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Relating Fibonacci Numbers to Bernoulli Numbers via Balancing Polynomials

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Abstract

We present new identities involving Fibonacci and Bernoulli numbers, and Lucas and Euler numbers, respectively. To achieve this, we derive general relations between Bernoulli (Euler) polynomials and balancing (Lucas-balancing) polynomials. The derivations make use of elementary methods including generating functions and functional equations. Evaluating these polynomial relations at specific points, we get several new identities for the Fibonacci-Bernoulli and Lucas-Euler pairs. We also state some identities involving Bernoulli and balancing numbers.

1 Introduction and Preliminaries

Bernoulli numbers $(B_n)_{n\geq 0}$ and Fibonacci numbers $(F_n)_{n\geq 0}$ are regarded as the most important and fascinating sequences in mathematics. Relationships between them and between their companion sequences, Euler numbers $(E_n)_{n\geq 0}$ and Lucas numbers $(L_n)_{n\geq 0}$, have been studied by some authors in the past. Zhang and Ma proved [11] a relation between Fibonacci polynomials and Bernoulli numbers. The following identity is a special case of their result:

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} F_k B_{n-k} = n\beta^{n-1}, \tag{1}$$

where $\beta = (1-\sqrt{5})/2$. Byrd [3] derived a result of similar structure involving Lucas and Euler numbers:

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{5}{4}\right)^{\frac{n-k}{2}} L_k E_{n-k} = 2^{1-n}. \tag{2}$$

Still other relations are contained in the articles [2, 4, 7, 8], congruences are studied in [10]. Inspired by these elegant results, the present paper is devoted to develop further relations between these famous number sequences. To achieve this goal, we study relations between Bernoulli (Euler) polynomials and balancing (Lucas-balancing) polynomials. The derivations are based on functional equations for the respective generating functions.

We begin by introducing some notation. As usual, $B_n(x)$ will be used for the nth Bernoulli polynomial and $E_n(x)$ will denote the nth Euler polynomial, where the argument x is assumed to be a complex number. These classical polynomials are defined by [1]

$$H(x,z) = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{z e^{xz}}{e^z - 1} \qquad (|z| < 2\pi) \tag{3}$$

and

$$I(x,z) = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2 e^{xz}}{e^z + 1} \qquad (|z| < \pi). \tag{4}$$

The numbers $B_n(0) = B_n$ and $E_n = 2^n E_n(1/2)$ are the famous Bernoulli and Euler numbers, respectively. Bernoulli numbers are rational numbers starting with $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_4 = -1/30$ and so on. Also, $B_{2n+1} = 0$ for $n \geq 1$. In contrast, Euler numbers are integers. We have $E_0 = 1, E_2 = -1, E_4 = 5$ and $E_{2n+1} = 0$ for $n \geq 0$. Bernoulli and Euler numbers (polynomials) play a distinguished role in many mathematical branches, such as, number theory, combinatorics and analysis. There is a countless number of articles in which their properties are studied. The basic properties can be found in the textbook [1], for instance. For more material we refer to a bibliography of Bernoulli numbers (polynomials) [5]. The following properties of Bernoulli and Euler polynomials will be employed in a sequel: The reciprocal relations are

$$B_n(1-x) = (-1)^n B_n(x) \quad \text{and} \quad E_n(1-x) = (-1)^n E_n(x),$$

and the difference equations are given by

$$n x^{n-1} = (-1)^n B_n(-x) - B_n(x) \;\; \text{and} \;\; 2x^n = E_n(x) + (-1)^n E_n(-x).$$

Euler polynomials can be expressed in terms of Bernoulli polynomials via

$$E_{n-1}(x) = \frac{2}{n}(B_n(x) - 2^n B_n(x/2)).$$

In this article, we choose balancing polynomials $B_n^*(x)$ and Lucas-balancing polynomials $C_n(x)$ to serve as the linking objects between Fibonacci and Bernoulli numbers and Lucas and Euler numbers, respectively. Balancing polynomials are defined by the recurrence (see [9])

$$B_n^*(x) = 6xB_{n-1}^*(x) - B_{n-2}^*(x), \qquad n \geq 2, \tag{5}$$

with the initial terms $B_0^*(x) = 0$ and $B_1^*(x) = 1$. Similarly, Lucas-balancing polynomials are defined by

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x), \qquad n \geq 2, \tag{6}$$

with the initial terms $C_0(x) = 1$ and $C_1(x) = 3x$. The numbers $B_n^*(1) = B_n^*$ and $C_n(1) = C_n$ are called balancing and Lucas-balancing numbers, respectively. It is a routine task to derive the Binet forms for these polynomials. We have

$$B_n^*(x) = \frac{\lambda^n(x) - \lambda^{-n}(x)}{\lambda(x) - \lambda^{-1}(x)} \quad \text{and} \quad C_n(x) = \frac{1}{2}\Big(\lambda^n(x) + \lambda^{-n}(x)\Big), \tag{7}$$

where $\lambda(x) = 3x + \sqrt{9x^2-1}$ and $\lambda^{-1}(x) = 3x - \sqrt{9x^2-1}$. The polynomials are connected to Chebyshev and Legendre polynomials (see [6]). We will exploit two other results about balancing polynomials to prove many of our discoveries. The first result relates balancing polynomials to Fibonacci and Lucas numbers.

Lemma 1. *Balancing and Lucas-balancing polynomials admit the following evaluations:*

$$B_n^*\Big(\frac{L_{2m}}{6}\Big) = \frac{F_{2mn}}{F_{2m}}\,, \qquad C_n\Big(\frac{L_{2m}}{6}\Big) = \frac{L_{2mn}}{2}, \tag{8}$$

$$B_n^*\Big(\frac{i}{6}L_{2m+1}\Big) = i^{n-1}\frac{F_{(2m+1)n}}{F_{2m+1}}\,, \qquad C_n\Big(\frac{i}{6}L_{2m+1}\Big) = i^n\frac{L_{(2m+1)n}}{2}, \tag{9}$$

where m is an integer, $i = \sqrt{-1}$ is the imaginary unit, and F_n and L_n denote Fibonacci and Lucas numbers, respectively.

The second lemma deals with exponential generating functions for balancing and Lucas-balancing polynomials.

Lemma 2. *Let $F(x,z)$ and $G(x,z)$ be the exponential generating functions of $(B_n^*(x))_{n\geq 0}$ and $(C_n(x))_{n\geq 0}$, respectively. Then*

$$F(x,z) = \sum_{n=0}^{\infty} B_n^*(x)\frac{z^n}{n!} = \frac{1}{\sqrt{9x^2-1}}e^{3xz}\sinh(\sqrt{9x^2-1}z), \tag{10}$$

and

$$G(x,z) = \sum_{n=0}^{\infty} C_n(x)\frac{z^n}{n!} = e^{3xz}\cosh(\sqrt{9x^2-1}z), \tag{11}$$

where $\sinh(\cdot)$ *and* $\cosh(\cdot)$ *are the hyperbolic sine and cosine functions.*

2 Results

Our first result is the following theorem.

Theorem 3. *Let $n \geq 1$. Then*

$$\sum_{\substack{k=0 \\ n\equiv k \pmod 2}}^{n} \binom{n}{k} B_k^*(x)(2\sqrt{9x^2-1})^{n-k} B_{n-k} = nC_{n-1}(x), \tag{12}$$

and

$$\sum_{\substack{k=0 \\ n\equiv k \pmod 2}}^{n} \binom{n}{k} C_k(x)(2\sqrt{9x^2-1})^{n-k}(2^{n-k}-1) B_{n-k} = n(9x^2-1)B_{n-1}^*(x). \tag{13}$$

Proof. By (10) and (11) we have

$$zG(x,z) = \sqrt{9x^2-1}z\coth(\sqrt{9x^2-1}z)F(x,z).$$

The left hand side is

$$zG(x,z) = \sum_{n=1}^{\infty} C_{n-1}(x) n \frac{z^n}{n!}.$$

Next, form the power series expansion

$$\coth(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n} z^{2n-1},$$

we immediately get

$$\sqrt{9x^2-1}z\coth(\sqrt{9x^2-1}z) = \sum_{n=0}^{\infty} (2\sqrt{9x^2-1})^{2n} B_{2n} \frac{z^{2n}}{(2n)!}.$$

Now, the relation follows from multiplying the series on the right using Cauchy's rule and comparing coefficients. The second statement is proved analogously utilizing

$$(9x^2-1)zF(x,z) = \sqrt{9x^2-1}z\tanh(\sqrt{9x^2-1}z)G(x,z),$$

in combination with

$$\tanh(z) = \sum_{n=0}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} z^{2n-1}.$$

□

Corollary 4. *Let $n \geq 1$ be an integer. Then we have the following relations between balancing numbers and Bernoulli numbers:*

$$\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n} \binom{n}{k} 32^{\frac{n-k}{2}} B_k^* B_{n-k} = nC_{n-1}, \tag{14}$$

and

$$\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n} \binom{n}{k} 32^{\frac{n-k}{2}} (2^{n-k}-1) C_k B_{n-k} = 8nB_{n-1}^*. \tag{15}$$

Also, for each $m \geq 0$ (or ≥ 1), the following identities are valid:

$$\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} F_{(2m+1)k} F_{2m+1}^{n-1-k} B_{n-k} = \frac{n}{2} L_{(2m+1)(n-1)}, \tag{16}$$

$$\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} (2^{n-k}-1) L_{(2m+1)k} F_{2m+1}^{n-1-k} B_{n-k} = \frac{5n}{2} F_{(2m+1)(n-1)}, \tag{17}$$

$$\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} F_{2mk} F_{2m}^{n-1-k} B_{n-k} = \frac{n}{2} L_{2m(n-1)}, \tag{18}$$

and

$$\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} (2^{n-k}-1) L_{2mk} F_{2m}^{n-1-k} B_{n-k} = \frac{5n}{2} F_{2m(n-1)}. \tag{19}$$

Proof. Evaluate (12) and (13) at the points $x = 1$, $x = i/6L_{2m+1}$ and $x = 1/6L_{2m}$, respectively and use Lemma 1. To simplify the square root recall that $L_n^2 = 5F_n^2 + (-1)^n 4$. □

Special cases of the corollary are the following sums which should be compared with (1):

$$\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} F_k B_{n-k} = \frac{n}{2} L_{n-1}, \tag{20}$$

$$\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} (2^{n-k}-1) L_k B_{n-k} = \frac{5n}{2} F_{n-1}, \tag{21}$$

$$\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\, 2)}}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} F_{2k} B_{n-k} = \frac{n}{2} L_{2n-2}, \tag{22}$$

and

$$\sum_{\substack{k=0\\ n\equiv k \,(\mathrm{mod}\,2)}}^{n} \binom{n}{k} 5^{\frac{n-k}{2}}(2^{n-k}-1)L_{2k}B_{n-k} = \frac{5n}{2}F_{2n-2}. \tag{23}$$

The next theorem deals with sums of weighted products of balancing and Bernoulli polynomials and Lucas-balancing and Euler polynomials, respectively.

Theorem 5. *Let $n \geq 1$. Then*

$$\sum_{k=0}^{n} \binom{n}{k} (2\sqrt{9x^2-1})^{n-k} B_k^*(x) B_{n-k}(x) = n\Big(3x + (2x-1)\sqrt{9x^2-1}\Big)^{n-1}. \tag{24}$$

Also, for $n \geq 0$, we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} (2\sqrt{9x^2-1})^{n-k} C_k(x) E_{n-k}(x) = \Big(3x + (2x-1)\sqrt{9x^2-1}\Big)^{n}. \tag{25}$$

Proof. Since

$$H(x,z) = \frac{ze^{(x-1/2)z}}{e^{z/2}-e^{-z/2}} = \frac{z}{2}\frac{e^{(x-1/2)z}}{\sinh(z/2)},$$

we have the functional relation

$$F(x,z)H(x,2\sqrt{9x^2-1}z) = ze^{(3x+(2x-1)\sqrt{9x^2-1})z}.$$

Expanding both sides in form of power series and comparing the coefficients of z^n yields the first formula. The second formula corresponds to the similar functional relation

$$G(x,z)I(x,2\sqrt{9x^2-1}z) = e^{(3x+(2x-1)\sqrt{9x^2-1})z}.$$

$\square$

Theorem 5 also contains some interesting sums as special cases. In order to give a more detailed exposition, we state each example as a separate corollary.

Corollary 6. *For $n \geq 1$*

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 2^{n-2k-1} B_{n-2k-1} = n(-1)^{n-1}, \tag{26}$$

and for $n \geq 0$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \frac{B_{n-2k+1}}{n-2k+1} 2^{n-2k+1}(1-2^{n-2k+1}) = (-1)^n. \tag{27}$$

Proof. Both expressions follow from (24) and (25) evaluated at $x = 0$ in combination with

$$B_n^*(0) = \begin{cases} 0, & n \text{ even}; \\ (-1)^{\frac{n-1}{2}} & n \text{ odd}; \end{cases} \quad \text{and} \quad C_n(0) = \begin{cases} (-1)^{\frac{n}{2}}, & n \text{ even}; \\ 0 & n \text{ odd}. \end{cases}$$

To complete the proof of the second expression we have used

$$E_n(0) = \frac{2}{n+1} B_{n+1}(1 - 2^{n+1}).$$

□

Corollary 7. *Let $m \geq 1$. Then, for each $n \geq 1$*

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} F_{2mk} F_{2m}^{n-k-1} B_{n-k}\Big(\frac{L_{2m}}{6}\Big) = n\Big(\frac{3L_{2m} + \sqrt{5}F_{2m}(L_{2m} - 3)}{6}\Big)^{n-1}, \tag{28}$$

and for $n \geq 0$

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} L_{2mk} F_{2m}^{n-k} E_{n-k}\Big(\frac{L_{2m}}{6}\Big) = 2\Big(\frac{3L_{2m} + \sqrt{5}F_{2m}(L_{2m} - 3)}{6}\Big)^{n}. \tag{29}$$

Proof. Evaluate (24) and (25) at $x = L_{2m}/6$. □

It is worth to remark that the following two results are contained in the corollary as special cases for $m = 1$:

$$\sum_{k=0}^{n} \binom{n}{k} \Big(\frac{5}{4}\Big)^{\frac{n-k}{2}} \Big(1 - 2^{n-k-1}\Big) F_{2k} B_{n-k} = \frac{n}{2}\Big(\frac{3}{2}\Big)^{n-1}, \tag{30}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \Big(\frac{5}{4}\Big)^{\frac{n-k}{2}} L_{2k} E_{n-k} = 2\Big(\frac{3}{2}\Big)^{n}. \tag{31}$$

Compare with (1) and (2). Also, using the relations $B_n^*(-x) = (-1)^{n+1} B_n^*(x)$ and $C_n(-x) = (-1)^n C_n(x)$ in combination with the difference relations for $B_n(x)$ and $E_n(x)$, explicit evaluations at negative points can be derived easily. For instance, setting $x = -1/2$ in (24) and (25) leads to

$$\sum_{k=0}^{n} \binom{n}{k} \Big(\frac{5}{4}\Big)^{\frac{n-k}{2}} F_{2k}\Big(\Big(1 - 2^{n-k-1}\Big) B_{n-k} + n - k\Big) = \frac{n}{2}\Big(\alpha^2 + \frac{\sqrt{5}}{2}\Big)^{n-1}, \tag{32}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \Big(\frac{5}{4}\Big)^{\frac{n-k}{2}} L_{2k}\Big(1 - \frac{E_{n-k}}{2}\Big) = \Big(\alpha^2 + \frac{\sqrt{5}}{2}\Big)^{n}, \tag{33}$$

where $\alpha = (1 + \sqrt{5})/2$ is the golden ratio.

Corollary 8. *Let $m \geq 0$ and $i = \sqrt{-1}$. Then, for each $n \geq 1$*

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} F_{(2m+1)k} F_{2m+1}^{n-k-1} B_{n-k}\Big(i\frac{L_{2m+1}}{6}\Big) = n\Big(\frac{3L_{2m+1} + \sqrt{5}F_{2m+1}(iL_{2m+1} - 3)}{6}\Big)^{n-1}, \tag{34}$$

and for $n \geq 0$

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} L_{(2m+1)k} F_{2m+1}^{n-k} E_{n-k}\Big(i\frac{L_{2m+1}}{6}\Big) = 2\Big(\frac{3L_{2m+1} + \sqrt{5}F_{2m+1}(iL_{2m+1} - 3)}{6}\Big)^{n}. \tag{35}$$

Proof. Evaluate (24) and (25) at $x = iL_{2m+1}/6$. □

When $m = 0$, then

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} F_k B_{n-k}\Big(\frac{i}{6}\Big) = n\Big(\beta + \frac{\sqrt{5}i}{6}\Big)^{n-1}, \tag{36}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} L_k E_{n-k}\Big(\frac{i}{6}\Big) = 2\Big(\beta + \frac{\sqrt{5}i}{6}\Big)^{n}, \tag{37}$$

where $\beta = (1-\sqrt{5})/2 = -1/\alpha$.

Indeed, we can prove more for these special sums.

Theorem 9. *For $n \geq 1$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} F_k B_{n-k}(x) = n\Big(\beta + \sqrt{5}x\Big)^{n-1}, \tag{38}$$

and for $n \geq 0$ the analogue relation for the Lucas-Euler pair is

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} L_k E_{n-k}(x) = 2\Big(\beta + \sqrt{5}x\Big)^{n}. \tag{39}$$

Proof. We start with

$$H(x, \sqrt{5}z) = \frac{\sqrt{5}z}{2} \frac{e^{(x-1/2)\sqrt{5}z}}{\sinh(\sqrt{5}z/2)}$$

and

$$\sum_{n=0}^{\infty} F_n \frac{z^n}{n!} = \frac{2}{\sqrt{5}} e^{z/2} \sinh(\sqrt{5}z/2).$$

Hence,

$$\left(\sum_{n=0}^{\infty} F_n \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n(x) 5^{n/2} \frac{z^n}{n!}\right) = z e^{z/2} e^{(x-1/2)\sqrt{5}z} = z e^{(\beta+\sqrt{5}x)z},$$

and the proof of the first identity is completed. To prove the second identity we use

$$I(x, \sqrt{5}z) = \frac{e^{(x-1/2)\sqrt{5}z}}{\cosh(\sqrt{5}z/2)}$$

and

$$\sum_{n=0}^{\infty} L_n \frac{z^n}{n!} = 2e^{z/2}\cosh(\sqrt{5}z/2).$$

□

If we set $x = 0$ in (38), then we recover (1). Setting $x = 0$ in (39) gives

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} \frac{1 - 2^{n-k+1}}{n-k+1} L_k B_{n-k+1} = \beta^n. \tag{40}$$

Similarly, setting $x = 1/2$ in (38) results in

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} \left(2^{1-(n-k)} - 1\right) F_k B_{n-k} = n 2^{1-n}, \tag{41}$$

whereas setting $x = 1/2$ in (39) gives (2). Also, using the reciprocal relations for $B_n(x)$ and $E_n(x)$ and $\alpha - \beta = \sqrt{5}$, we immediately get the alternating version of Theorem 9:

Corollary 10. *The alternating variants for the Fibonacci-Bernoulli and Lucas-Euler pairs equal*

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} (-1)^{n-k} F_k B_{n-k}(x) = n\left(\alpha - \sqrt{5}x\right)^{n-1}, \tag{42}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} (-1)^{n-k} L_k E_{n-k}(x) = 2\left(\alpha - \sqrt{5}x\right)^{n}. \tag{43}$$

Now, the value $x = 0$ produces

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} (-1)^{n-k} F_k B_{n-k} = n\alpha^{n-1}, \tag{44}$$

which is the alternating version of (1) and

$$\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} (-1)^{n-k} \frac{1 - 2^{n-k+1}}{n-k+1} L_k B_{n-k+1} = \alpha^n. \tag{45}$$

Corollary 11. *Let n and q be two integers with $n \geq 1$ and $q \geq 2$. Then it holds that*

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} \left(q^{1-(n-k)} - 1\right) F_k B_{n-k} \\
= &\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} (-1)^{n-k} \left(q^{1-(n-k)} - 1\right) F_k B_{n-k} \\
= &\; nq^{1-n} \sum_{r=1}^{q-1} \left(\beta q + \sqrt{5} r\right)^{n-1} = nq^{1-n} \sum_{r=1}^{q-1} \left(\alpha q - \sqrt{5} r\right)^{n-1}. \qquad (46)
\end{aligned}$$

Proof. The statement follows from (38) and (42) combined with Raabe's formula

$$\frac{1}{q} \sum_{r=0}^{q-1} B_n\left(x + \frac{r}{q}\right) = q^{-n} B_n(qx).$$

□

When $q = 2$, then (46) gives (41). When $q = 3$, then we obtain

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} \left(3^{1-(n-k)} - 1\right) F_k B_{n-k} \\
= &\sum_{k=0}^{n} \binom{n}{k} 5^{\frac{n-k}{2}} (-1)^{n-k} \left(3^{1-(n-k)} - 1\right) F_k B_{n-k} = n 3^{1-n} L_{2n-2}. \qquad (47)
\end{aligned}$$

Theorem 12. *The following relation between even-indexed Fibonacci numbers and Bernoulli polynomials holds for $n \geq 1$:*

$$\sum_{k=0}^{n} \binom{2n}{2k} 5^{n-k} F_{2k} B_{2(n-k)}(x) = 2^{2(1-n)} \sum_{k=0}^{n} \binom{2n}{2k} k 5^{n-k} (2x-1)^{2(n-k)}. \qquad (48)$$

For $n \geq 0$, the analogue relation involving even-indexed Lucas numbers and Euler polynomials is

$$\sum_{k=0}^{n} \binom{2n}{2k} 5^{n-k} L_{2k} E_{2(n-k)}(x) = 2^{1-2n} \sum_{k=0}^{n} \binom{2n}{2k} 5^{n-k} (2x-1)^{2(n-k)}. \qquad (49)$$

Proof. We start with (48). By (3) we have

$$\sum_{n=0}^{\infty} B_{2n}(x) \frac{z^{2n}}{(2n)!} = \frac{1}{2}(H(x,z) + H(x,-z)) = \frac{z}{2} \frac{\cosh((x-1/2)z)}{\sinh(z/2)}.$$

Also, it is easy to show that

$$\sum_{n=0}^{\infty} F_{2n} \frac{z^{2n}}{(2n)!} = \frac{2}{\sqrt{5}} \sinh(\sqrt{5}z/2) \sinh(z/2).$$

Thus, applying Cauchy's rule, we arrive at

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n}{2k}5^{n-k}F_{2k}B_{2(n-k)}(x)\frac{z^{2n}}{(2n)!}=z\sinh(z/2)\cosh((2x-1)\sqrt{5}z/2).$$

To finish the proof of (48) we must apply Cauchy's rule a second time and keep in mind that

$$z\sinh(z/2)=\sum_{n=0}^{\infty}(2n)2^{1-2n}\frac{z^{2n}}{(2n)!},$$

and

$$\cosh((2x-1)\sqrt{5}z/2)=\sum_{n=0}^{\infty}5^{n}2^{-2n}(2x-1)^{2n}\frac{z^{2n}}{(2n)!}.$$

The proof of (49) uses similar arguments. The key components are

$$\sum_{n=0}^{\infty}E_{2n}(x)\frac{z^{2n}}{(2n)!}=\frac{\cosh((x-1/2)z)}{\cosh(z/2)},$$

which follows from (4) and

$$\sum_{n=0}^{\infty}L_{2n}\frac{z^{2n}}{(2n)!}=2\cosh(\sqrt{5}z/2)\cosh(z/2).$$

□

A slight modification in the above proof leads to the next theorem.

Theorem 13.

$$\sum_{k=0}^{n}\binom{2n}{2k}F_{2k}B_{2(n-k)}(x)=2^{2(1-n)}\sum_{k=0}^{n}\binom{2n}{2k}k5^{k-1}(2x-1)^{2(n-k)}, \tag{50}$$

and

$$\sum_{k=0}^{n}\binom{2n}{2k}L_{2k}E_{2(n-k)}(x)=2^{1-2n}\sum_{k=0}^{n}\binom{2n}{2k}5^{k}(2x-1)^{2(n-k)}. \tag{51}$$

Corollary 14. *For each $n\geq 1$ we have the following relations:*

$$\sum_{k=0}^{n}\binom{2n}{2k}5^{n-k}F_{2k}B_{2(n-k)}=2^{2(1-n)}\sum_{k=0}^{n}\binom{2n}{2k}k5^{n-k}, \tag{52}$$

$$\sum_{k=0}^{n}\binom{2n}{2k}5^{n-k}\left(2^{1-2(n-k)}-1\right)F_{2k}B_{2(n-k)}=n2^{2(1-n)}, \tag{53}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} 5^{n-k} \big(3^{1-2(n-k)} - 1\big) F_{2k} B_{2(n-k)} = 2^{2(1-n)+1} \sum_{k=0}^{n} \binom{2n}{2k} k \Big(\frac{5}{9}\Big)^{n-k}, \tag{54}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} F_{2k} B_{2(n-k)} = 2^{2(1-n)} \sum_{k=0}^{n} \binom{2n}{2k} k 5^{k-1}, \tag{55}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} \big(2^{1-2(n-k)} - 1\big) F_{2k} B_{2(n-k)} = n 5^{n-1} 2^{2(1-n)}, \tag{56}$$

and

$$\sum_{k=0}^{n} \binom{2n}{2k} \big(3^{1-2(n-k)} - 1\big) F_{2k} B_{2(n-k)} = \frac{8}{5} 6^{-2n} \sum_{k=0}^{n} \binom{2n}{2k} k 45^{k}. \tag{57}$$

In addition, we have the identities

$$L_{2n} = 2^{1-2n} \sum_{k=0}^{n} \binom{2n}{2k} 5^{n-k}, \tag{58}$$

$$\sum_{k=0}^{n} \binom{2n}{2k} \Big(\frac{5}{4}\Big)^{n-k} L_{2k} E_{2(n-k)} = 2^{1-2n}, \tag{59}$$

and

$$\sum_{k=0}^{n} \binom{2n}{2k} 2^{-2(n-k)} L_{2k} E_{2(n-k)} = 5^{n} 2^{1-2n}. \tag{60}$$

Proof. Equations (52), (53) and (54) follow from (48) for $x = 0$ (or $x = 1$), $x = 1/2$ and for $x = 1/3$, respectively, where we have also used $B_{2n}(1/3) = 2^{-1}(3^{1-2n} - 1)B_{2n}$. To get equations (55), (56) and (57) set $x = 0$ (or $x = 1$), $x = 1/2$ and for $x = 1/3$ in (50). Identity (58) is (49) (or (51)) evaluated at $x = 0$ (or $x = 1$). Identities (59) and (60) are (49) and (51) evaluated at $x = 1/2$, respectively. $\square$

Interestingly, it is known that

$$\sum_{k=0}^{n} \binom{2n}{2k} = 2^{2n-1}.$$

This leads to somewhat curious versions of the previous results. For instance, L_{2n} can be expressed as

$$L_{2n} = \frac{\sum_{k=0}^{n} \binom{2n}{2k} 5^{k}}{\sum_{k=0}^{n} \binom{2n}{2k}}. \tag{61}$$

We conclude this study with the following theorem.

Theorem 15. *Let $n \geq 1$. Then*

$$\sum_{k=1}^{n} \binom{n}{k} \frac{B_k^*(x) B_{n-k}(x)}{(2\sqrt{9x^2-1})^k} = \sum_{k=1}^{n} \binom{n}{k} k \frac{C_{k-1}(x) E_{n-k}(x)}{(2\sqrt{9x^2-1})^k}. \tag{62}$$

Proof. From (3) and (4) we have

$$\frac{H(x,z)}{I(x,z)} = \frac{z}{2} \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = \frac{z}{2} \coth(z/2).$$

This gives the functional relation

$$F(x,z) H(x, 2\sqrt{9x^2-1}z) = zG(x,z) I(x, 2\sqrt{9x^2-1}z).$$

Expanding both sides in form of power series and comparing the coefficients of z^n yields

$$\sum_{k=0}^{n} \binom{n}{k} \frac{B_k^*(x) B_{n-k}(x)}{(2\sqrt{9x^2-1})^k} = n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{C_k(x) E_{n-1-k}(x)}{(2\sqrt{9x^2-1})^{k+1}}.$$

The proof is completed noting that $B_0^*(x) = 0$ and

$$n \binom{n-1}{k} = (k+1) \binom{n}{k+1}.$$

□

Corollary 16. *The following identities are immediate consequences of Theorem 15:*

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} 32^{-\frac{k}{2}} B_k^* B_{n-k} = \sum_{k=1}^{n} \binom{n}{k} (-1)^{n-k} 32^{-\frac{k}{2}} \frac{2k}{n-k+1} \big(1 - 2^{n-k+1}\big) C_{k-1} B_{n-k+1}, \tag{63}$$

$$\sum_{k=1}^{n} \binom{n}{k} 5^{-\frac{k}{2}} \big(2^{1-(n-k)} - 1\big) F_{2k} B_{n-k} = \sum_{k=1}^{n} \binom{n}{k} k 5^{-\frac{k}{2}} 2^{-(n-k+1)} L_{2k-2} E_{n-k}, \tag{64}$$

and more generally

$$\sum_{k=1}^{n} \binom{n}{k} 5^{-\frac{k}{2}} F_{2mk} F_{2m}^{-(k+1)} B_{n-k}\Big(\frac{L_{2m}}{6}\Big) = \sum_{k=1}^{n} \binom{n}{k} \frac{k}{2} 5^{-\frac{k}{2}} F_{2m}^{-k} L_{2m(k-1)} E_{n-k}\Big(\frac{L_{2m}}{6}\Big), \tag{65}$$

and

$$\sum_{k=1}^{n} \binom{n}{k} 5^{-\frac{k}{2}} F_{(2m+1)k} F_{2m+1}^{-(k+1)} B_{n-k}\Big(i\frac{L_{2m+1}}{6}\Big) = \sum_{k=1}^{n} \binom{n}{k} \frac{k}{2} 5^{-\frac{k}{2}} F_{2m+1}^{-k} L_{(2m+1)(k-1)} E_{n-k}\Big(i\frac{L_{2m+1}}{6}\Big). \tag{66}$$

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