]).

What is remarkable about this situation is that while u_n is composite for every $n \geq 3$, it seems likely that there is no finite set of primes $p_1, p_2, \ldots p_t$ such that every u_n is divisible by some p_i, $1 \leq i \leq t$. In fact, we suspect the following is true.

Conjecture 12. Let $(u_n)_{n\geq 0}$ be the Lucas sequence of the first kind associated with some $|a| \geq 3$ and $b = -1$. Then for any two different primes p and q, u_p and u_q are relatively prime.

If true, this conjecture would immediately imply that there is no finite set of primes $p_1, p_2, \ldots p_t$ such that every u_n is divisible by some p_i, $1 \leq i \leq t$.

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