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Congruent Number Elliptic Curves Related to Integral Solutions of $m^2 = n^2 + nl + n^2$

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Abstract

We construct an infinite family of congruent number elliptic curves, each with rank at least two, which are related to integral solutions of $m^2 = n^2 + nl + l^2$.

1 Introduction

Elliptic curves and their geometric and algebraic structure have been a flourishing field of research in the past. They find prominent applications in cryptography and played a key role in the proof of Fermat's Last Theorem. A salient feature of the algebraic structure of an elliptic curve is its rank. Among general elliptic curves, congruent number curves of high rank are of particular interest (see, for example, [2]). More difficult than finding an individual congruent number curve of high rank is to find infinite families of such curves. Johnstone and Spearman [7] constructed such a family with rank at least three, which is related to rational points on the biquadratic curve $w^2 = t^4 + 14t^2 + 4$. In the present paper, we show an elementary construction for an infinite family of congruent number curves of rank at least two that are related to the quadratic diophantine equation $m^2 = n^2 + nl + l^2$, and which have three integral points with positive y-coordinate on a straight line. Incidentally, some members of the family exhibit surprisingly high individual rank, namely rank five (whereas the members of the family given by Johnstone and Spearman [7] all have rank three). We start by recalling some basic results on congruent numbers.

A positive integer A is called a *congruent number* if A is the area of a right-angled triangle with three rational sides. So A is congruent if and only if there exists a rational Pythagorean triple (a,b,c) (i.e., $a,b,c\in\mathbb{Q}$, $a^2+b^2=c^2$, and $ab\neq 0$) such that $\frac{ab}{2}=A$. The sequence of integer congruent numbers starts with

$$5,6,7,13,14,15,20,21,22,23,24,28,29,30,31,34,37,\ldots$$

(see, for example, the On-Line Encyclopedia of Integer Sequences [10, A003273]).

It is well-known that A is a congruent number if and only if the cubic curve

$$C_A:\ y^2=x^3-A^2x$$

has a rational point (x_0,y_0) with $y_0\neq 0$. The cubic curve C_A is called a *congruent number elliptic curve* or just *congruent number curve*. With respect to some congruent number A, the correspondence between rational points (x,y) with $y\neq 0$ on the congruent number curve C_A on the one hand, and rational Pythagorean triples (a,b,c) with $ab=2A$ on the other hand, is given by

$$\big(a,b,c\big)\ \mapsto\ \left(\frac{b(b+c)}{2},\frac{b^2(b+c)}{2}\right), \tag{1}$$

and

$$\big(x,y\big)\ \mapsto\ \left(\frac{2xA}{y},\frac{x^2-A^2}{y},\frac{x^2+A^2}{y}\right). \tag{2}$$

For a positive integer A, a triple (a,b,c) of non-zero rational (integral) numbers such that $a^2+b^2=c^2$ and $A=\left|\frac{ab}{2}\right|$ is called a *rational (integral) Pythagorean A-triple*. Notice that if (a,b,c) is a rational Pythagorean A-triple, then A is a congruent number and $|a|,|b|,|c|$ are the lengths of the sides of a right-angled triangle with area A. Notice also that we allow a,b,c to be negative. In particular, for any positive integers m and n with $m>n$, the triple

$$\big(\underbrace{2mn}_{a},\underbrace{m^2-n^2}_{b},\underbrace{m^2+n^2}_{c}\big)$$

is an integral Pythagorean A-triple. In this case, we obtain $A=mn(m^2-n^2)$ and by (1)

$$(a,b,c)\mapsto\big(\underbrace{m^2(m^2-n^2)}_{x},\underbrace{m^2(m^2-n^2)^2}_{y}\big)\,. \tag{3}$$

Notice that in this case, the point (x,y) on C_A that corresponds to the integral Pythagorean A-triple (a,b,c) is an integral point.

Concerning the equations

$$m=n^2+nl+l^2 \tag*{eq(m)}$$

and

$$m^2 = n^2 + nl + l^2 \qquad \text{eq}(m^2)$$

we first prove the following result (for a geometric representation of integral solutions of $x^2 + xy + y^2 = m^2$, see Halbeisen and Hungerbühler [5]):

Proposition 1. *Let $p_1 < p_2 < \cdots < p_j$ be primes, such that $p_i \equiv 1 \pmod 6$ for $1 \le i \le j$, and let*

$$m = \prod_{i=1}^{j} p_i.$$

(a) The number of positive, integral solutions (n, l) of $\text{eq}(m)$ with $l < n$ is 2^j.

(b) For each integral solution of $\text{eq}(m)$, n and l are relatively prime and neither n nor l is a multiple of p_i (for $1 \le i \le j$).

(c) The number of positive, integral solutions (n, l) of $\text{eq}(m^2)$ with $l < n$ is $\frac{3^j - 1}{2}$.

(d) Among the $\frac{3^j-1}{2}$ integral solutions solutions (n, l) of $\text{eq}(m^2)$ with $l < n$ we find 2^{j-1} solutions (n, l) such that n and l are relatively prime. In particular, if $j = 1$ and $p \equiv 1 \pmod 6$, then the solution in positive integers $l < n$ of

$$p^2 = n^2 + nl + l^2,$$

such that n and l are relatively prime, is unique.

Proof.
(a) By Dickson [1, Exercises XXII.2, p. 80], the number of integral solutions of $\text{eq}(m)$ is $6E(m)$, where $E(m)$ is the excess of the number of divisors $3h + 1$ of m over the number of divisors of the form $3h+2$. By definition of m, $E(m) = 2^j$. Now, with each positive, integral solution (n, l) of $\text{eq}(m)$ with $0 < l < n$ we obtain the following 12 pairwise different integral solutions:

$$\begin{gathered}(n, l),\ (l, n),\ (-n, -l),\ (-l, -n),\\ (-n, n + l),\ (n + l, -n),\ (n, -n - l),\ (-n - l, n),\\ (-l, n + l),\ (n + l, -l),\ (l, -n - l),\ (-n - l, l).\end{gathered}$$

So, if $e(m)$ denotes the number of positive, integral solutions (n, l) of $\text{eq}(m)$ with $l < n$, then $6E(m) \ge 12e(m)$. On the other hand, every integral solution of $m = n^2 + nl + l^2$ corresponds to a unique positive, integral solution (n, l) with $l < n$, which implies that $6E(m) = 12e(m)$ and consequently we obtain $e(m) = 2^{j-1}$.

(b) This follows immediately from the definition of m.

(c) Again by Dickson [1, Exercises XXII.2, p. 80], the number of integral solutions of eq(m^2) is $6E(m^2)$, where $E(m^2) = 3^j$. Let $e(m^2)$ denote the number of positive, integral solutions (n,l) of $m^2 = n^2+nl+l^2$ with $l < n$. In addition to the $12e(m^2)$ integral solutions of eq(m^2), we have the 6 solutions

$$(m,0),\ (0,m),\ (-m,0),\ (0,-m),\ (-m,m),\ (m,-m).$$

So, $6E(m) = 12e(m) + 6$ and consequently we obtain $e(m) = \frac{3^j-1}{2}$.

(d) For the sake of simplicity, let us call a positive, integral solution (n,l) of eq(m^2) with $l < n$ a *normal solution*. Among the normal solutions (n,l) of eq(m^2), we distinguish between the ones with n and l relatively prime, which we call *primitive solutions*, and the other ones, which we call *composite*. For a given m, there is a one-to-one correspondence between composite solutions of eq(m^2) with $(n,l) = d > 1$ and primitive solutions of eq($(m/d)^2$) via the correspondence

$$m^2 = n^2 + nl + l^2 \iff (m/d)^2 = (n/d)^2 + (n/d)(l/d) + (l/d)^2\,.$$

The proof is now by induction on j. For $j = 1$, the statement follows from the definition of m. Suppose the statement holds when m has $j-1$ or less prime factors. Then, let m have j prime factors. We observe by the above correspondence that the number of composite solutions to eq(m^2) is

$$\sum_{\substack{r|m \\ 0<r<m}} P(r^2)$$

where $P(r^2)$ is the number of primitive solutions to $r^2 = n^2+nl+l^2$. By the inductive step, if r is comprised of i prime factors of m then $P(r^2) = 2^{i-1}$. Furthermore, the number of distinct factors r of m with i prime factors is $\binom{j}{i}$, and therefore, by the binomial theorem, the number of composite solutions to eq(m^2) is

$$\sum_{\substack{r|m \\ 0<r<m}} P(r^2) = \sum_{i=1}^{j-1} \binom{j}{i} 2^{i-1} = \frac{1}{2}\big((1+2)^j - 2^j - 1\big) = \frac{3^j-1}{2} - 2^{j-1}\,.$$

Since the total number of solutions to eq(m^2) is $\frac{3^j-1}{2}$, by subtracting the number of composite solutions we finally obtain $P(m^2) = 2^{j-1}$, which completes the proof. □

Let us now consider the relationship between positive, integral solutions of $m^2 = n^2 + nl + l^2$ and integral Pythagorean triples.

If m, n, l are positive integers such that $m^2 = n^2 + nl + l^2$, then, for $k := n + l$, each of the following three triples

$$\big(\underbrace{2mn}_{a_1},\ \underbrace{m^2-n^2}_{b_1},\ \underbrace{m^2+n^2}_{c_1}\big)\,,$$

$$\big(\underbrace{2ml}_{a_2},\ \underbrace{m^2-l^2}_{b_2},\ \underbrace{m^2+l^2}_{c_2}\big),$$

$$\big(\underbrace{2mk}_{a_3},\ \underbrace{k^2-m^2}_{b_3},\ \underbrace{k^2+m^2}_{c_3}\big),$$

is an integral Pythagorean A-triple for

$$A = mn(m^2-n^2) = ml(m^2-l^2) = km(k^2-m^2) = klmn$$

(see Hungerbühler [6]). In particular, with m, n, l and (3) we obtain three distinct integral points on C_A.

As a matter of fact we would like to mention that the three integral points on C_A that correspond to an integral solution of $m^2 = n^2 + nl + l^2$ lie on a straight line.

Let us now turn back to the curve C_A, where A is a congruent number. One can readily check that the three points $(0, 0)$ and $(\pm A, 0)$ are the only points on C_A of order 2 with respect to the group law of elliptic curves. Moreover, one can show that these three points, together with the point at infinity, are the only points of finite order (for an elementary proof of this result, which is based on a theorem of Fermat; see Halbeisen and Hungerbühler [4]). This implies that if A is a congruent number and (x_0, y_0) is a rational point on C_A with $y_0 \neq 0$, then the order of (x_0, y_0) is infinite. So, Mordell's theorem (which states that the group of rational points on C_A is finitely generated) and the fundamental theorem of finitely generated abelian groups imply that the group of rational points on a congruent number curve C_A is isomorphic to

$$\underbrace{\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}_{\text{torsion group}}\times\mathbb{Z}^r,$$

where $r > 0$ is the aforementioned *rank* of C_A.

2 Rank at least two

Based on integral solutions of $m^2 = n^2 + nl + l^2$, we will show that there are infinitely many congruent number curves C_A with rank at least two — where many of the curves C_A have rank three or four and several curves have even rank five (see Section 3).

The following result, which can be found in Silverman and Tate [9, Chapter III.6.], allows us to compute the rank — or at least a lower bound of the rank — of certain elliptic curves. For simplicity, we state the result just for congruent number curves.

Proposition 2. *Let b be a non-zero integer and let $\bar{b} := -4b$. Furthermore, let*

$$B := \big\{b_1 \in \mathbb{Z} : b_1 \mid b,\ \ \textit{and}\ \ b_1\ \textit{is square-free}\big\}$$

and

$$\bar{B} := \big\{\bar{b}_1 \in \mathbb{Z} : \bar{b}_1 \mid \bar{b},\ \ \textit{and}\ \ \bar{b}_1\ \textit{is square-free}\big\}.$$

Finally, let β_b and $\beta_{\bar{b}}$ be the number of integers $b_1 \in B$ and $\bar{b}_1 \in \bar{B}$, respectively, such that the corresponding equation

$$\begin{aligned} b_1 M^4 + \tfrac{b}{b_1} e^4 &= N^2 \quad (4) \\ \bar{b}_1 \bar{M}^4 + \tfrac{\bar{b}}{\bar{b}_1} \bar{e}^4 &= \bar{N}^2 \quad (5) \end{aligned}$$

has integral solutions, where $e \neq 0$, $\bar{e} \neq 0$, and $(M, e) = (\bar{M}, \bar{e}) = 1$.

Then the rank r of the curve $y^2 = x^3 + bx$ satisfies the equation

$$2^r = \frac{\beta_b \cdot \beta_{\bar{b}}}{4}.$$

Moreover, if (x, y) is a rational point on the curve $y^2 = x^3 + bx$ with $y \neq 0$, then one can write that point in the form

$$x = \frac{b_1 M^2}{e^2}, \qquad y = \frac{b_1 M N}{e^3}, \tag{6}$$

where M, e, N is an integral solution of (4) with $N > 0$ and $(M, e) = 1$, and vice versa. The analogous statement holds for rational points on the curve $y^2 = x^3 + \bar{b}x$ with respect to equations of the form (5).

Now we are ready to prove

Theorem 3. *Let m, n, l be pairwise relatively prime positive integers, where $m = \prod_{i=1}^{j} p_i$ is a product of pairwise distinct primes $p_i \equiv 1 \pmod 6$ and $m^2 = n^2 + nl + l^2$. Furthermore, let $k := n + l$ and let $A := klmn$. Then the rank of the congruent number curve*

$$C_A : y^2 = x^3 - A^2 x$$

is at least two.

Proof. Since we have at least one rational point (x, y) on C_A with $y \neq 0$, we know that the rank r of C_A is positive. So, to show that the rank of the curve C_A is at least two, it would be enough to show that $\beta_{-A^2} \geq 9$. For this, we have to show that there are integral solutions for (4) for at least 9 distinct square-free integers b_1 dividing $-A^2$, or equivalently, we have to find at least 9 rational points on C_A, such that the 9 corresponding integers b_1 are pairwise distinct. Even though it would be enough to find integral solutions for (4) for at least 9 distinct square-free integers b_1, we shall give 16 solutions, such that a single additional solution for (4) would give us a rank of at least three (see Proposition 5).

Notice that because of (6), to compute b_1 from a rational point $P = (x, y)$ on C_A with $x \neq 0$, it is enough to know the x-coordinate of P and then compute $x \bmod \mathbb{Q}^{*2}$ (i.e., we compute x modulo squares of rationals). The x-coordinates of the three integral points we get by (1) from the three integral Pythagorean A-triples (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) generated by m, n, l, k, are

$$x_1 = m^2(m^2 - n^2) = m^2 kl\,, \quad x_2 = m^2(m^2 - l^2) = m^2 kn\,, \quad x_3 = k^2(k^2 - m^2) = k^2 nl\,,$$

and modulo squares, this gives us three values for b_1 modulo squares, namely

$$b_{1,1} \equiv kl\,, \quad b_{1,2} \equiv kn\,, \quad b_{1,3} \equiv nl\,.$$

Now, exchanging in each of the three integral Pythagorean A-triples the two catheti a_i and b_i (for $i = 1, 2, 3$), we obtain again three distinct integral points on C_A, whose x-coordinates gives us again three values for b_1 modulo squares, namely

$$b_{1,4} \equiv mn\,, \quad b_{1,5} \equiv ml\,, \quad b_{1,6} \equiv mk\,.$$

Finally, if we replace each hypotenuse c_j of these six integral Pythagorean A-triples with $-c_j$, we obtain again six distinct integral points on C_A, whose x-coordinates give us six values for b_1 modulo squares, namely

$$b_{1,7} \equiv -kl\,, \qquad b_{1,8} \equiv -kn\,, \quad b_{1,9} \equiv -nl$$
$$b_{1,10} \equiv -mn\,, \quad b_{1,11} \equiv -ml\,, \quad b_{1,12} \equiv -mk\,.$$

In addition to these 12 integral points on C_A (and the corresponding b_1's), we have the two integral points $(\pm A, 0)$ on C_A, which give us two more values for b_1 modulo squares, namely

$$b_{1,13} \equiv klmn \qquad \text{and} \qquad b_{1,14} \equiv -klmn.$$

Now, it remains to show that the square-free parts of the $b_{1,j}$'s are pairwise distinct. By assumption, m is square-free and k, l, n are pairwise relatively prime. Therefore, if for some i, j with $1 \leq i < j \leq 14$, $b_{1,i} \equiv b_{1,j} \pmod{\mathbb{Q}^{*2}}$, at least two of the integers k, l, n are squares, say $n = u^2$, and $l = v^2$ or $k = v^2$. Then

$$m^2 = u^4 + u^2v^2 + v^4 \quad \text{(in the case when } l = v^2\text{)},$$

or

$$m^2 = u^4 - u^2v^2 + v^4 \quad \text{(in the case when } k = v^2\text{)}.$$

If $l = v^2$, this implies that $u^2 = 1$ and $v = 0$, or $u = 0$ and $v^2 = 1$, and if $k = v^2$, this implies that $u^2 = 1$ and $v = 0$, $u = 0$ and $v^2 = 1$, or $u^2 = v^2 = 1$ (see, for example, Mordell [8, p. 19f] or Euler [3, p. 16]). So, at most one of the integers k, l, n is a square, which shows that at least 14 equations of the form (4) — for different square-free integers b_1 dividing A^2 — have integral solutions. Notice that so far, because the corresponding points on C_A are integral, we always had $|e| = 1$.

Now, we show that there are also solutions for (4) with $b_{1,15} = 1$ and $b_{1,16} = -1$. Assume first that there is a solution for (4) with $b_{1,15} = 1$ and $e = 1$, i.e., there are positive integers M and N such that

$$M^4 - A^2 = N^2.$$

Then we also have

$$A^2M^4 - A^4 = (AN)^2,$$

which shows that $\tilde{M} := A$, $\tilde{e} := M$, and $\tilde{N} := AN$, satisfy

$$-\tilde{M}^4 + A^2\tilde{e}^4 = \tilde{N}^2,$$

and hence, there is a solution for (4) with $b_{1,16} = -1$. Notice that since $|\tilde{e}| \neq 1$, the corresponding point on C_A is not an integral point.

It remains to find a solution for (4) with $b_{1,15} = 1$ and $e = 1$. Since m, n, l, k are pairwise relatively prime positive integers and $k = n + l$, exactly one of n, l, k is even, i.e., at least one of n and l is odd. Without loss of generality, assume that n is odd. Furthermore, by definition of m, m is odd. Let $p := \frac{m+n}{2}$ and $q := \frac{m-n}{2}$. Then p and q are positive integers. Now, $m = p + q$, $n = p - q$, and $m^2 - n^2 = 4pq$, and since $m^2 - n^2 = kl$, we have

$$A = klmn = 4pq(p + q)(p - q)\,.$$

Notice that since $pq(p+q)$ is even, we have $A \equiv 0 \pmod 8$. An easy calculation shows that $M := p^2 + q^2$ and $N := (p^2 - q^2)^2 - (2pq)^2$ satisfy

$$M^4 - A^2 = N^2.$$

So, there is a solution for (4) with $b_{1,15} = 1$ and $e = 1$, which gives us again an integral point on C_A

This shows that $\beta_{-A^2} \geq 16$ and completes the proof. □

As an immediate consequence we get the following

Corollary 4. *Let m, n, l be as in Theorem 3 and let q be a non-zero integer. Then the rank of the curve C_{Aq^4} is at least two.*

Proof. Notice that if m, n, l are such that $m^2 = n^2 + nl + l^2$, then, for mq, nq, lq, we have $(mq)^2 = (nq)^2 + nq \cdot lq + (lq)^2$, which implies that for $\tilde{A} = kq \cdot lq \cdot mq \cdot nq = Aq^4$, the rank of the curve $C_{\tilde{A}}$ is at least two. □

3 Rank at least three

Proposition 5. *For $A = 341\,880$, the rank of the curve C_A is at least three.*

Proof. For $k = 40$, $l = 7$, $m = 37$, $n = 33$, we have $A = klmn$, $m^2 = n^2 + nl + l^2$, and $k = n+l$. Thus, by Theorem 3, the rank of the curve C_A is a least two. Now, for $b_{1,17} = -30$, which is distinct from the square-free values of $b_{1,1}, \ldots, b_{1,16}$, we get that $M = 98$, $e = 1$, $N = 33\,600$ is an integral solution of (4), which implies that the rank of C_A is at least three. In fact, with the help of sage one can show that the rank of C_A is equal to three. $\square$

As a final remark concerning the rank of congruent number curves, we would like to mention that with the help of sage we found that many of the curves which correspond to an integral solution of $m^2 = n^2 + nl + l^2$ have rank 3 or higher. In fact, we found plenty of curves of rank 3 or 4, as well as the following curves of rank 5:

$A = klmn$	$m = \prod p_i$	l	n	$k = n + l$
237 195 512 400	$7 \cdot 127$	464	561	1 025
8 813 542 297 560	$7 \cdot 13 \cdot 37$	232	3 245	3 477
10 280 171 942 040	$37 \cdot 67$	741	2 024	2 765
81 096 660 783 600	$37 \cdot 103$	2 139	2 261	4 400
225 722 120 463 840	$13 \cdot 19 \cdot 31$	505	7 392	7 897
457 485 316 904 280	$7 \cdot 31 \cdot 37$	895	7 544	8 439
5 117 352 889 729 080	$67 \cdot 223$	1 551	14 105	15 656
281 692 457 452 791 000	$79 \cdot 409$	9 064	26 811	35 875
24 666 188 870 481 576 600	$13 \cdot 31 \cdot 223$	46 169	57 400	103 569

Of special interest might be values of A which are related to an m with few factors, especially to primes m. Recall that if m is prime, then the integers $n, l, n + l$ such that $m^2 = n^2 + nl + l^2$ are unique, and therefore, $A(m) := (n+l)lmn$ is determined by m. Among the 666 prime numbers $m \leq 11\,113$ with $m \equiv 1 \pmod 6$, we found the following 30 values of m such that $C_{A(m)}$ has rank 4:

$$127, 139, 181, 277, 337, 709, 769, 823, 829, 883, 1051, 1087, 1213,$$
$$1747, 1777, 1873, 2137, 2287, 2377, 2467, 2521, 3529, 3877, 3931,$$
$$4129, 4999, 5521, 7573, 9601, 10711.$$

However, we did not find any prime $m \equiv 1 \pmod 6$ such that $C_{A(m)}$ has rank 5.

The preceding observations might indicate that the congruent number curves C_A constructed in Theorem 3 are candidates for high rank congruent number elliptic curves (for another approach, see Dujella, Janfada, Salami [2]).

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