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Consecutive Patterns in Inversion Sequences II: Avoiding Patterns of Relations

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Abstract

Inversion sequences are integer sequences $e = e_1 e_2 \cdots e_n$ such that $0 \leq e_i < i$ for each i. The study of patterns in inversion sequences was initiated by Corteel-Martinez-Savage-Weselcouch and Mansour-Shattuck in the classical (non-consecutive) case, and later by Auli-Elizalde in the consecutive case, where the entries of a pattern are required to occur in adjacent positions. In this paper we continue this investigation by considering *consecutive patterns of relations*, in analogy to the work of Martinez-Savage in the classical case. Specifically, given two binary relations $R_1, R_2 \in \{\leq, \geq, <, >, =, \neq\}$, we study inversion sequences e with no subindex i such that $e_i R_1 e_{i+1} R_2 e_{i+2}$. By enumerating such inversion sequences according to their length, we obtain well-known quantities such as Catalan numbers, Fibonacci numbers and central polynomial numbers, relating inversion sequences to other combinatorial structures. We also classify consecutive patterns of relations into Wilf equivalence classes, according to the number of inversion sequences avoiding them, and into more restrictive classes that consider the positions of the occurrences of the patterns.

As a byproduct of our techniques, we obtain a simple bijective proof of a result of Baxter-Shattuck and Kasraoui about Wilf-equivalence of vincular patterns, and we prove a conjecture of Martinez and Savage, as well as related enumeration formulas for inversion sequences satisfying certain unimodality conditions.

1 Introduction

A common encoding of permutations is by their inversion sequences. Specifically, denoting by S_n the set of permutations of $[n] = \{1, 2, \ldots, n\}$, and by $\mathbf{I}_n$ the set of inversion sequences of length n—that is, integer sequences $e = e_1 e_2 \cdots e_n$ with $0 \leq e_i < i$ for each i—one can define a bijection $\Theta : S_n \to \mathbf{I}_n$ that assigns to each $\pi \in S_n$ its inversion sequence

$$\Theta(\pi) = e = e_1 e_2 \cdots e_n, \quad \text{where} \quad e_i = |\{j : j < i \text{ and } \pi_j > \pi_i\}|. \tag{1}$$

Clearly, $e_1 + \cdots + e_n$ is the number of inversions of π, namely, pairs (i, j) with $i < j$ and $\pi_i > \pi_j$.

In analogy to patterns in permutations, a research area that has received a lot of attention in the last few decades, one can study patterns in inversion sequences. In this context, a *pattern* is a sequence $p = p_1 p_2 \cdots p_r$ with $p_i \in \{0, 1, \ldots, r-1\}$ for each i, where any value $j > 0$ can appear in p only if $j - 1$ appears as well. Given a word $w = w_1 w_2 \cdots w_k$ over the integers, define its *reduction* to be the word obtained by replacing all the occurrences of the ith smallest entry of w with $i-1$ for all i. Then, an inversion sequence e *contains* the classical pattern $p = p_1 p_2 \cdots p_r$ if there exists a subsequence $e_{i_1} e_{i_2} \cdots e_{i_r}$ of e (where $i_1 < \cdots < i_r$) with reduction p. Otherwise, we say that e *avoids* p. For instance, the inversion sequence $e = 00014224$ avoids the pattern 210, but it contains the pattern 101 because $e_5 e_6 e_8 = 424$ has reduction 101.

The study of classical patterns in inversion sequences was started by Corteel, Martinez, Savage, and Weselcouch [10], and Mansour and Shattuck [19]. Their work connected classical patterns in inversion sequences to other combinatorial structures, which inspired a growing body of research on classical patterns in inversion sequences [5, 7, 17, 16, 18, 20, 25].

Motivated by the work of Corteel et al. [10], and Mansour and Shattuck [19], and by the growing interest in consecutive patterns in permutations [13, 12], we introduced consecutive patterns in inversion sequences and initiated an analogous study [1]. In the definition below, the entries of a consecutive pattern are underlined to distinguish it from a classical pattern.

Definition 1. An inversion sequence e *contains* the consecutive pattern $p = \underline{p_1 p_2 \cdots p_r}$ if there is a consecutive subsequence $e_i e_{i+1} \cdots e_{i+r-1}$ of e whose reduction is p. In this case, we call $e_i e_{i+1} \cdots e_{i+r-1}$ an *occurrence* of p in position i. If e does not contain p, then we say that e *avoids* p. Denote by $\mathbf{I}_n(p)$ the set of inversion sequences of length n that avoid p.

Example 2. The inversion sequence $e = 002241250 \in \mathbf{I}_9$ avoids the consecutive pattern $\underline{210}$, even though it contains the classical pattern 210. On the other hand, e contains $\underline{201}$ because $e_5 e_6 e_7$ is an occurrence of $\underline{201}$ in position 5.

It is often useful to represent an inversion sequence e as an underdiagonal lattice path from the origin to the line $x = n$, consisting of unit horizontal steps $E = (1, 0)$ and unit vertical steps $N = (0, 1)$ and $S = (0, -1)$. Each entry e_i of e is represented by a horizontal step: a segment between the points $(i - 1, e_i)$ and (i, e_i). Any necessary vertical steps are then inserted to make the path connected (see Figure 1).

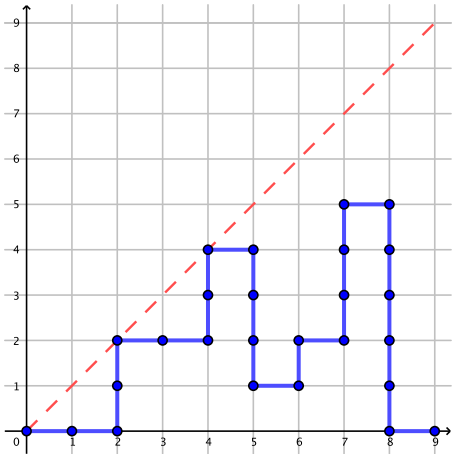


Figure 1: Representation of $e = 002241250 \in \mathbf{I}_9$ as a lattice path.

Extending the systematic study of Corteel et al. [10] for classical patterns in inversion sequences, Martinez and Savage [20] reframe the notion of a pattern of length 3 to instead consider a triple of binary relations between the entries of the occurrence. Given a fixed triple of binary relations (R_1, R_2, R_3), where $R_i \in \{\leq, \geq, <, >, =, \neq, -\}$ for all i, they study the set $\mathbf{I}_n(R_1, R_2, R_3)$ consisting of those $e \in \mathbf{I}_n$ with no subindices $i < j < k$ such that $e_i R_1 e_j$, $e_j R_2 e_k$ and $e_i R_3 e_k$. The symbol $-$ denotes the trivial relation where all elements are related, that is, $x - y$ for all x, y.

Example 3. $\mathbf{I}_n(\geq, \leq, \leq)$ is the set of inversion sequences $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i \geq e_j$, $e_j \leq e_k$ and $e_i \leq e_k$; equivalently, the set of inversion sequences in $\mathbf{I}_n$ avoiding all the patterns in the set $\{000, 001, 101, 102\}$. On the other hand, $\mathbf{I}_n(\geq, \leq, -)$ denotes the set of inversion sequences avoiding all the patterns in the set $\{000, 001, 100, 101, 102, 201\}$.

In this paper, we continue the work we initiated on consecutive patterns in inversion sequences [1], by considering consecutive analogues of the notions introduced by Martinez and Savage [20]. Specifically, we focus on the consecutive analogues of the sets $\mathbf{I}_n(R_1, R_2, -)$. These are the most natural ones to study, since the general case would impose a restrictive relation R_3 between non-consecutive entries.

Definition 4. Let $R_1, R_2 \in \{\leq, \geq, <, >, =, \neq\}$. An inversion sequence e *contains* the *consecutive pattern of relations* $\underline{(R_1, R_2)}$ if there is an i such that $e_i R_1 e_{i+1}$ and $e_{i+1} R_2 e_{i+2}$. In this case, we call $e_i e_{i+1} e_{i+2}$ an *occurrence* of $\underline{(R_1, R_2)}$ in position i. If e does not contain $\underline{(R_1, R_2)}$, then we say that e *avoids* $\underline{(R_1, R_2)}$. Denote by $\mathbf{I}_n\underline{(R_1, R_2)}$ the set of inversion sequences of length n that avoid $\underline{(R_1, R_2)}$.

Example 5. The inversion sequence $e = 002241250$ contains $\underline{(>, <)}$ because $e_5 e_6 e_7 = 412$ is an occurrence of this pattern. However, e avoids $\underline{(=, >)}$, and so $e \in \mathbf{I}_9\underline{(=, >)}$.

Note that an occurrence of a consecutive pattern $\left(\underline{R_1, R_2}\right)$ in an inversion sequence is also an occurrence of some consecutive pattern of length 3. Thus, every set $\mathbf{I}_n\left(\underline{R_1, R_2}\right)$ can be expressed as an intersection $\bigcap_p \mathbf{I}_n(p)$ where p ranges over the consecutive patterns $p = \underline{p_1 p_2 p_3}$ satisfying $p_1 R_1 p_2$ and $p_2 R_2 p_3$. For instance, since occurrences of $\left(\underline{\geq, =}\right)$ are occurrences of either $\underline{100}$ or $\underline{000}$, we can write $\mathbf{I}_n\left(\underline{\geq, =}\right) = \mathbf{I}_n\left(\underline{000}\right) \cap \mathbf{I}_n\left(\underline{100}\right)$.

This paper focuses on the enumeration of the sets $\mathbf{I}_n\left(\underline{R_1, R_2}\right)$. These sets often exhibit more structure than the sets $\mathbf{I}_n(p)$ avoiding a single consecutive pattern of length 3. Consequently, the sequences $\left|\mathbf{I}_n\left(\underline{R_1, R_2}\right)\right|$ are often simpler than the sequences $|\mathbf{I}_n(p)|$, which we studied before [1], and they provide more connections to other combinatorial objects and well-known integer sequences.

In addition to providing enumeration formulas, we also introduce several notions of equivalence between consecutive patterns of relations. These definitions are analogous to those for consecutive patterns in inversion sequences [1], which in turn are based on standard notions of equivalence between consecutive patterns in permutations [11, 12].

Definition 6. Let $\left(\underline{R_1, R_2}\right)$ and $\left(\underline{R_1', R_2'}\right)$ be consecutive patterns of relations in inversion sequences. We say that $\left(\underline{R_1, R_2}\right)$ and $\left(\underline{R_1', R_2'}\right)$ are

- *Wilf equivalent*, denoted by $\left(\underline{R_1, R_2}\right) \sim \left(\underline{R_1', R_2'}\right)$, if $\left|\mathbf{I}_n\left(\underline{R_1, R_2}\right)\right| = \left|\mathbf{I}_n\left(\underline{R_1', R_2'}\right)\right|$ for all n;
- *strongly Wilf equivalent*, denoted by $\left(\underline{R_1, R_2}\right) \overset{s}{\sim} \left(\underline{R_1', R_2'}\right)$, if for each n and m, the number of inversion sequences in $\mathbf{I}_n$ containing m occurrences of $\left(\underline{R_1, R_2}\right)$ is the same as for $\left(\underline{R_1', R_2'}\right)$;
- *super-strongly Wilf equivalent*, denoted by $\left(\underline{R_1, R_2}\right) \overset{ss}{\sim} \left(\underline{R_1', R_2'}\right)$, if the above condition holds for any set of prescribed positions for the m occurrences.

We use the term *generalized Wilf equivalence* to refer to an equivalence of any one of the these three types.

2 Summary of results

We will show that the 36 consecutive patterns of relations of the form $\left(\underline{R_1, R_2}\right)$, with $R_1, R_2 \in \{\leq, \geq, <, >, =, \neq\}$ fall into 30 Wilf equivalence classes, and into 31 strong Wilf equivalence classes, which are also super-strong equivalence classes. The next result provides this classification. Patterns are listed from least avoided to most avoided in inversion sequences of length 10.

Theorem 7. *A complete list of the generalized Wilf equivalences between consecutive patterns of relations* $\left(\underline{R_1, R_2}\right)$ *in inversion sequences is as follows:*

(i) $\left(\underline{\geq,<}\right) \overset{ss}{\sim} \left(\underline{<,\geq}\right) \sim \left(\underline{\neq,\geq}\right)$.

(ii) $\left(\underline{\geq,\geq}\right) \overset{ss}{\sim} \left(\underline{<,<}\right)$.

(iii) $\left(\underline{\geq,=}\right) \overset{ss}{\sim} \left(\underline{=,\geq}\right)$.

(iv) $\left(\underline{\geq,>}\right) \overset{ss}{\sim} \left(\underline{>,\geq}\right)$.

(v) $\left(\underline{>,=}\right) \overset{ss}{\sim} \left(\underline{=,>}\right)$.

It is worth pointing out that the patterns $\left(\underline{\geq,<}\right)$ and $\left(\underline{\neq,\geq}\right)$ (similarly, $\left(\underline{<,\geq}\right)$ and $\left(\underline{\neq,\geq}\right)$) are Wilf equivalent but not strongly Wilf equivalent.

Corollary 8. *Wilf equivalence and strong Wilf equivalence classes of consecutive patterns of relations in inversion sequences do not coincide in general.*

This result is interesting for two reasons. On the one hand, Wilf equivalence and strong Wilf equivalence classes of single consecutive patterns are conjectured to coincide, both in the setting of permutations, see Nakamura's conjecture [21, Conj. 5.6], and in the setting of inversion sequences [1, Conj. 2.3]. Corollary 8 shows that, perhaps surprisingly, the analogous statement for consecutive patterns of relations does not hold. On the other hand, when considering consecutive patterns of relations in the setting of permutations, by defining $\pi_i\pi_{i+1}\pi_{i+2}$ to be an occurrence of $\left(\underline{R_1,R_2}\right)$ in $\pi \in S_n$ if $\pi_i R_1 \pi_{i+1}$ and $\pi_{i+1} R_2 \pi_{i+2}$, Wilf equivalence and strong Wilf equivalence classes of patterns of the form $\left(\underline{R_1,R_2}\right)$ in permutations coincide. In fact, all such equivalences are obtained from trivial symmetries, unlike in the case of consecutive patterns of relations in inversion sequences.

As a consequence of Theorem 7(iv), we will deduce the following result about permutation patterns, originally conjectured by Baxter and Pudwell [3, Conj. 17], and later proved by Baxter and Shattuck [4, Cor. 11] and by Kasraoui [15, Cor. 1.9(a)]. Here we present a direct bijective proof based on consecutive patterns of relations in inversion sequences, which is simpler than the previously known proofs. We write $\underline{1243}$ and $\underline{4213}$ to denote *vincular* (also called *generalized*) permutation patterns, where entries in underlined positions are required to be adjacent in an occurrence [2]. We use $S_n(\sigma)$ to denote the set of permutations in S_n that avoid a pattern σ, and we say that two permutation patterns σ and τ are Wilf equivalent if $|S_n(\sigma)| = |S_n(\tau)|$ for all n.

Corollary 9. *The vincular permutation patterns* $\underline{1243}$ *and* $\underline{4213}$ *are Wilf equivalent.*

Aside from the classification provided in Theorem 7, the other central result in this paper is the enumeration of inversion sequences avoiding consecutive patterns of relations. We will show that, for many patterns (R_1, R_2), the sequence $\left|\mathbf{I}_n\left(\underline{R_1,R_2}\right)\right|$ matches an existing sequence in the On-line Encyclopedia of Integer Sequences (OEIS) [22] enumerating other combinatorial objects. In most cases, we will prove this bijectively. These results are summarized in Table 1. For other patterns of relations, even though we have no closed-form formulas for $\left|\mathbf{I}_n\left(\underline{R_1,R_2}\right)\right|$, we can obtain recurrences to compute these numbers.

As a byproduct of our enumeration, we will prove the following result involving non-consecutive triples of relations, which was conjectured by Martinez and Savage [20, Sec. 2.19]. After presenting our solution at the conference *Permutation Patterns 2018*, we learned

Pattern $(\underline{R_1, R_2})$	OEIS [22]	Description	Initial terms $\left\lvert \mathbf{I}_n\left(\underline{R_1, R_2}\right)\right\rvert$ for $1 \leq n \leq 9$
$(\underline{\leq, \neq})$	A040000	2 (for $n > 1$)	$1, 2, 2, 2, 2, 2, 2, 2, 2$
$(\underline{\leq, \geq})$	A000027	n	$1, 2, 3, 4, 5, 6, 7, 8, 9$
$(\underline{\geq, \neq})$	A000124	$\binom{n}{2} + 1$	$1, 2, 4, 7, 11, 16, 22, 29, 37$
$(\underline{\geq, \leq})$	A000045	F_{n+1}	$1, 2, 3, 5, 8, 13, 21, 34, 55$
$(\underline{\neq, \leq})$	A000071	$F_{n+2} - 1$	$1, 2, 4, 7, 12, 20, 33, 54, 88$
$(\underline{\geq, <}) \overset{ss}{\sim} (\underline{<, \geq}) \sim (\underline{\neq, \geq})$	A000079	2^{n-1}	$1, 2, 4, 8, 16, 32, 64, 128, 256$
$(\underline{\neq, \neq})$	A000085	Number of involutions of $[n]$	$1, 2, 4, 10, 26, 76, 232, 764, 2620$
$(\underline{\leq, >})$	A000108	C_n (Catalan)	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$(\underline{>, \leq})$	A071356	Underdiagonal paths of from the origin to $x = n$ with steps $(0, 1)$, $(1, 0)$, $(1, 2)$	$1, 2, 6, 20, 72, 272, 1064, 4272, 17504$
$(\underline{=, \neq})$	A003422	$0!+1!+2!+\cdots+(n-1)!$	$1, 2, 4, 10, 34, 154, 874, 5914, 46234$
$(\underline{\geq, \geq}) \overset{ss}{\sim} (\underline{<, <})$	A049774	$\lvert S_n\left(\underline{321}\right)\rvert$	$1, 2, 5, 17, 70, 349, 2017, 13358, 99377$
$(\underline{\neq, =})$	A000522	$\sum_{i=0}^{n-1}(n-1)!/i!$	$1, 2, 5, 16, 65, 326, 1957, 13700, 109601$
$(\underline{\geq, >}) \overset{ss}{\sim} (\underline{>, \geq})$	A200403	$\lvert S_n\left(\underline{1243}\right)\rvert$	$1, 2, 6, 23, 107, 584, 3660, 25910, 204564$
$(\underline{=, =})$	A052169	$\frac{(n+1)!-d_{n+1}}{n}$	$1, 2, 5, 19, 91, 531, 3641, 28673, 254871$

Table 1: Consecutive patterns of relations $\left(\underline{R_1, R_2}\right)$ for which $\left\lvert \mathbf{I}_n\left(\underline{R_1, R_2}\right)\right\rvert$ appears in the OEIS [22] and has an existing alternative combinatorial interpretation. Here F_n denotes the nth Fibonacci number, and d_n is the number of derrangements of $[n]$. The patterns are listed from least avoided to most avoided in inversion sequences of length 10.

that it has also been proved independently using different methods by Cao, Jin, and Lin [9, Thm. 5.1] and by Hossain [14].

Theorem 10. *The sequence* $|\mathbf{I}_n(>, \leq, -)|$ *has ordinary generating function (OGF)*

$$\sum_{n \geq 0} |\mathbf{I}_n(>, \leq, -)| \ z^n = \frac{1 + 2z - \sqrt{1 - 4z - 4z^2}}{4z}.$$

The paper is organized as follows. Section 3 is devoted to the classification of consecutive patterns of relations into generalized Wilf equivalence classes, as given in Definition 6, proving Theorem 7. We also provide a bijective proof of Corollary 9. In Section 4, we prove the enumerative results summarized in Table 1, as well as Theorem 10. Finally, in Section 5 we generalize the method used in the proof of Theorem 10 and apply it to enumerate different types of unimodal inversion sequences.

3 Generalized Wilf equivalences

3.1 Proof of Theorem 7

In this section we will find all the generalized Wilf equivalences between consecutive patterns of relations. We start proving part (v) of Theorem 7. Occurrences of $\underline{(>,=)}$ and $\underline{(=,>)}$ are simply occurrences of $\underline{100}$ and $\underline{110}$, respectively. Hence, the equivalence $\underline{(>,=)} \overset{ss}{\sim} \underline{(=,>)}$ is a restatement of the equivalence $\underline{100} \overset{ss}{\sim} \underline{110}$ (defined to mean that the number of inversion sequences of any given length with occurrences in any prescribed positions is the same for both patterns), which is already known [1, Prop. 3.12]. We repeat the proof here because the same method will be useful in proving several equivalences between consecutive patterns of relations.

First, we introduce some notation. Given $R_1, R_2 \in \{\leq, \geq, <, >, =, \neq\}$ and $e \in \mathbf{I}_n$, define

$$\mathrm{Em}\left(\underline{(R_1, R_2)}, e\right) = \left\{i : e_i e_{i+1} e_{i+2} \text{ is an occurrence of } \underline{(R_1, R_2)}\right\}.$$

Note that, by definition, $\underline{(R_1, R_2)} \overset{ss}{\sim} \underline{\left(R_1', R_2'\right)}$ if and only if

$$\left|\left\{e \in \mathbf{I}_n : \mathrm{Em}\left(\underline{(R_1, R_2)}, e\right) = S\right\}\right| = \left|\left\{e \in \mathbf{I}_n : \mathrm{Em}\left(\underline{\left(R_1', R_2'\right)}, e\right) = S\right\}\right|$$

for all n and all $S \subseteq [n]$. The next lemma, which is analogous to [1, Lem. 3.11], has a straightforward proof using the principle of inclusion-exclusion.

Lemma 11. *Let $\underline{(R_1, R_2)}$ and $\underline{\left(R_1', R_2'\right)}$ be two consecutive patterns of relations such that*

$$\left|\left\{e \in \mathbf{I}_n : \mathrm{Em}\left(\underline{(R_1, R_2)}, e\right) \supseteq S\right\}\right| = \left|\left\{e \in \mathbf{I}_n : \mathrm{Em}\left(\underline{\left(R_1', R_2'\right)}, e\right) \supseteq S\right\}\right|$$

for all positive integers n and all $S \subseteq [n]$. Then $\underline{(R_1, R_2)} \overset{ss}{\sim} \underline{\left(R_1', R_2'\right)}$.

Now we are ready to prove Theorem 7(v).

Proposition 12. *The patterns $\underline{(>,=)}$ and $\underline{(=,>)}$ are super-strongly Wilf equivalent.*

Proof. By Lemma 11, it suffices to show that, for all positive integers n and all $S \subseteq [n]$,

$$\left|\left\{e \in \mathbf{I}_n : \mathrm{Em}\left(\underline{(=,>)}, e\right) \supseteq S\right\}\right| = \left|\left\{e \in \mathbf{I}_n : \mathrm{Em}\left(\underline{(>,=)}, e\right) \supseteq S\right\}\right|.$$

To show this, we construct a bijection

$$\Phi_S : \left\{e' \in \mathbf{I}_n : \mathrm{Em}\left(\underline{(=,>)}, e'\right) \supseteq S\right\} \to \left\{e \in \mathbf{I}_n : \mathrm{Em}\left(\underline{(>,=)}, e\right) \supseteq S\right\}$$

that replaces the occurrences of $\underline{(=,>)}$ (equivalently, of $\underline{110}$) in e in positions S with occurrences of $\underline{(>,=)}$ (equivalently, of $\underline{100}$). For $e \in \mathbf{I}_n$ with $\mathrm{Em}\left(\underline{(=,>)}, e'\right) \supseteq S$, define $\Phi_S(e) = e'$ by setting

$$e'_j = \begin{cases} e_{j+1}, & \text{if } j-1 \in S; \\ e_j, & \text{otherwise;} \end{cases}$$

for $1 \leq j \leq n$. The sequence e' is an inversion sequence because if $j-1 \in S$, then $e'_j = e_{j+1} < e_j < j$.

To see that Φ_S is a bijection, we describe its inverse Ψ_S. For $e' \in \mathbf{I}_n$ with $\mathrm{Em}\left(\underline{(>,=)}, e\right) \supseteq S$, define $\Psi_S(e') = e$ by

$$e_j = \begin{cases} e'_{j-1}, & \text{if } j-1 \in S; \\ e'_j, & \text{otherwise;} \end{cases}$$

for $1 \leq j \leq n$. Note that $e \in \mathbf{I}_n$ because if $j-1 \in S$, then $e_j = e'_{j-1} < j-1 < j$. Since no two occurrences of $\underline{(=,>)}$ (respectively, $\underline{(>,=)}$) can overlap in more than one entry, the maps Φ_S and Ψ_S are inverses of each other. □

The next result proves Theorem 7(iii). It again relies on Lemma 11.

Proposition 13. *The patterns* $\underline{(\geq,=)}$ *and* $\underline{(=,\geq)}$ *are super-strongly Wilf equivalent.*

Proof. Let n be a positive integer and $S \subseteq [n]$. By Lemma 11, it suffices to construct a bijection

$$\Phi_S : \left\{e \in \mathbf{I}_n : \mathrm{Em}\left(\underline{(=,\geq)}, e\right) \supseteq S\right\} \to \left\{e' \in \mathbf{I}_n : \mathrm{Em}\left(\underline{(\geq,=)}, e'\right) \supseteq S\right\}.$$

We can write S uniquely as a disjoint union of *blocks*, which we define as maximal subsets whose entries are consecutive. Explicitly, write

$$S = \bigsqcup_{j=1}^{m} B_j,$$

where $B_j = \{i_j, i_j+1, \ldots, i_j+l_j\}$, with $l_j \geq 0$ and $i_j + l_j + 1 < i_{j+1}$ for all j.

Let $e \in \mathbf{I}_n$ be such that $\mathrm{Em}\left(\underline{(=,\geq)}, e\right) \supseteq S$. Then, for each $1 \leq j \leq m$, we have

$$e_{i_j} = e_{i_j+1} = \cdots = e_{i_j+l_j+1} \geq e_{i_j+l_j+2}.$$

Define $\Phi_S(e) = e'$ by setting

$$e'_i = \begin{cases} e_{i_j+l_j+2}, & \text{if } i-1 \in B_j \text{ for some } j \text{ and } e_{i_j+l_j+1} > e_{i_j+l_j+2}; \\ e_i, & \text{otherwise;} \end{cases}$$

for $1 \leq i \leq n$, as described schematically in Figure 2. In other words, we have

$$e_{i_j} = \ e'_{i_j} \geq e'_{i_j+1} = \cdots = e'_{i_j+l_j+1} = e'_{i_j+l_j+2} \ = e_{i_j+l_j+2}$$

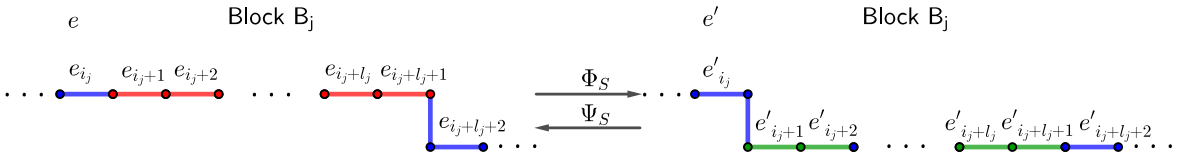


Figure 2: Schematic diagram of the behavior of Φ_S and Ψ_S from Proposition 13.

for each j, and so $\mathrm{Em}\left(\left(\underline{\geq,=}\right),e'\right) \supseteq S$. Additionally, if i is such that $e'_i \neq e_i$, then $e'_i = e_{i_j+l_j+2} \leq e_i < i$. Hence $e' \in \mathbf{I}_n$.

To see that Φ_S is a bijection, we describe its inverse map Ψ_S as follows. Given $e' \in \mathbf{I}_n$ such that $\mathrm{Em}\left(\left(\underline{\geq,=}\right),e'\right) \supseteq S$, let $\Psi_S(e') = e$, where

$$e_i = \begin{cases} e'_{i_j}, & \text{if } i-1 \in B_j \text{ for some } j \text{ and } e'_{i_j} > e'_{i_j+1}; \\ e'_i, & \text{otherwise;} \end{cases}$$

for $1 \leq i \leq n$. If i is such that $e_i \neq e'_i$, then $e_i = e'_{i_j} < i_j < i$, and so $e \in \mathbf{I}_n$. Additionally, $\mathrm{Em}\left(\left(\underline{=,\geq}\right),e\right) \supseteq S$. By construction, the maps Φ_S and Ψ_S are inverses of each other. □

Next we prove Theorem 7(iv), using a similar argument.

Proposition 14. *The patterns* $\left(\underline{\geq,>}\right)$ *and* $\left(\underline{>,\geq}\right)$ *are super-strongly Wilf equivalent.*

Proof. Let n be a positive integer and $S \subseteq [n]$. By Lemma 11, it is enough to describe a bijection

$$\Phi_S : \left\{e \in \mathbf{I}_n : \mathrm{Em}\left(\left(\underline{\geq,>}\right),e\right) \supseteq S\right\} \to \left\{e' \in \mathbf{I}_n : \mathrm{Em}\left(\left(\underline{>,\geq}\right),e'\right) \supseteq S\right\}.$$

Again, we write S as a disjoint union of blocks, $S = \bigsqcup_{j=1}^{m} B_j$, where $B_j = \{i_j, i_j+1, \ldots, i_j+l_j\}$, with $l_j \geq 0$ and $i_j + l_j + 1 < i_{j+1}$ for all j.

Given $e \in \mathbf{I}_n$ such that $\mathrm{Em}\left(\left(\underline{\geq,>}\right),e\right) \supseteq S$, we have

$$e_{i_j} \geq e_{i_j+1} > \cdots > e_{i_j+l_j+1} > e_{i_j+l_j+2}$$

for each j. Define $\Phi_S(e) = e'$ by setting

$$e'_i = \begin{cases} e_{i+1}, & \text{if } i-1 \in B_j \text{ for some } j \text{ and } e_{i_j} = e_{i_j+1}; \\ e_i, & \text{otherwise;} \end{cases}$$

for $1 \leq i \leq n$, as shown in Figure 3. Then

$$e'_{i_j} > e'_{i_j+1} > \cdots > e'_{i_j+l_j+1} \geq e'_{i_j+l_j+2}$$

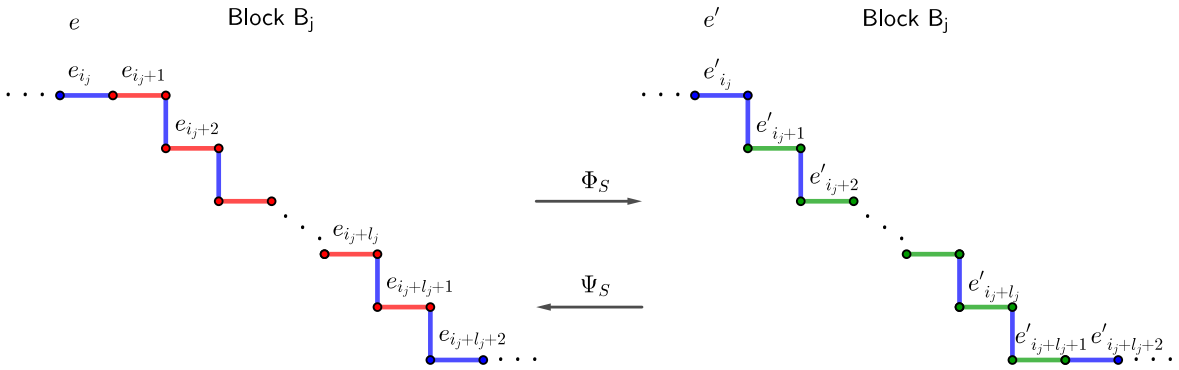


Figure 3: Schematic diagram of the behavior of Φ_S and Ψ_S from Proposition 14.

for each j, and so $\mathrm{Em}\left(\left(\underline{\geq,\geq}\right), e'\right) \supseteq S$. Additionally, since $e'_i = e_{i+1} < e_i < i$ whenever $e'_i \neq e_i$, we have that $e' \in \mathbf{I}_n$.

To see that Φ_S is a bijection, note that its inverse Ψ_S can be described as follows. Given $e' \in \mathbf{I}_n$ such that $\mathrm{Em}\left(\left(\underline{\geq,\geq}\right), e'\right) \supseteq S$, let $\Psi_S(e') = e$, where

$$e_i = \begin{cases} e'_{i-1}, & \text{if } i-1 \in B_j \text{ for some } j \text{ and } e'_{i_j+l_j+1} = e'_{i_j+l_j+2}; \\ e'_i, & \text{otherwise.} \end{cases}$$

□

Given $e = e_1 e_2 \cdots e_n \in \mathbf{I}_n$, we define its *complement* to be the inversion sequence $e^C = e_1^C e_2^C \cdots e_n^C$, where $e_i^C = i - 1 - e_i$ for $1 \leq i \leq n$. This notion is useful in proving Theorem 7(i)(ii), which we do next.

Proposition 15. *The following equivalences hold.*

(i₁) $\left(\underline{\geq,<}\right) \overset{ss}{\sim} \left(\underline{<,\geq}\right)$. *(ii)* $\left(\underline{\geq,\geq}\right) \overset{ss}{\sim} \left(\underline{<,<}\right)$.

Proof. To prove part (i₁), note that $e_i \geq e_{i+1}$ if and only if $e_i^C < e_{i+1}^C$. Indeed, the first equality is equivalent to $i - 1 - e_i^C \geq i - e_{i+1}^C$, which is in turn equivalent to the second. We deduce that

$$e_i \geq e_{i+1} < e_{i+2} \quad \text{if and only if} \quad e_i^C < e_{i+1}^C \geq e_{i+2}^C.$$

It follows that the involution $e \to e^C$ replaces occurrences of $\left(\underline{\geq,<}\right)$ by occurrences of $\left(\underline{<,\geq}\right)$, and vice versa. We conclude that $\left(\underline{\geq,<}\right)$ and $\left(\underline{<,\geq}\right)$ are super-strongly Wilf equivalent.

A very similar argument proves part (ii). □

The next proposition proves the other equivalence in Theorem 7(i). Corollary 8 is an immediate consequence.

Proposition 16. *The patterns* $\left(\underline{<,\geq}\right)$ *and* $\left(\underline{\neq,\geq}\right)$ *are Wilf equivalent, but not strongly Wilf equivalent.*

Proof. To prove that $\left(\underline{<,\geq}\right) \sim \left(\underline{\neq,\geq}\right)$, we will show that, in fact, $\mathbf{I}_n\left(\underline{<,\geq}\right) = \mathbf{I}_n\left(\underline{\neq,\geq}\right)$. Occurrences of $\left(\underline{<,\geq}\right)$ are occurrences of one of the following consecutive patterns: $\underline{010}$, $\underline{011}$, $\underline{021}$, $\underline{120}$. Similarly, occurrences of $\left(\underline{\neq,\geq}\right)$ are occurrences of one of the same four patterns, together with $\underline{100}$ or $\underline{210}$. We deduce that $\mathbf{I}_n\left(\underline{\neq,\geq}\right) \subseteq \mathbf{I}_n\left(\underline{<,\geq}\right)$.

Suppose that $e \in \mathbf{I}_n\left(\underline{<,\geq}\right)$ has an occurrence $e_ie_{i+1}e_{i+2}$ of $(\neq,\geq)$. Then it must be an occurrence of $\underline{100}$ or $\underline{210}$, so $e_i > e_{i+1} \geq 0$. Let j be the largest index such that $1 \leq j < i$ and $e_j < e_{j+1}$. Such a j must exist because $e_1 = 0$ and $e_i > 0$. But then $e_j < e_{j+1} \geq e_{j+2}$, that is, $e_je_{j+1}e_{j+2}$ is an occurrence of $\left(\underline{<,\geq}\right)$, which is a contradiction. We conclude that $\mathbf{I}_n\left(\underline{<,\geq}\right) = \mathbf{I}_n\left(\underline{\neq,\geq}\right)$.

To show that $\left(\underline{<,\geq}\right)$ and $\left(\underline{\neq,\geq}\right)$ are not strongly Wilf equivalent, it is enough to look at the number of occurrences of these patterns in inversion sequences of length 4. There is an inversion sequence in $\mathbf{I}_4$ with two occurrences of $\left(\underline{\neq,\geq}\right)$, namely 0100, but none with two occurrences of $\left(\underline{<,\geq}\right)$. □

Proof of Theorem 7. Propositions 12, 13, 14, 15 and 16 prove all the equivalences listed. A brute force computation of the first 10 terms of the sequences $\left|\mathbf{I}_n\left(\underline{R_1,R_2}\right)\right|$ for $R_1, R_2 \in \{\leq, \geq, <, >, =, \neq\}$ confirms that there are no other Wilf equivalences between such consecutive patterns of relations, showing that the list is complete. □

3.2 An application to permutation patterns

Inversion sequences avoiding the patterns $\left(\underline{\geq,>}\right)$ and $\left(\underline{>,\geq}\right)$ are closely related to permutations avoiding vincular patterns. We will exploit this connection to deduce Corollary 9 from Theorem 7(iv), using the following lemma.

Lemma 17. *Let* $\pi \in S_n$ *and* $e = \Theta(\pi)$. *Then:*

(a) $e_i \geq e_{i+1}$ *if and only if* $\pi_i < \pi_{i+1}$;

(b) $e_i > e_{i+1}$ *if and only if there exists* $j < i$ *such that* $\pi_i < \pi_j < \pi_{i+1}$.

Proof. Let $E_i = \{j \in [n] : j < i \text{ and } \pi_j > \pi_i\}$, so that the entries of e satisfy $e_i = |E_i|$.

To prove (a), first suppose that $\pi_i < \pi_{i+1}$. Then every $j < i+1$ such that $\pi_j > \pi_{i+1}$ satisfies $j < i$ and $\pi_j > \pi_i$. It follows that $E_{i+1} \subseteq E_i$, and so $e_i \geq e_{i+1}$. Conversely, suppose that $\pi_i > \pi_{i+1}$. Then every $j < i$ such that $\pi_j > \pi_i$ satisfies $\pi_j > \pi_{i+1}$, so $E_i \subseteq E_{i+1}$. Furthermore, $i \in E_{i+1}\backslash E_i$, so this inclusion is strict and, consequently, $e_i < e_{i+1}$.

To prove (b), suppose that $e_i > e_{i+1}$. Then $\pi_i < \pi_{i+1}$ by part (a), and so $E_{i+1} \subseteq E_i$. If for every $j < i$ we had either $\pi_j < \pi_i$ or $\pi_{i+1} < \pi_j$, then $E_i \subseteq E_{i+1}$, from where $E_i = E_{i+1}$ and $e_i = e_{i+1}$, which is a contradiction. Hence, there exists $j < i$ such that $\pi_i < \pi_j < \pi_{i+1}$.

Conversely, if $j < i$ is such that $\pi_i < \pi_j < \pi_{i+1}$, then $E_{i+1} \subseteq E_i$ and $j \in E_i\backslash E_{i+1}$, so this inclusion is strict. Thus, $e_i > e_{i+1}$. □

The following result connects the vincular permutation patterns $\underline{124}3$ and $\underline{421}3$ to the consecutive patterns of relations $\left(\underline{>,\geq}\right)$ and $\left(\underline{\geq,>}\right)$, respectively. In the proof we will use the next two definitions. For a permutation $\pi \in S_n$, define its *reverse* π^R to be the permutation with entries $\pi_i^R = \pi_{n+1-i}$, and its *reverse-complement* π^{RC} to be the permutation such that $\pi_i^{RC} = n + 1 - \pi_i^R$ for all i.

Proposition 18. *Let $\pi \in S_n$ and $e = \Theta(\pi)$. Then:*

(a) π avoids the vincular pattern $2\underline{134}$ if and only if e avoids the consecutive pattern of relations $\left(\underline{>,\geq}\right)$. Consequently, $|S_n(\underline{124}3)| = |S_n(2\underline{134})| = \left|\mathbf{I}_n\left(\underline{>,\geq}\right)\right|$.

(b) π avoids the vincular pattern $3\underline{124}$ if and only if e avoids the consecutive pattern of relations $\left(\underline{\geq,>}\right)$. Consequently, $|S_n(\underline{421}3)| = |S_n(3\underline{124})| = \left|\mathbf{I}_n\left(\underline{\geq,>}\right)\right|$.

Proof. The permutation π contains the vincular pattern $2\underline{134}$ if and only if there exist indices $j < i$ such that $\pi_i < \pi_j < \pi_{i+1} < \pi_{i+2}$. By Lemma 17, these inequalities are equivalent to $e_i > e_{i+1} \geq e_{i+2}$, namely, to e having an occurrence of $\left(\underline{>,\geq}\right)$ in position i. It follows that π avoids $2\underline{134}$ if and only if e avoids $\left(\underline{>,\geq}\right)$. Since the map $\pi \to \pi^{RC}$ induces a bijection between $S_n(\underline{124}3)$ and $S_n(2\underline{134})$, this proves part (a).

Similarly, π contains the vincular pattern $3\underline{124}$ if and only if there exist indices $j < i$ such that $\pi_i < \pi_{i+1} < \pi_j < \pi_{i+2}$. By Lemma 17, this is equivalent e having an occurrence of $\left(\underline{\geq,>}\right)$ in position i. Thus, π avoids $3\underline{124}$ if and only if e avoids $\left(\underline{\geq,>}\right)$. Since the map $\pi \to \pi^R$ induces a bijection between $S_n(\underline{421}3)$ and $S_n(3\underline{124})$, part (b) follows. □

Theorem 7(iv) and Proposition 18 imply that $|S_n(\underline{124}3)| = |S_n(\underline{421}3)|$, which proves Corollary 9. This result was originally conjectured by Baxter and Pudwell [3, Conj. 17], and later proved by Baxter and Shattuck [4, Cor. 11] and by Kasraoui [15, Cor. 1.9(a)], who showed that the vincular permutation patterns $\underline{124}3$ and $\underline{421}3$ are, in fact, strongly Wilf equivalent.

However, none of these proofs provides a simple bijection between $S_n(\underline{124}3)$ and $S_n(\underline{421}3)$. Next we present such a bijection. Its easiest description is at the level of inversion sequences. Note that, even though we proved in Proposition 14 that $\left(\underline{\geq,>}\right)$ and $\left(\underline{>,\geq}\right)$ are super-strongly Wilf equivalent, our proof did not give a bijection between the corresponding pattern-avoiding sets.

Proposition 19. *For $n \geq 1$, there is an explicit bijection $\Phi : \mathbf{I}_n\left(\underline{\geq,>}\right) \to \mathbf{I}_n\left(\underline{>,\geq}\right)$ with the property that if $e' = \Phi(e)$, then $e'_n = e_n$.*

Proof. Given $e \in \mathbf{I}_n\left(\underline{\geq,>}\right)$, define $\Phi(e) = e'$ to be the inversion sequence obtained by replacing each maximal occurrence of a consecutive pattern $\underline{10^r}$ (for $r \geq 2$) in e with an occurrence of $\underline{1^r0}$. In other words, each maximal subsequence of the form $e_j > e_{j+1} = e_{j+2} = \cdots = e_{j+r}$ (with $r \geq 2$) becomes a subsequence $e'_j = e'_{j+1} = \cdots = e'_{j+r-1} > e'_{j+r}$, where $e'_j = e_j$ and $e'_{j+r} = e_{j+r}$. Note that Φ acts on these subsequences in analogy to how the map Ψ_S, described in the schematic diagram in Figure 2, acted on blocks.

We now show that $e' \in \mathbf{I}_n\left(\underline{>,\geq}\right)$. Since $e \in \mathbf{I}_n\left(\underline{\geq,>}\right)$, it avoids $\left(\underline{>,>}\right)$ as well. Given that Φ removes all occurrences of $\left(\underline{>,=}\right)$ in e, it suffices to prove that Φ introduces no new occurrences of $\left(\underline{>,\geq}\right)$ at the edges of the changed subsequences; that is, that $e'_{j-1}e'_je'_{j+1}$ and $e'_{j+r-1}e'_{j+r}e'_{j+r+1}$ are not occurrences of $\left(\underline{>,\geq}\right)$ for j and r as above.

We argue by contradiction. Suppose that $e'_{j-1} > e'_j \geq e'_{j+1}$. Then either $e_{j-1} = e'_{j-1}$ or $e_{j-1} = e'_j$. Thus, $e_{j-1} \geq e'_j = e_j > e_{j+1}$, so $e_{j-1}e_je_{j+1}$ would have been an occurrence of $\left(\underline{\geq,>}\right)$.

Now, suppose that $e'_{j+r-1} > e'_{j+r} \geq e'_{j+r+1}$. Then $e_{j+r-1} = e_{j+r} = e'_{j+r} \geq e'_{j+r+1} \geq e_{j+r+1}$. However, at least one of these two inequalities must be strict because $e_{j+r} = e_{j+r+1}$ would contradict the maximality of r. But then $e_{j+r-1}e_{j+r}e_{j+r+1}$ would have been an occurrence of $\left(\underline{\geq,>}\right)$.

It is clear that Φ is a bijection, since the inverse map can be obtained by replacing each maximal occurrence of a consecutive pattern $\underline{1^r0}$ (for $r \geq 2$) with an occurrence of $\underline{10^r}$. Also, by construction, Φ preserves the last entry of e. □

Combining Propositions 18 and 19, we see that the map $\pi \mapsto \Theta^{-1}\left(\Phi(\Theta(\pi^{RC}))\right)^R$ is a bijection between $S_n\left(\underline{1243}\right)$ and $S_n\left(\underline{4213}\right)$.

4 Enumerative results

In this section, we show that for consecutive patterns of relations $\left(\underline{R_1,R_2}\right)$ in 14 of the 30 Wilf equivalence classes, the sequence $\left|\mathbf{I}_n\left(\underline{R_1,R_2}\right)\right|$ matches a sequence in the OEIS [22] that is known to enumerates another combinatorial object. These results are summarized in Table 1.

This leaves 16 Wilf equivalence classes for which the first few terms of the sequence $\left|\mathbf{I}_n\left(\underline{R_1,R_2}\right)\right|$ do not match any previously existing sequence in the OEIS [22], see Table 2. For 4 of these 16 classes, avoidance of a pattern of relations equates to avoidance of a single consecutive pattern. Specifically, the patterns of relations $\left(\underline{\leq,=}\right)$, $\left(\underline{=,<}\right)$, $\left(\underline{\geq,=}\right) \overset{ss}{\sim} \left(\underline{=,>}\right)$ and $\left(\underline{>,>}\right)$ correspond to the consecutive patterns $\underline{011}$, $\underline{001}$, $\underline{100} \overset{ss}{\sim} \underline{110}$ and $\underline{210}$, respectively. Recurrences for these 4 classes are known, as they were treated in our systematic study of consecutive patterns in inversion sequences [1]. The remaining 12 Wilf equivalence classes of consecutive patterns of relations, listed from least avoided to most avoided in inversion sequences of length 10, are $\left(\underline{\leq,\leq}\right)$, $\left(\underline{<,\neq}\right)$, $\left(\underline{\neq,<}\right)$, $\left(\underline{\leq,<}\right)$, $\left(\underline{\neq,>}\right)$, $\left(\underline{<,>}\right)$, $\left(\underline{<,\leq}\right)$, $\left(\underline{>,\neq}\right)$, $\left(\underline{>,<}\right)$, $\left(\underline{=,\leq}\right)$, $\left(\underline{\leq,=}\right)$, and $\left(\underline{\geq,=}\right) \overset{ss}{\sim} \left(\underline{=,\geq}\right)$.

For each pattern $\left(\underline{R_1,R_2}\right)$ above, even though we do not have closed-form formulas for $\left|\mathbf{I}_n\left(\underline{R_1,R_2}\right)\right|$, we can obtain recurrences to compute these quantities. The recurrences are similar to the one in Proposition 47, and can be proved using analogous arguments.

Pattern (R_1, R_2)	New OEIS [22] entry	Initial terms $\left\vert \mathbf{I}_n\left(R_1, R_2\right)\right\vert$ for $1 \leq n \leq 10$
$(\leq, \leq)$	A328357	$1, 2, 1, 4, 6, 36, 117, 804, 4266, 33768$
$(<, \neq)$	A328358	$1, 2, 4, 10, 30, 100, 376, 1566, 7094, 34751$
$(\neq, <)$	A328429	$1, 2, 5, 14, 46, 170, 691, 3073, 14809, 76666$
$(\leq, <)$	A328430	$1, 2, 3, 7, 18, 70, 317, 1825, 11805, 88212$
$(\neq, >)$	A328431	$1, 2, 5, 15, 53, 214, 960, 4701, 24873, 141147$
$(<, >)$	A328432	$1, 2, 5, 15, 53, 216, 994, 5076, 28403, 172538$
$(<, \leq)$	A328433	$1, 2, 4, 11, 37, 157, 791, 4676, 31490, 238814$
$(>, \neq)$	A328434	$1, 2, 6, 21, 81, 346, 1630, 8350, 45958, 269815$
$(>, <)$	A328435	$1, 2, 6, 21, 83, 368, 1814, 9837, 58095, 370499$
$(=, \leq)$	A328436	$1, 2, 3, 9, 37, 190, 1181, 8564, 70914, 659810$
$(=, <)$	A328437	$1, 2, 4, 11, 42, 210, 1292, 9352, 77505, 722294$
$(\leq, =)$	A328438	$1, 2, 4, 13, 57, 304, 1937, 14315, 120264, 1131896$
$(<, =)$	A328439	$1, 2, 5, 17, 75, 407, 2621, 19524, 165090, 1561900$
$(\geq, =) \overset{ss}{\sim} (=, \geq)$	A328440	$1, 2, 5, 18, 81, 448, 2920, 21955, 186981, 1779170$
$(>, =) \overset{ss}{\sim} (=, >)$	A328441	$1, 2, 6, 23, 109, 618, 4098, 31173, 267809, 2565520$
$(>, >)$	A328442	$1, 2, 6, 24, 118, 684, 4554, 34192, 285558, 2624496$

Table 2: Consecutive patterns of relations $\left(R_1, R_2\right)$ for which $\left\vert \mathbf{I}_n\left(R_1, R_2\right)\right\vert$ did not previously appear in the OEIS [22]. The patterns are listed from least avoided to most avoided in inversion sequences of length 10.

4.1 The pattern $(\leq, \neq)$

The following lemma gives a very simple description of the set $\mathbf{I}_n\left(\leq, \neq\right)$. The notation a^s indicates repetition s times of the entry a.

Lemma 20. *For $n \geq 2$,*

$$\mathbf{I}_n\left(\leq, \neq\right) = \left\{0^n, 01^{n-1}\right\}.$$

Proof. We proceed by induction on n. The statement is trivially true for the base case $n = 2$. Now let $n \geq 3$, suppose the statement holds for $n-1$, and let $e \in \mathbf{I}_n\left(\leq, \neq\right)$. Then $e_1 e_2 \cdots e_{n-1} \in \mathbf{I}_{n-1}\left(\leq, \neq\right)$, so either $e = 0^{n-1}$ or $e = 01^{n-2}$. In any case, $e_{n-2} \leq e_{n-1}$, so it must be that $e_{n-1} = e_n$. Hence, either $e = 0^n$ or $e = 01^{n-1}$. Since both of these inversion sequences avoid $\left(\leq, \neq\right)$, we deduce that $\mathbf{I}_n\left(\leq, \neq\right) = \{0^n, 01^{n-1}\}$. □

It follows from Lemma 20 that $\left\vert \mathbf{I}_1\left(\leq, \neq\right)\right\vert = 1$ and $\left\vert \mathbf{I}_n\left(\leq, \neq\right)\right\vert = 2$ for $n \geq 2$.

4.2 The pattern $(\leq, \geq)$

The next lemma provides a characterization of the set $\mathbf{I}_n\left(\leq, \geq\right)$ in terms of a monotonicity condition. This will be a recurrent idea in this paper, as it is often convenient to describe

the inversion sequences in $\mathbf{I}_n\left(\underline{R_1, R_2}\right)$ in terms of a monotonicity or unimodal condition, in order to enumerate them.

Lemma 21. *For $n \geq 1$,*

$$\mathbf{I}_n\left(\underline{\leq, \geq}\right) = \{e \in \mathbf{I}_n : e_2 < e_3 < \cdots < e_n\}.$$

Proof. The inclusion to the left is immediate. To prove the inclusion to the right, let $e \in \mathbf{I}_n\left(\underline{\leq, \geq}\right)$. We will show that $e_j < e_{j+1}$ for $2 \leq j < n$ by induction on j. For the base case $j = 2$, note that $e_1 = 0 \leq e_2$ implies $e_2 < e_3$, otherwise $e_1e_2e_3$ would be an occurrence of $\left(\underline{\leq, \geq}\right)$. Now suppose that $e_j < e_{j+1}$ for some $2 \leq j < n-1$. Then $e_{j+1} < e_{j+2}$, because otherwise $e_je_{j+1}e_{j+2}$ would be an occurrence of $\left(\underline{\leq, \geq}\right)$. □

Proposition 22. *For $n \geq 1$, $\left|\mathbf{I}_n\left(\underline{\leq, \geq}\right)\right| = n$.*

Proof. It follows from Lemma 21 and the definition of inversion sequence that $\mathbf{I}_n\left(\underline{\leq, \geq}\right)$ is the set of sequences $e_1 \ldots e_n$ with $0 = e_1 \leq e_2 < e_3 < \cdots < e_n \leq n-1$. Disregarding the forced entry $e_1 = 0$, this condition in equivalent to

$$0 \leq e_2 \leq e_3 - 1 \leq e_4 - 2 \leq \cdots \leq e_n - (n-2) \leq 1, \tag{2}$$

and so these entries are determined by the choice of which one of the n inequalities in (2) is strict. In other words, these entries are given by

$$e_i = \begin{cases} i-2, & \text{if } 2 \leq i \leq j; \\ i-1, & \text{if } j < i \leq n; \end{cases}$$

for some fixed $1 \leq j \leq n$. Thus, there are n such sequences. □

4.3 The pattern $\left(\underline{\geq, \neq}\right)$

The next lemma characterizes inversion sequences avoiding this pattern.

Lemma 23. *For $n \geq 1$,*

$$\mathbf{I}_n\left(\underline{\geq, \neq}\right) = \{e \in \mathbf{I}_n : e_1 < e_2 < \cdots < e_j \geq e_{j+1} = e_{j+2} = \cdots = e_n \text{ for some } 1 \leq j \leq n\}.$$

Proof. The inclusion to the left is straightforward. For the inclusion to the right, let $e \in \mathbf{I}_n\left(\underline{\geq, \neq}\right)$, and let j be the largest integer such that $e_1 < e_2 < \cdots < e_j$. If $j < n$, then $e_j \geq e_{j+1}$ by construction, and since e avoids $\left(\underline{\geq, \neq}\right)$, it must be that $e_{j+1} = e_{j+2}$ (if these entries are defined). Repeating the same argument, we have $e_{j+1} = e_{j+2} = \cdots = e_n$. □

Next we obtain a formula for the number of inversion sequences that avoid $\left(\underline{\geq, \neq}\right)$. This sequence appears as A000124 in the OEIS [22]; it is sometimes referred to as *central polygonal numbers* or as the *lazy caterer's sequence*.

Proposition 24. *For $n \geq 1$, $\left|\mathbf{I}_n\left(\underline{\geq, \neq}\right)\right| = \binom{n}{2} + 1$.*

Proof. Let $\mathcal{A}_n$ be the collection of subsets of $\{0, 1, \ldots, n-2\}$ with at most two elements. Clearly, $|\mathcal{A}_n| = \binom{n-1}{2} + (n-1) + 1 = \binom{n}{2} + 1$. We define a bijection $\Gamma : \mathbf{I}_n\left(\underline{\geq, \neq}\right) \to \mathcal{A}_n$ by letting, for each $e \in \mathbf{I}_n\left(\underline{\geq, \neq}\right)$,

$$\Gamma(e) = \begin{cases} \{\max_i\{e_i\}, e_n\}, & \text{if } e_n \neq n-1; \\ \varnothing, & \text{otherwise;} \end{cases}$$

where $\{a, a\}$ is simply the set $\{a\}$. If j is the index such that $e_1 < e_2 < \cdots < e_j \geq e_{j+1} = e_{j+2} = \cdots = e_n$, which exists by Lemma 23 and is unique by definition, then $e_i = i - 1$ for $1 \leq i \leq j$, so the above definition can be restated as $\Gamma(e) = \{e_j, e_n\}$ if $j < n$ and $\Gamma(e) = \varnothing$ otherwise.

To see that Γ is a bijection, we describe its inverse. Given any $A \in \mathcal{A}_n$, we have

$$\Gamma^{-1}(A) = \begin{cases} 012\cdots abb\cdots b, & \text{if } A = \{a, b\} \text{ with } a > b; \\ 012\cdots aa\cdots a, & \text{if } A = \{a\}; \\ 012\cdots(n-1), & \text{if } A = \varnothing. \end{cases}$$

□

Example 25. Applying the map Γ from the proof of Proposition 24 to $e = 012322 \in \mathbf{I}_6\left(\underline{\geq, \neq}\right)$, we get $\Gamma(e) = \{3, 2\}$. Similarly, if $e = 012344 \in \mathbf{I}_6\left(\underline{\geq, \neq}\right)$, then $\Gamma(e) = \{4\}$.

One can also obtain the formula in Proposition 24 as follows. Martinez and Savage [20, Observation 11] note that the set $\mathbf{I}_n(\geq, \neq, -)$ is characterized by the same inequalities from Lemma 23, so we deduce that $\mathbf{I}_n\left(\underline{\geq, \neq}\right) = \mathbf{I}_n(\geq, \neq, -)$. Moreover, they show that $\mathbf{I}_n(\geq, \neq, -) = \Theta\left(S_n(213, 321)\right)$, where Θ is given by (1) and $S_n(213, 321)$ denotes the set of permutations in S_n avoiding both of the (classical) permutation patterns 213 and 321 [20, Thm. 9]. It was shown by Simion and Schmidt [23, Prop. 11] that $|S_n(213, 321)| = \binom{n}{2} + 1$. We conclude that

$$\left|\mathbf{I}_n\left(\underline{\geq, \neq}\right)\right| = |\mathbf{I}_n(\geq, \neq, -)| = |S_n(213, 321)| = \binom{n}{2} + 1.$$

4.4 The pattern $\left(\underline{\geq, \leq}\right)$

The next lemma characterizes the set $\mathbf{I}_n\left(\underline{\geq, \leq}\right)$ as strictly unimodal inversion sequences.

Lemma 26. *For $n \geq 1$,*

$$\mathbf{I}_n\left(\underline{\geq, \leq}\right) = \{e \in \mathbf{I}_n : e_1 < e_2 < \cdots < e_j \geq e_{j+1} > e_{j+2} > \cdots > e_n \text{ for some } 1 \leq j \leq n\}.$$

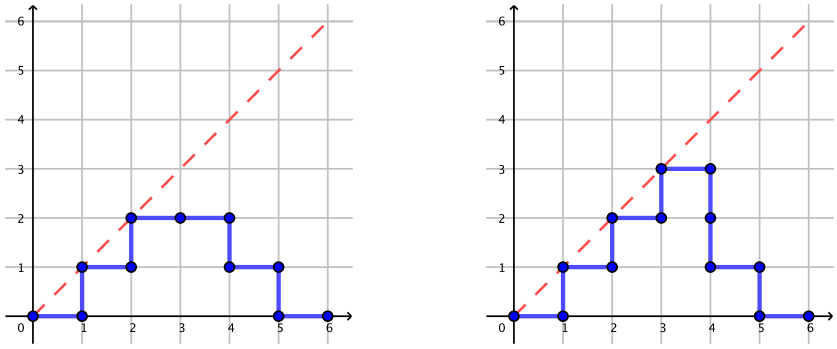


Figure 4: Two inversion sequences $e \in \mathbf{I}_6\left(\underline{\geq, \leq}\right)$: $e = 012210$ (left) and $e = 012310$ (right).

Proof. The inclusion to the left is immediate, so let us prove the inclusion to the right. Suppose that $e \in \mathbf{I}_n\left(\underline{\geq, \leq}\right)$, and let j be the smallest integer such that $e_j \geq e_{j+1}$, or $j = n$ if no such integer exists. Since e avoids $\left(\underline{\geq, \leq}\right)$, it must be that $e_{j+1} > e_{j+2}$ (if these entries are defined). Repeating the same argument, we see that $e_{j+1} > e_{j+2} > \cdots > e_n$. On the other hand, $e_{j-1} < e_j$ (if $j \geq 2$) by definition of j. Since e avoids $\left(\underline{\geq, \leq}\right)$, we must then have $e_1 < e_2 < \cdots < e_j$. We conclude that e satisfies the stated unimodality condition. □

Let $\mathcal{C}_n$ be the set of compositions of n with parts 1 and 2, that is, sequences $(a_1, a_2, \ldots, a_j)$ such that $a_1 + \cdots + a_j = n$ and $a_i \in \{1, 2\}$ for $1 \leq i \leq j$. It is well-known that $|\mathcal{C}_n| = F_{n+1}$, the $(n+1)$th Fibonacci number, defined by the recurrence $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The integer sequence sequence F_n is listed as A000045 in the OEIS [22].

Proposition 27. *There is a bijective correspondence between* $\mathbf{I}_n\left(\underline{\geq, \leq}\right)$ *and* $\mathcal{C}_n$. *In particular,* $\left|\mathbf{I}_n\left(\underline{\geq, \leq}\right)\right| = F_{n+1}$.

Proof. Given $e \in \mathbf{I}_n\left(\underline{\geq, \leq}\right)$, we know by Lemma 26 that

$$e_1 < e_2 < \cdots < e_j \geq e_{j+1} > e_{j+2} > \cdots > e_n$$

for some j. Since e is an inversion sequence, it follows that $e_i = i - 1$ for $1 \leq i \leq j$.

Define $f(e) = (a_1, a_2, \ldots, a_j)$, where a_i is the number of entries in e that are equal to $i - 1$, for $1 \leq i \leq j$. Note that $f(e) \in \mathcal{C}_n$ because, for each $1 \leq i \leq j$, the sequence e has one or two entries equal to $i - 1$, and it has no entries larger than $j - 1$.

Let us show that f is a bijection by describing its inverse. Given $(a_1, a_2, \ldots, a_j) \in \mathcal{C}_n$, we recover the unique $e \in \mathbf{I}_n\left(\underline{\geq, \leq}\right)$ such that $f(e) = (a_1, a_2, \ldots, a_j)$ as follows. First, set $e_i = i - 1$ for $1 \leq i \leq j$. To define the remaining entries, let i range from 1 to j, and whenever $a_i = 2$, set the rightmost entry of e that has not been defined equal to e_i. □

Example 28. Applying the bijection f from the above proof to the inversion sequences in Figure 4, we get $f(012210) = (2, 2, 2)$ and $f(012310) = (2, 2, 1, 1)$.

4.5 The pattern $\left(\underline{\neq, \leq}\right)$

As in previous subsections, we start by characterizing the inversion sequences avoiding this pattern.

Lemma 29. *For $n \geq 1$,*

$$\mathbf{I}_n\left(\underline{\neq, \leq}\right) = \{e \in \mathbf{I}_n : e_1 = e_2 = \cdots = e_{j-1} < e_j > e_{j+1} > \cdots > e_n \text{ for some } 2 \leq j \leq n+1\}.$$

Proof. The inclusion to the left is straightforward. For the inclusion to the right, take $e \in \mathbf{I}_n\left(\underline{\neq, \leq}\right)$ and let $2 \leq j \leq n+1$ be the largest integer such that $0 = e_1 = e_2 = \cdots = e_{j-1}$. If $j \leq n$, then $e_{j-1} < e_j$. Furthermore, if $j \leq n-1$, then $e_j > e_{j+1}$ because e avoids the pattern $\left(\underline{\neq, \leq}\right)$. The same argument shows that $e_j > e_{j+1} > \cdots > e_n$. □

Martinez and Savage [20, Thm. 12] proved that the inversion sequences in $\mathbf{I}_n(\neq, \leq, -)$ are characterized by the same inequalities from Lemma 29, so we deduce that $\mathbf{I}_n\left(\underline{\neq, \leq}\right) = \mathbf{I}_n(\neq, \leq, -)$. They also show that $|\mathbf{I}_n(\neq, \leq, -)| = F_{n+2} - 1$ for $n \geq 1$. This sequence is listed as A000071 in the OEIS [22]. The next corollary follows.

Corollary 30. *For $n \geq 1$, $\left|\mathbf{I}_n\left(\underline{\neq, \leq}\right)\right| = |\mathbf{I}_n(\neq, \leq, -)| = F_{n+2} - 1$.*

4.6 The patterns $\left(\underline{\geq, <}\right) \overset{ss}{\sim} \left(\underline{<, \geq}\right) \sim \left(\underline{\neq, \geq}\right)$

We have the following characterizations of inversion sequences avoiding these three patterns.

Lemma 31. *For $n \geq 1$,*

$$\mathbf{I}_n\left(\underline{<, \geq}\right) = \mathbf{I}_n\left(\underline{\neq, \geq}\right) = \{e \in \mathbf{I}_n : e_1 = e_2 = \cdots = e_j < e_{j+1} < \cdots < e_n \text{ for some } 1 \leq j \leq n\},$$
$$\mathbf{I}_n\left(\underline{\geq, <}\right) = \{e \in \mathbf{I}_n : e_1 < e_2 < \cdots < e_j \geq e_{j+1} \geq \cdots \geq e_n \text{ for some } 1 \leq j \leq n\}.$$

Proof. We showed in the proof of Proposition 16 that $\mathbf{I}_n\left(\underline{<, \geq}\right) = \mathbf{I}_n\left(\underline{\neq, \geq}\right)$. Given $e \in \mathbf{I}_n\left(\underline{<, \geq}\right)$, let $1 \leq j \leq n$ be the largest integer such that $0 = e_1 = e_2 = \cdots = e_j$. If $j < n$, then $e_j < e_{j+1}$, and if $j < n-1$, then $e_{j+1} < e_{j+2}$ because e avoids the pattern $\left(\underline{<, \geq}\right)$. An inductive argument shows that $e_j < e_{j+1} < \cdots < e_n$.

The characterization of $e \in \mathbf{I}_n\left(\underline{\geq, <}\right)$ can be obtained similarly by letting j be the largest integer such that $e_1 < e_2 < \cdots < e_j$. Alternatively, it follows from the fact, shown in the proof of Proposition 15(i_1), that the map $e \to e^C$ sending each inversion sequence to its complement induces a bijection between $\mathbf{I}_n\left(\underline{\geq, <}\right)$ and $\mathbf{I}_n\left(\underline{<, \geq}\right)$. □

Martinez and Savage [20, Thm. 15] showed that the set $\mathbf{I}_n(\neq, \leq, -)$ is also characterized by the same condition as in the first part of Lemma 31, and so $\mathbf{I}_n\left(\underline{<, \geq}\right) = \mathbf{I}_n\left(\underline{\neq, \geq}\right) = \mathbf{I}_n(\neq, \leq, -)$. In addition, there is a bijection between this set and the set of all subsets of $[n-1]$, obtained by mapping an inversion sequence to the set of its nonzero values. The next result now follows using Theorem 7(i).

Corollary 32. *For $n \geq 1$, $\left|\mathbf{I}_n\left(\underline{\geq, <}\right)\right| = \left|\mathbf{I}_n\left(\underline{<, \geq}\right)\right| = \left|\mathbf{I}_n\left(\underline{\neq, \geq}\right)\right| = 2^{n-1}$.*

4.7 The pattern $\left(\underline{\neq,\neq}\right)$

Let $\mathcal{I}_n$ denote the set of involutions of $[n]$, that is, permutations $\pi \in S_n$ such that $\pi^{-1} = \pi$. We will show that there is a bijection between $\mathbf{I}_n\left(\underline{\neq,\neq}\right)$ and $\mathcal{I}_n$.

Proposition 33. *There is a bijection $\Upsilon : \mathbf{I}_n\left(\underline{\neq,\neq}\right) \to \mathcal{I}_n$. In particular, the number of inversion sequences avoiding $\left(\underline{\neq,\neq}\right)$ satisfies the recurrence*

$$\left|\mathbf{I}_n\left(\underline{\neq,\neq}\right)\right| = \left|\mathbf{I}_{n-1}\left(\underline{\neq,\neq}\right)\right| + (n-1)\left|\mathbf{I}_{n-2}\left(\underline{\neq,\neq}\right)\right| \tag{3}$$

for $n \geq 2$, with initial conditions $\left|\mathbf{I}_0\left(\underline{\neq,\neq}\right)\right| = \left|\mathbf{I}_1\left(\underline{\neq,\neq}\right)\right| = 1$, and its exponential generating function is

$$\sum_{n\geq 0}\left|\mathbf{I}_n\left(\underline{\neq,\neq}\right)\right|\frac{z}{n!} = \exp\left(z + \frac{z^2}{2}\right).$$

Proof. We start by proving the recurrence (3), which will also inform the construction of the bijection Υ. The initial conditions are trivially satisfied. Let $n \geq 2$, and let $e \in \mathbf{I}_n\left(\underline{\neq,\neq}\right)$. If $e_{n-1} = e_n$, then e is obtained from an arbitrary inversion sequence in $\mathbf{I}_{n-1}\left(\underline{\neq,\neq}\right)$ by repeating the last entry, so there are $\left|\mathbf{I}_{n-1}\left(\underline{\neq,\neq}\right)\right|$ inversion sequences with $e_{n-1} = e_n$. If $e_{n-1} \neq e_n$, then we must have $e_{n-2} = e_{n-1}$ (unless $n = 2$, in which case $e = 01$), since e avoids $\left(\underline{\neq,\neq}\right)$. In this case, e is obtained from an arbitrary inversion sequence in $\mathbf{I}_{n-2}\left(\underline{\neq,\neq}\right)$ by repeating the last entry, and then appending any element from $\{0, 1, \ldots, n-1\} \setminus \{e_{n-2}\}$, for which there are $n-1$ choices. Thus, there are $(n-1)\left|\mathbf{I}_{n-2}\left(\underline{\neq,\neq}\right)\right|$ inversion sequences with $e_{n-1} \neq e_n$. The recurrence is now proved.

It is well known that the number of involutions $|\mathcal{I}_n|$ satisfies the same recurrence. It follows that $\left|\mathbf{I}_n\left(\underline{\neq,\neq}\right)\right| = |\mathcal{I}_n|$, and that their exponential generating function is $\exp\left(z + \frac{z^2}{2}\right)$.

To construct an explicit bijection Υ, let $e \in \mathbf{I}_n\left(\underline{\neq,\neq}\right)$, and define $\Upsilon(e) \in S_n$ recursively as follows. For $n = 0$, define the image of the empty inversion sequence to be the empty permutation; for $n = 1$, define $\Upsilon(0) = 1$. For $n \geq 2$, let $i \in [n]$ be such that $e_n - e_{n-1} \equiv i \pmod{n}$.

- If $i = n$, let $\sigma = \Upsilon\left(e_1 e_2 \cdots e_{n-1}\right)$, and define $\Upsilon(e) = \sigma_1\sigma_2\cdots\sigma_{n-1}\, n$.
- If $i \neq n$, let σ be the permutation of $[n-1]\setminus\{i\}$ with reduction $\Upsilon\left(e_1 e_2 \cdots e_{n-2}\right)$, and define $\Upsilon(e) = \sigma_1\sigma_2\cdots\sigma_{i-1}\, n\, \sigma_i\sigma_{i+1}\cdots\sigma_{n-2}\, i$.

The inverse map Υ^{-1} is defined recursively as follows. Given $\sigma \in \mathcal{I}_n$ with $n \geq 2$ (the cases $n = 0$ and $n = 1$ are trivial), let $i = \sigma_n$.

- If $i = n$, let $e_1 e_2 \cdots e_{n-1} = \Upsilon^{-1}\left(\sigma_1\sigma_2\cdots\sigma_{n-1}\right)$, and define $\Upsilon^{-1}(\sigma) = e_1 e_2 \cdots e_{n-1} e_{n-1}$.
- If $i \neq n$, let σ' be the reduction of $\sigma_1\sigma_2\cdots\sigma_{i-1}\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{n-1}$, and let $e_1 e_2 \cdots e_{n-2} = \Upsilon^{-1}(\sigma')$. Define $\Upsilon^{-1}(\sigma) = e_1 e_2 \cdots e_{n-2} e_{n-2} k$, where $0 \leq k < n$ is the unique integer satisfying $k \equiv e_{n-2} + i \pmod{n}$. ∎

Example 34. Consider $e = 00114$. Computing $\Upsilon(e_1 e_2 \cdots e_i)$ recursively yields: $\Upsilon(0) = 1$, $\Upsilon(00) = 12$, $\Upsilon(001) = 321$, $\Upsilon(0011) = 3214$, and $\Upsilon(e) = 42513$.

Both maps Υ^{-1} (from the proof of Proposition 33) and Θ (from Equation (1)) give encodings of involutions as inversion sequences, but they do not coincide in general. For instance, if $\pi = 42513$, then $\Upsilon^{-1}(\pi) = 00114 \neq 01032 = \Theta(\pi)$.

4.8 The pattern $(\underline{\leq, >})$

We show that inversion sequences avoiding this pattern are those that are weakly increasing, and that they are in bijection with Dyck paths.

Lemma 35. *For $n \geq 1$,*

$$\mathbf{I}_n(\underline{\leq, >}) = \{e \in \mathbf{I}_n : e_1 \leq e_2 \leq \cdots \leq e_n\}.$$

Proof. If $e \in \mathbf{I}_n(\underline{\leq, >})$, the fact that $0 = e_1 \leq e_2$ implies that $e_2 \leq e_3$, assuming these entries are defined. The remaining inequalities follow by iterating the same argument. □

A *Dyck path* of semilength n is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) consisting of vertical steps $N = (0,1)$ and horizontal steps $E = (1,0)$, which never goes above the line $y = x$. Denoting by $\mathcal{D}_n$ the set of such paths, it is well-known that $|\mathcal{D}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the *$n$-th Catalan number*, and that

$$C(z) := \sum_{n \geq 0} C_n z^n = \frac{1 - \sqrt{1-4z}}{2z}. \tag{4}$$

The sequence C_n is one of the most ubiquitous sequences in enumerative combinatorics [24], and it is listed as A000108 in the OEIS [22].

Proposition 36. *Let $n \geq 1$ and $0 \leq k < n$. Then*

$$\left|\mathbf{I}_n(\underline{\leq, >})\right| = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Using Lemma 35, we obtain a straightforward bijection between $\mathbf{I}_n(\underline{\leq, >})$ and $\mathcal{D}_n$ by appending $n - e_n$ steps N to our usual representation of an inversion sequence $e \in \mathbf{I}_n(\underline{\leq, >})$ as an underdiagonal lattice path, see Figure 5. □

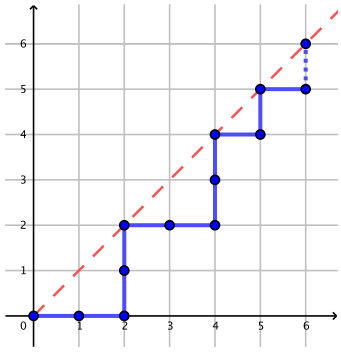


Figure 5: Representation of $e = 001123 \in \mathbf{I}_6\left(\underline{\leq, >}\right)$ as an underdiagonal lattice path. Adding the last vertical step we obtain the Dyck path $EENNEENNENEN$.

4.9 The pattern $\left(\underline{>, \leq}\right)$

We characterize inversion sequences avoiding this pattern in terms of an asymmetric unimodality condition.

Lemma 37. *For $n \geq 1$,*

$$\mathbf{I}_n\left(\underline{>, \leq}\right) = \{e \in \mathbf{I}_n : e_1 \leq e_2 \leq \cdots \leq e_j > e_{j+1} > \cdots > e_n \text{ for some } 1 \leq j \leq n\}.$$

Proof. Given $e \in \mathbf{I}_n\left(\underline{>, \leq}\right)$, let $1 \leq j \leq n$ be the smallest index such that $e_j > e_{j+1}$, or let $j = n$ if there is no such index. Since e avoids $\left(\underline{>, \leq}\right)$, we must have $e_j > e_{j+1} > \cdots > e_n$. The characterization follows. □

Martinez and Savage [20, Sec. 2.19] showed that inversion sequences in $\mathbf{I}_n(>, \leq, -)$ are those satisfying the same inequalities from Lemma 37, and so

$$\mathbf{I}_n\left(\underline{>, \leq}\right) = \mathbf{I}_n(>, \leq, -). \tag{5}$$

They also conjecture that $|\mathbf{I}_n(>, \leq, -)|$ is given by sequence A071356 in the OEIS [22]; equivalently, that Theorem 10 holds. To prove their conjecture, we introduce certain lattice paths.

Definition 38. A *marked Dyck path* P is an underdiagonal lattice path from $(0, 0)$ to some point in the diagonal, with horizontal steps $E = (1, 0)$ and two possible kinds of vertical steps $(0, 1)$, denoted by N and N^* (the latter are called *marked* steps). Denoting by $E(P)$, $N(P)$ and $N^*(P)$, the number of E, N and N^* steps in P, respectively, the size of P is defined as $N^*(P) + E(P) = N(P) + 2N^*(P)$. Let $\mathcal{P}_n$ denote the set of marked Dyck paths of size n.

If a marked Dyck path P has at least one N^* step in the last run of vertical steps, then we say that P has a *marked tail*. Otherwise, we say that P has an *unmarked tail*. We denote by $\mathcal{R}_n$ the set of paths in $\mathcal{P}_n$ with an unmarked tail.

Example 39. The marked Dyck path $P_1 = ENEN^*$ has size 3 and a marked tail, so $P_1 \in \mathcal{P}_3 \setminus \mathcal{R}_3$. On the other hand, $P = ENEN^*EEN^*EENNN$ has size 8 and an unmarked tail, so $P \in \mathcal{R}_8$. These paths are drawn in Figure 6, with a $*$ indicating which vertical steps are N^* steps.

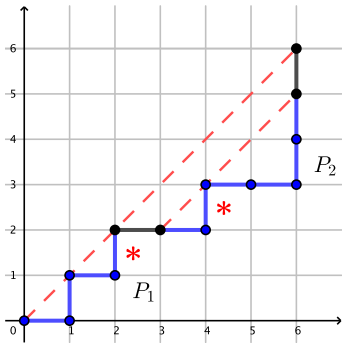


Figure 6: The marked Dyck path $R = ENEN^*EEN^*EENNN \in \mathcal{R}_8$ may be decomposed as $R = P_1EP_2N$, with $P_1 = ENEN^* \in \mathcal{P}_3$ and $P_2 = EN^*EENN \in \mathcal{R}_4$.

Next we prove Theorem 10, which gives a generating function for the sequence $\left|\mathbf{I}_n\left(\underline{\geq, \leq}\right)\right| = |\mathbf{I}_n(>, \leq, -)|$.

Proof of Theorem 10. First, we describe a bijection φ between $\mathbf{I}_n\left(\underline{\geq, \leq}\right)$ (which, by Equation (5), equals $\mathbf{I}_n(>, \leq, -)$) and $\mathcal{R}_n$. Let $e \in \mathbf{I}_n\left(\underline{\geq, \leq}\right)$. By Lemma 37, there exists $1 \leq j \leq n$ such that $e_1 \leq e_2 \leq \cdots \leq e_j > e_{j+1} > \cdots > e_n$. Let P be the corresponding underdiagonal lattice path from $(0, 0)$ to the line $x = n$, using steps $N = (0, 1)$, $S = (0, -1)$ and $E = (1, 0)$, having n steps E at heights given by $e_1, \ldots, e_n$. We construct $\varphi(e) \in \mathcal{R}_n$ as follows; see Figure 7(a)(b) for an example.

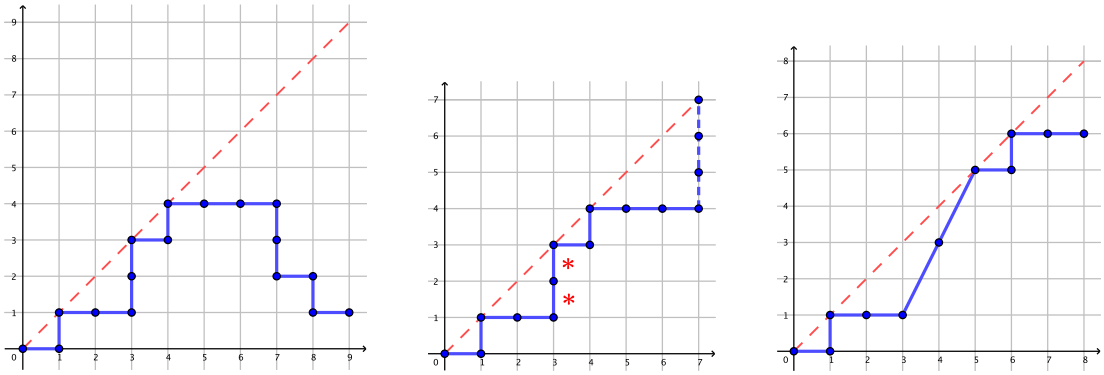
1) For every E step in the descending portion of P (that is, to the right of the line $x = j$), which corresponds to an entry e_i with $i > j$, mark the N step in the ascending portion of P going from height e_i to height $e_i + 1$, turning it into an N^* step.

2) Erase the descending portion of P, and instead append $j - e_j$ N steps. Let $\varphi(e) \in \mathcal{R}_n$ be the resulting path from the origin to (j, j).

It is clear that $\varphi : \mathbf{I}_n\left(\underline{\geq, \leq}\right) \to \mathcal{R}_n$ is a bijection, so it suffices to enumerate $\mathcal{R}_n$.

For $n \geq 1$, every $R \in \mathcal{R}_n$ can be decomposed uniquely as $R = P_1EP_2N$, where $P_1 \in \mathcal{P}_j$ and $P_2 \in \mathcal{R}_{n-j-1}$ for some $0 \leq j \leq n - 1$ (see Figure 6 for an example). It follows that if we define $P(z) = \sum_{n \geq 0} |\mathcal{P}_n| z^n$ and $R(z) = \sum_{n \geq 0} |\mathcal{R}_n| z^n$, these generating functions are related by $R(z) = 1 + z\,P(z)R(z)$, and so

$$R(z) = \frac{1}{1 - z\,P(z)}.$$



(a) (b) (c)

Figure 7: (a) The inversion sequence $e = 011344421 \in \mathbf{I}_9\left(>,\leq\right)$ represented as the underdiagonal lattice path $P = ENEENNNENEEESSESE$. (b) Its corresponding marked Dyck path $\varphi(e) = ENEEN^*N^*ENEEENNN \in \mathcal{R}_9$. (c) The path $R' = ENEEDDENEEE \in \mathcal{R}'_8$ corresponding to e under the bijection φ'.

To find $P(z)$, think of marked Dyck paths as Dyck paths with two kinds of vertical steps: N (contributing 1 to the size) and N^* (contributing 2 to the size), whereas E steps do not contribute to the size. If follows that $P(z) = C(z + z^2)$, with $C(z)$ given by Equation (4). Indeed, the monomials z and z^2 account for the choice of N and N^* steps, respectively. We deduce that

$$R(z) = \frac{1}{1 - z\,C(z + z^2)} = \frac{2 + 2z}{1 + 2z + \sqrt{1 - 4z - 4z^2}} = \frac{1 + 2z - \sqrt{1 - 4z - 4z^2}}{4z}. \qquad \square$$

Martinez and Savage's original conjecture [20, Sec. 2.19] stated that $|\mathbf{I}_n\left(>,\leq,-\right)|$ coincides with sequence A071356 in the OEIS [22], which enumerates the sets $\mathcal{R}'_n$ consisting of underdiagonal paths from $(0,0)$ to the line $x = n$ with steps N, E and $D = (1,2)$. Even though Theorem 10 already proves their conjecture, we can also give a direct bijective proof, by exhibiting a bijection between $\mathbf{I}_n\left(>,\leq,-\right)$ and $\mathcal{R}'_{n-1}$. Indeed, given $e \in \mathbf{I}_n\left(>,\leq,-\right)$, we can construct its corresponding marked Dyck path with an unmarked tail, $R = \varphi(e) \in \mathcal{R}_n$. Replacing each marked step N^* in R by a D step and deleting the last run of N steps as well as the E step immediately preceding it, we obtain a path $R' \in \mathcal{R}'_{n-1}$ (see Figure 7(c) for an example). Defining $\varphi'(e) = R'$, it is straightforward to verify that φ' is a bijection between $\mathbf{I}_n\left(>,\leq,-\right)$ and $\mathcal{R}'_{n-1}$.

For $e \in \mathbf{I}_n$, define the statistic $\mathrm{dist}(e) = |\{e_1, e_2, \ldots, e_n\}|$, that is, the number of distinct entries of e. Using numerical evidence, Martinez and Savage [20, Sec. 2.19] also conjecture that the distribution of this statistic is symmetric on $\mathbf{I}_n\left(>,\leq,-\right)$. Next we prove this conjecture by interpreting the statistic dist in terms of marked Dyck paths, and finding the corresponding bivariate generating function refining Theorem 10.

Theorem 40.

$$\sum_{n\geq 0}\sum_{e\in \mathbf{I}_n\left(\underline{>,\leq}\right)} z^n t^{\mathrm{dist}(e)} = \frac{1+z(3-t)-\sqrt{1-z\left(2+2t-z+6zt-zt^2\right)}}{4z}.$$

Proof. Let $e \in \mathbf{I}_n\left(\underline{>,\leq}\right)$, and let $R = \varphi(e)$, where φ is the bijection in the proof of Theorem 10. Define an *elbow* of a path in $\mathcal{P}_n$ to be an occurrence of a horizontal step E immediately followed by a vertical step N or N^*. Then $\mathrm{dist}(e)$ becomes the following statistic on R, which we also denote by dist with some abuse of notation:

$$\mathrm{dist}(R) = \#\left\{\text{elbows in } R\right\} + \#\left\{N^* \text{ steps in } R \text{ that are not part of an elbow}\right\}.$$

For instance, the marked Dyck path $R = ENEEN^*N^*ENEEENNN$ in Figure 7(b) has 4 elbows and 1 marked step that is not part of an elbow, so $\mathrm{dist}(R) = 5$. Define the bivariate generating functions

$$R(z,t) = \sum_{n\geq 0}\sum_{R\in\mathcal{R}_n} z^n t^{\mathrm{dist}(R)} = \sum_{n\geq 0}\sum_{e\in \mathbf{I}_n\left(\underline{>,\leq}\right)} z^n t^{\mathrm{dist}(e)} \quad \text{and} \quad P(z,t) = \sum_{n\geq 0}\sum_{P\in\mathcal{P}_n} z^n t^{\mathrm{dist}(P)}.$$

Decomposing $R \in \mathcal{R}_n$ for $n \geq 1$ as $R = P_1EP_2N$, where $P_1 \in \mathcal{P}_j$ and $P_2 \in \mathcal{R}_{n-j-1}$ for some $0 \leq j \leq n-1$, and noting that the last two steps of R form an elbow when P_2 is empty, we get the equation

$$R(z,t) = 1 + zt\,P(z,t) + z\,P(z,t)\left(R(z,t)-1\right),$$

and so

$$R(z,t) = \frac{1-z(1-t)P(z,t)}{1-z\,P(z,t)}. \tag{6}$$

To find $P(z,t)$, note that if a marked Dyck path P is *irreducible* (i.e., it is nonempty and it returns to the diagonal only at the end), then $P = EP'N$ or $P = EP'N^*$, where P' is a marked Dyck path. When P' is empty, such paths P contribute $zt + z^2t$ to the generating function, since they consist of an elbow; when P' is nonempty, they contribute $(z+z^2t)\left(P(z,t)-1\right)$. Since every path in $\mathcal{P}_n$ can be decomposed uniquely as a sequence of irreducible paths, we deduce that

$$P(z,t) = \frac{1}{1-\left[zt+z^2t+(z+z^2t)\left(P(z,t)-1\right)\right]}.$$

Solving for $P(z,t)$ and taking the solution without negative powers in its series expansion, we find that

$$P(z,t) = \frac{1+z(1-t)-\sqrt{\left(1+z(1-t)\right)^2-4\left(z+z^2t\right)}}{2\left(z+z^2t\right)}.$$

It then follows from Equation (6) that

$$R(z,t) = \frac{1+z(3-t)-\sqrt{1-z\left(2+2t-z+6zt-zt^2\right)}}{4z}.$$

□

In Section 5 we will show how the argument in the proof of Theorem 40 can be generalized to find bivariate generating functions for the number of inversion sequences e satisfying certain unimodality conditions, with respect to the length of e and $\mathrm{dist}(e)$.

Now we can finally show that the statistic dist has a symmetric distribution on $\mathbf{I}_n(>,\leq,-) = \mathbf{I}_n\left(\underline{>,\leq}\right)$, proving Martinez and Savage's conjecture.

Corollary 41. *For all $n \geq d \geq 1$,*

$$|\{e \in \mathbf{I}_n\left(\underline{>,\leq}\right) : \mathrm{dist}(e) = d\}| = |\{e \in \mathbf{I}_n\left(\underline{>,\leq}\right) : \mathrm{dist}(e) = n+1-d\}|.$$

Proof. Write $R(z,t) = \sum_{n\geq 0} U_n(t) z^n$, where $U_n(t) = \sum_{e\in\mathbf{I}_n\left(\underline{>,\leq}\right)} t^{\mathrm{dist}(e)} = \sum_{i=1}^{n} u_{i,n} t^i$. Then

$$\begin{aligned} t\,R\left(zt, \frac{1}{t}\right) &= t + t\sum_{n\geq 1}\left(u_{1,n}\frac{1}{t} + u_{2,n}\frac{1}{t^2} + \cdots + u_{n,n}\frac{1}{t^n}\right)(zt)^n \\ &= t + \sum_{n\geq 1}\left(u_{1,n}t^n + u_{2,n}t^{n-1} + \cdots + u_{n,n}t\right) z^n. \end{aligned}$$

Thus, in order to prove that $u_{d,n} = u_{n+1-d,n}$ for $n \geq d \geq 1$, it is enough to show that $R(z,t) - 1 = t\,R\left(zt, \frac{1}{t}\right) - t$. This is now immediate from Theorem 40, since

$$t\,R\left(zt, \frac{1}{t}\right) - t = \frac{1 - z(1+t) - \sqrt{1 - z\left(2 + 2t - z + 6zt - zt^2\right)}}{4z} = R(z,t) - 1. \qquad \square$$

A different proof of this corollary, showing γ-positivity of $U_n(t)$, has recently been given by Cao, Jin, and Lin [9, Them. 5.1].

4.10 The pattern $\left(\underline{=,\neq}\right)$

The characterization of inversion sequences avoiding this pattern is relatively simple.

Lemma 42. *For $n \geq 1$,*

$$\mathbf{I}_n\left(\underline{=,\neq}\right) = \{e \in \mathbf{I}_n : e_1 \neq e_2 \neq \cdots \neq e_j = e_{j+1} = \cdots = e_n \text{ for some } 1 \leq j \leq n\}.$$

Proof. Given $e \in \mathbf{I}_n\left(\underline{=,\neq}\right)$, let $1 \leq j \leq n-1$ be the smallest index such that $e_j = e_{j+1}$, or let $j = n$ if there is not such index. Since e avoids $\left(\underline{=,\neq}\right)$, we must have $e_j = e_{j+1} = \cdots = e_n$. $\square$

Next we use this characterization to find a formula for $\left|\mathbf{I}_n\left(\underline{=,\neq}\right)\right|$. The corresponding sequence, which is listed as A003422 in the OEIS [22], is sometimes referred to as the *left factorial* of n.

Proposition 43. *For $n \geq 1$, $\left|\mathbf{I}_n\left(\underline{=,\neq}\right)\right| = 0! + 1! + 2! + \cdots + (n-1)!$.*

Proof. By Lemma 42, we can express $\mathbf{I}_n\left(\underline{=,\neq}\right)$ as a disjoint union $\bigsqcup_{j=1}^{n} A_{n,j}$, where $A_{n,j} = \{e \in \mathbf{I}_n : e_1 \neq e_2 \neq \cdots \neq e_j = e_{j+1} = \cdots = e_n\}$.

To specify an element $e \in A_{n,j}$, after setting $e_1 = 0$, we have $i-1$ choices for e_i for each $2 \leq i \leq j$, because $e_i \in \{0, 1, \ldots, i-1\} \setminus \{e_{i-1}\}$. Once e_j is chosen, the entries $e_{j+1}, e_{j+2}, \ldots, e_n$ are forced. It follows that $|A_{n,j}| = (j-1)!$, and consequently,

$$\left|\mathbf{I}_n\left(\underline{=,\neq}\right)\right| = \sum_{j=1}^{n} |A_{n,j}| = 0! + 1! + 2! + \cdots + (n-1)!. \qquad \square$$

4.11 The patterns $\left(\underline{\geq,\geq}\right) \overset{ss}{\sim} \left(\underline{<,<}\right)$

An occurrence of $\left(\underline{<,<}\right)$ in an inversion sequence is precisely an occurrence of the consecutive pattern $\underline{012}$, so $\mathbf{I}_n\left(\underline{<,<}\right) = \mathbf{I}_n\left(\underline{012}\right)$. It is known that if $\Theta : S_n \to \mathbf{I}_n$ is the map defined by (1) and $\pi \in S_n$, then π avoids the consecutive permutation pattern $\underline{321}$ if and only $\Theta(\pi)$ avoids $\underline{012}$ [1, Prop. 3.19]. Using Theorem 7(ii) and [13, Thm. 4.1], the next corollary follows.

Corollary 44 ([1, 13])**.** *Let $\pi \in S_n$, and let $e = \Theta(\pi)$ be its corresponding inversion sequence. Then π avoids $\underline{321}$ if and only e avoids $\left(\underline{<,<}\right)$. In particular, $\left|\mathbf{I}_n\left(\underline{\geq,\geq}\right)\right| = \left|\mathbf{I}_n\left(\underline{<,<}\right)\right| = \left|\mathbf{I}_n\left(\underline{012}\right)\right| = \left|S_n\left(\underline{321}\right)\right|$ for all n, and the corresponding exponential generating function is*

$$\sum_{n\geq 0} \left|\mathbf{I}_n\left(\underline{<,<}\right)\right| \frac{z^n}{n!} = \frac{\sqrt{3}}{2} \frac{\exp\left(z/2\right)}{\cos\left(\pi/6 + \sqrt{3}z/2\right)}.$$

4.12 The pattern $\left(\underline{\neq,=}\right)$

The techniques in this section are similar to those in Section 4.10.

Lemma 45. *For $n \geq 1$,*

$$\mathbf{I}_n\left(\underline{\neq,=}\right) = \{e \in \mathbf{I}_n : e_1 = e_2 = \cdots = e_j \neq e_{j+1} \neq \cdots \neq e_n \text{ for some } 1 \leq j \leq n\}.$$

Proof. Given $e \in \mathbf{I}_n\left(\underline{\neq,=}\right)$, let $1 \leq j \leq n-1$ be the smallest index such that $e_j \neq e_{j+1}$, or let $j = n$ if there is no such index. Since e avoids $\left(\underline{\neq,=}\right)$, we have $e_j \neq e_{j+1} \neq \cdots \neq e_n$. $\square$

We can use the description of inversion sequences avoiding the pattern $\left(\underline{\neq,=}\right)$ given by Lemma 45 to enumerate them. The corresponding sequence is listed as A000522 in the OEIS [22].

Proposition 46. *For $n \geq 1$, $\left|\mathbf{I}_n\left(\underline{\neq,=}\right)\right| = \sum_{i=0}^{n-1} (n-1)!/i!$.*

Proof. By Lemma 45, we can express $\mathbf{I}_n\left(\underline{\neq,=}\right)$ as a disjoint union $\bigsqcup_{j=1}^{n} A_{n,j}$, where $A_{n,j} = \{e \in \mathbf{I}_n : e_1 = e_2 = \cdots = e_j \neq e_{j+1} \neq \cdots \neq e_n\}$.

To specify an element $e \in A_{n,j}$, after setting $e_1 = \cdots = e_j = 0$, we have $i-1$ choices for e_i for each $j+1 \le i \le n$, because $e_i \in \{0, 1, \ldots, i-1\} \setminus \{e_{i-1}\}$. Thus, $|A_{n,j}| = j(j+1)\cdots(n-1) = (n-1)!/(j-1)!$, and

$$\left|\mathbf{I}_n\left(\underline{\neq, =}\right)\right| = \sum_{j=1}^{n} |A_{n,j}| = \sum_{i=0}^{n-1} \frac{(n-1)!}{i!}. \qquad \square$$

4.13 The patterns $\left(\underline{\geq, >}\right) \overset{ss}{\sim} \left(\underline{>, \geq}\right)$

We showed in Proposition 18 that $\left|\mathbf{I}_n\left(\underline{>, \geq}\right)\right| = \left|\mathbf{I}_n\left(\underline{\geq, >}\right)\right| = |S_n\left(\underline{1243}\right)|$. This sequence appears as A200403 in the OEIS [22], but no closed-form formula for it is known.

We can obtain a recurrence to compute $\left|\mathbf{I}_n\left(\underline{>, \geq}\right)\right|$ by introducing some refinements. Let $\mathbf{I}_{n,k} = \{e \in \mathbf{I}_n : e_n = k\}$, and define $\mathbf{I}_{n,k}\left(\underline{R_1, R_2}\right) = \mathbf{I}_{n,k} \cap \mathbf{I}_n\left(\underline{R_1, R_2}\right)$ for any consecutive pattern of relations $\left(\underline{R_1, R_2}\right)$. Note that $\mathbf{I}_n\left(\underline{R_1, R_2}\right) = \bigsqcup_{k=0}^{n-1} \mathbf{I}_{n,k}\left(\underline{R_1, R_2}\right)$, and that $\mathbf{I}_{n,k}\left(\underline{R_1, R_2}\right) = \varnothing$ unless $0 \le k < n$.

Proposition 19 shows that $\left|\mathbf{I}_{n,k}\left(\underline{>, \geq}\right)\right| = \left|\mathbf{I}_{n,k}\left(\underline{\geq, >}\right)\right|$. To compute these numbers, we define

$$\mathbf{I}^{>}_{n,k}\left(\underline{>, \geq}\right) = \left\{e \in \mathbf{I}_{n,k}\left(\underline{>, \geq}\right) : e_{n-1} > e_n\right\}.$$

Proposition 47. *For $n \ge 2$, the sequences $\left|\mathbf{I}_{n,k}\left(\underline{>, \geq}\right)\right|$ and $\left|\mathbf{I}^{>}_{n,k}\left(\underline{>, \geq}\right)\right|$ satisfy the recurrences*

$$\left|\mathbf{I}_{n,k}\left(\underline{>, \geq}\right)\right| = \begin{cases} \left|\mathbf{I}_{n-1}\left(\underline{>, \geq}\right)\right|, & \text{if } k = n-1; \\ \left|\mathbf{I}_{n,k+1}\left(\underline{>, \geq}\right)\right| - \left|\mathbf{I}^{>}_{n-1,k}\left(\underline{>, \geq}\right)\right|, & \text{if } 0 \le k \le n-2; \\ 0, & \text{otherwise;} \end{cases} \tag{7}$$

$$\left|\mathbf{I}^{>}_{n,k}\left(\underline{>, \geq}\right)\right| = \begin{cases} \left|\mathbf{I}^{>}_{n,k+1}\left(\underline{>, \geq}\right)\right| - \left|\mathbf{I}^{>}_{n-1,k+1}\left(\underline{>, \geq}\right)\right| + \left|\mathbf{I}_{n-1,k+1}\left(\underline{>, \geq}\right)\right|, & \text{if } 0 \le k \le n-2; \\ 0, & \text{otherwise;} \end{cases} \tag{8}$$

with initial conditions $\left|\mathbf{I}_1\left(\underline{>, \geq}\right)\right| = \left|\mathbf{I}_{1,0}\left(\underline{>, \geq}\right)\right| = 1$, $\left|\mathbf{I}_{1,k}\left(\underline{>, \geq}\right)\right| = 0$ for $k > 0$, and $\left|\mathbf{I}^{>}_{1,k}\left(\underline{>, \geq}\right)\right| = 0$ for all k.

Equations (7) and (8) allow us to compute the values $\left|\mathbf{I}_{n,k}\left(\underline{>, \geq}\right)\right|$ and $\left|\mathbf{I}^{>}_{n,k}\left(\underline{>, \geq}\right)\right|$ recursively, since an entry indexed by (n,k) depends only on entries indexed by (n,j) where $k < j < n$, or by $(n-1, j)$ for some j.

Proof. Note that $e \in \mathbf{I}_{n,k}\left(\underline{>, \geq}\right)$ if and only if $e_n = k$ and

$$e_1 e_2 \ldots e_{n-1} \in \mathbf{I}_{n-1}\left(\underline{>, \geq}\right) \setminus \bigsqcup_{j=k}^{n-2} \mathbf{I}^{>}_{n-1,j}\left(\underline{>, \geq}\right),$$

where the subtracted term guarantees that $e_{n-2}e_{n-1}e_n$ is not an occurrence of $\left(\underline{>,\geq}\right)$. Thus,

$$\left|\mathbf{I}_{n,k}\left(\underline{>,\geq}\right)\right| = \left|\mathbf{I}_{n-1}\left(\underline{>,\geq}\right)\right| - \sum_{j=k}^{n-2}\left|\mathbf{I}^{>}_{n-1,j}\left(\underline{>,\geq}\right)\right|. \tag{9}$$

In particular, this implies that $\left|\mathbf{I}_{n,n-1}\left(\underline{>,\geq}\right)\right| = \left|\mathbf{I}_{n-1}\left(\underline{>,\geq}\right)\right|$. If $0 \leq k \leq n-2$, then subtracting Equation (9) from the same equation with $k+1$ substituted for k, we may write

$$\left|\mathbf{I}_{n,k+1}\left(\underline{>,\geq}\right)\right| - \left|\mathbf{I}_{n,k}\left(\underline{>,\geq}\right)\right| = \left|\mathbf{I}^{>}_{n-1,k}\left(\underline{>,\geq}\right)\right|.$$

This proves the recurrence (7).

Now, $e \in \mathbf{I}^{>}_{n,k}\left(\underline{>,\geq}\right)$ if and only if $e_n = k$ and $e_1e_2\cdots e_{n-1} \in \bigsqcup_{j=k+1}^{n-2}\mathbf{I}_{n-1,j}\left(\underline{>,\geq}\right) \setminus \mathbf{I}^{>}_{n-1,j}\left(\underline{>,\geq}\right)$. Hence,

$$\left|\mathbf{I}^{>}_{n,k}\left(\underline{>,\geq}\right)\right| = \sum_{j=k+1}^{n-2}\left(\left|\mathbf{I}_{n-1,j}\left(\underline{>,\geq}\right)\right| - \left|\mathbf{I}^{>}_{n-1,j}\left(\underline{>,\geq}\right)\right|\right). \tag{10}$$

In particular, $\left|\mathbf{I}^{>}_{n,n-1}\left(\underline{>,\geq}\right)\right| = 0$. If $0 \leq k \leq n-2$, then subtracting Equation (10) from the same equation with $k+1$ substituted for k, we may write

$$\left|\mathbf{I}^{>}_{n,k+1}\left(\underline{>,\geq}\right)\right| - \left|\mathbf{I}^{>}_{n,k}\left(\underline{>,\geq}\right)\right| = -\left|\mathbf{I}_{n-1,k+1}\left(\underline{>,\geq}\right)\right| + \left|\mathbf{I}^{>}_{n-1,k+1}\left(\underline{>,\geq}\right)\right|.$$

This proves Equation (8). □

More generally, for any $R_1, R_2 \in \{\leq, \geq, <, >, =, \neq\}$, we define the refinement

$$\mathbf{I}^{R_1}_{n,k}\left(\underline{R_1,R_2}\right) = \left\{e \in \mathbf{I}_{n,k}\left(\underline{R_1,R_2}\right) : e_{n-1}R_1e_n\right\}. \tag{11}$$

Then, arguing as in the proof of Proposition 47, we can write

$$\left|\mathbf{I}_{n,k}\left(\underline{R_1,R_2}\right)\right| = \left|\mathbf{I}_{n-1}\left(\underline{R_1,R_2}\right)\right| - \sum_{\substack{j=0\\ jR_2k}}^{n-2}\mathbf{I}^{R_1}_{n-1,j}\left(\underline{R_1,R_2}\right)$$

and

$$\left|\mathbf{I}^{R_1}_{n,k}\left(\underline{R_1,R_2}\right)\right| = \sum_{\substack{j=0\\ jR_1k}}^{n-2}\left|\mathbf{I}_{n-1,j}\left(\underline{R_1,R_2}\right)\right| - \sum_{\substack{j=0\\ jR_1k,\, jR_2k}}^{n-2}\mathbf{I}^{R_1}_{n-1,j}\left(\underline{R_1,R_2}\right).$$

These two equations can be simplified, depending on R_1 and R_2, to obtain recurrences analogous to Equations (7) and (8), which allow us to compute the values $\left|\mathbf{I}_{n,k}\left(\underline{R_1,R_2}\right)\right|$ and $\left|\mathbf{I}^{R_1}_{n,k}\left(\underline{R_1,R_2}\right)\right|$ recursively. In particular, we can obtain such recurrences for each of the 12 Wilf equivalence classes of consecutive patterns of relations mentioned in the beginning of Section 4, for which $\left|\mathbf{I}_{n}\left(\underline{R_1,R_2}\right)\right|$ does not match any sequence in the OEIS [22].

We remark that, even though $\left|\mathbf{I}_{n,k}\left(\underline{>,\geq}\right)\right| = \left|\mathbf{I}_{n,k}\left(\underline{\geq,>}\right)\right|$ by Proposition 19, the resulting recurrences involving the refinements in (11) are different. In particular, while the quantities $\left|\mathbf{I}^{>}_{n,k}\left(\underline{>,\geq}\right)\right|$ and $\left|\mathbf{I}^{\geq}_{n,k}\left(\underline{\geq,>}\right)\right|$ are not equal in general, one can show that $\left|\mathbf{I}^{>}_{n,k}\left(\underline{>,\geq}\right)\right| = \left|\mathbf{I}^{\geq}_{n,k+1}\left(\underline{\geq,>}\right)\right|$ for all n and k.

4.14 The pattern $\left(\underline{=},=\right)$

An occurrence of $\left(\underline{=},=\right)$ in an inversion sequence is precisely an occurrence of the consecutive pattern $\underline{000}$, so $\mathbf{I}_n\left(\underline{=},=\right) = \mathbf{I}_n\left(\underline{000}\right)$. The next proposition rephrases a known result about inversion sequences that avoid $\underline{000}$, see [1, Prop. 3.1 and Cor. 3.3]. We use d_n to denote the number of derangements of length n, i.e., the number of permutations in S_n with no fixed points.

Proposition 48 ([1]). *The sequence* $\left|\mathbf{I}_n\left(\underline{=},=\right)\right|$ *satisfies the recurrence*

$$\left|\mathbf{I}_n\left(\underline{=},=\right)\right| = (n-1)\left|\mathbf{I}_{n-1}\left(\underline{=},=\right)\right| + (n-2)\left|\mathbf{I}_{n-2}\left(\underline{=},=\right)\right|$$

for $n \geq 3$, *with initial terms* $\left|\mathbf{I}_1\left(\underline{=},=\right)\right| = 1$ *and* $\left|\mathbf{I}_2\left(\underline{=},=\right)\right| = 2$. *It follows that*

$$\left|\mathbf{I}_n\left(\underline{=},=\right)\right| = \frac{(n+1)! - d_{n+1}}{n}.$$

5 Inversion sequences satisfying unimodality conditions

In Lemma 37 we enumerated inversion sequences satisfying a unimodality condition, which, as observed by Martinez and Savage [20, Sec. 2.19], are precisely those inversion sequences avoiding the pattern $(>, \leq, -)$. This is one of 10 triples of binary relations of the form (R_1, R_2, R_3) such that $\mathbf{I}_n(R_1, R_2, R_3)$ is characterized by some type of unimodality condition. Martinez and Savage [20] carried out the enumeration of the inversion sequences in each case, with the results summarized in Table 3.

Martinez and Savage [20] obtain these via ad hoc methods. In this section, we provide a unified approach, generalizing the method that we used in Section 4.9 to prove Theorem 10, in order to recover their results. Furthermore, we find bivariate generating functions keeping track of the statistic dist (recording the number of distinct entries), in analogy to Theorem 40, for each of the patterns in Table 3. For a triple of relations (R_1, R_2, R_3), define

$$I_{(R_1,R_2,R_3)}(z,t) = \sum_{n\geq 0} \sum_{e \in \mathbf{I}_n(R_1,R_2,R_3)} z^n t^{\mathrm{dist}(e)}.$$

We begin by considering the pattern $(>, <, -)$, and noting that $\mathbf{I}_n(>, \leq, -) \subseteq \mathbf{I}_n(>, <, -)$. We generalize Definition 38 by allowing different kinds of marked steps in a Dyck path.

Definition 49. A *multi-marked Dyck path* P is an underdiagonal path from $(0,0)$ to some point in the diagonal, with horizontal steps $E = (1,0)$ and infinitely many possible kinds of vertical steps $(0,1)$, denoted by N and N_t^* for $t \geq 2$ (the latter are called *marked* steps). Denoting by $E(P)$, $N(P)$ and $N_t^*(P)$ the number of E, N and N_t^* steps in P, respectively, the size of P is defined as

$$E(P) + \sum_{t\geq 2}(t-1)N_t^*(P) = N(P) + \sum_{t\geq 2} t\, N_t^*(P).$$

Pattern (R_1, R_2, R_3)	OEIS[22]	Unimodality condition for $e \in \mathbf{I}_n(R_1, R_2, R_3)$	$\lvert\mathbf{I}_n(R_1, R_2, R_3)\rvert$ counted by
$(<, -, <)$	A000124	$\exists j : e_1 = \cdots = e_j \leq e_{j+1} \geq 0 = \cdots = 0$	$1 + \binom{n}{2}$
$(\neq, <, -)$	A000325	$\exists j : e_1 = \cdots = e_j \leq e_{j+1} \geq e_{j+2} \geq \cdots \geq e_n$	$2^n - n$
$(\neq, \leq, -)$	A000071	$\exists j : e_1 = \cdots = e_j < e_{j+1} > e_{j+2} > \cdots > e_n$	$F_{n+2} - 1$
$(>, <, -)$	A033321	$\exists j : e_1 \leq \cdots \leq e_j > e_{j+1} \geq e_{j+2} \geq \cdots \geq e_n$	OGF is $\frac{1+z-\sqrt{1-6z+5z^2}}{2z(2-z)}$
$(>, \leq, -)$	A071356	$\exists j : e_1 \leq \cdots \leq e_j > e_{j+1} > e_{j+2} > \cdots > e_n$	OGF is $\frac{1+2z-\sqrt{1-4z-4z^2}}{4z}$
$(>, \neq, -)$	A279561	$\exists j : e_1 \leq \cdots \leq e_j \geq e_{j+1} = e_{j+2} = \cdots = e_n$	$1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}$
$(\geq, \neq, -)$	A000124	$\exists j : e_1 < \cdots < e_j \geq e_{j+1} = e_{j+2} = \cdots = e_n$	$1 + \binom{n}{2}$
$(=, <, -)$	A000079	$\exists j : e_1 < \cdots < e_j \geq e_{j+1} \geq e_{j+2} \geq \cdots \geq e_n$	2^{n-1}
$(=, \leq, -)$	A000045	$\exists j : e_1 < \cdots < e_j \geq e_{j+1} > e_{j+2} > \cdots > e_n$	F_{n+1}
$(\geq, \leq, \neq)$	A000071	$\exists (j \leq i) : e_1 < \cdots < e_j = \cdots = e_i > \cdots > e_n$	$F_{n+2} - 1$

Table 3: Patterns (R_1, R_2, R_3) for which $\mathbf{I}_n(R_1, R_2, R_3)$ is characterized by a unimodality condition. These enumerative results are due to Martinez and Savage [20]. Patterns are listed in the same order they use, rather than in our usual order (least avoided to most avoided).

We denote by $\widetilde{\mathcal{P}}_n$ the set of multi-marked Dyck paths of size n.

If $P \in \widetilde{\mathcal{P}}_n$ has at least one N_t^* step in the last run of vertical steps, then we say that P has a *marked tail*. Otherwise, we say that P has an *unmarked tail*. We denote by $\widetilde{\mathcal{R}}_n$ the set of paths in $\widetilde{\mathcal{P}}_n$ with an unmarked tail.

Multi-marked Dyck paths generalize marked Dyck paths from Definition 38, in the sense that any marked Dyck path in $\mathcal{P}_n$, after replacing N^* steps with N_2^* steps, can be seen as a multi-marked Dyck path in $\widetilde{\mathcal{P}}_n$ containing no steps N_t^* with $t \geq 3$. Hence, $\mathcal{P}_n \subseteq \widetilde{\mathcal{P}}_n$ and $\mathcal{R}_n \subseteq \widetilde{\mathcal{R}}_n$.

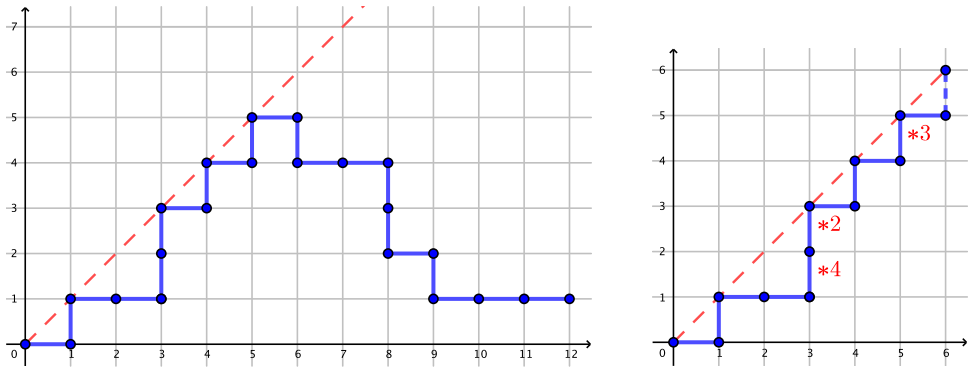
Example 50. The multi-marked Dyck path $R = ENEEN_4^*N_2^*ENEN_3^*EN$ has size 12, so $R \in \widetilde{\mathcal{P}}_{12}$. In fact, R has an unmarked tail, so $R \in \widetilde{\mathcal{R}}_{12}$. This path is drawn on the right of Figure 8(b).

Theorem 51.

$$I_{(>,<,-)}(z,t) = \frac{1 + z(2-t) - z^2(1-t) - (1-z)\sqrt{(1+z-zt)^2 - \frac{4z(1-z+zt)}{1-z}}}{2z(2-z)}.$$

Proof. We begin by describing a bijection $\varphi : \mathbf{I}_n(>, <, -) \to \widetilde{\mathcal{R}}_n$, generalizing the bijection in the proof of Theorem 10, which was between $\mathbf{I}_n(>, \leq, -)$ and $\mathcal{R}_n$.

Let $e \in \mathbf{I}_n(>, <, -)$, and let j be such that $e_1 \leq e_2 \leq \cdots \leq e_j > e_{j+1} \geq e_{j+2} \geq \cdots \geq e_n$. Let P be the corresponding underdiagonal lattice path from $(0,0)$ to the line $x = n$, with



(a) (b)

Figure 8: (a) The inversion sequence $e = 011345442111 \in \mathbf{I}_{12}(>,<,-)$ represented as the underdiagonal lattice path $P = ENEENNENENESEESSESEEE$. (b) Its corresponding multi-marked Dyck path $R = ENEEN_4^* N_2^* ENEN_3^* EN \in \widetilde{\mathcal{R}}_{12}$.

E steps at heights $e_1, \ldots, e_n$, and the necessary N and S steps in between. We construct $\varphi(e) \in \widetilde{\mathcal{R}}_n$ as follows; see Figure 8 for an example.

1) For every maximal run of consecutive E steps in the descending portion of P (to the right of $x = j$), which corresponds to entries $e_i = e_{i+1} = \cdots = e_{i+t-2}$ with $i > j$ and $t \geq 2$, turn the N step in the ascending portion of P going from height e_i to height $e_i + 1$ into an N_t^* step.

2) Erase the descending portion of P, and instead append $j - e_j$ N steps . Let $\varphi(e) \in \widetilde{\mathcal{R}}_n$ be the resulting path from the origin to (j, j).

It is clear that φ is a bijection, so $|\mathbf{I}_n(>,<,-)| = |\widetilde{\mathcal{R}}_n|$. In the rest of the proof we count multi-marked Dyck paths with an unmarked tail, mimicking the proof of Theorem 40.

Define an elbow of a path in $\widetilde{\mathcal{P}}_n$ to be a consecutive occurrence of EN or EN_t^* for some $t \geq 2$. If $e \in \mathbf{I}_n(>,<,-)$ and $R = \varphi(e)$, then $\mathrm{dist}(e)$ equals the following statistic on R, which we also denote by dist:

$$\mathrm{dist}(R) = \#\{\text{elbows in } R\} + \sum_{t \geq 2} \#\{N_t^* \text{ steps in } R \text{ that are not part of an elbow}\}.$$

Now we find expressions for the generating functions

$$\widetilde{R}(z,t) = \sum_{n \geq 0} \sum_{R \in \widetilde{\mathcal{R}}_n} z^n t^{\mathrm{dist}(R)} = I_{(>,<,-)}(z,t) \quad \text{and} \quad \widetilde{P}(z,t) = \sum_{n \geq 0} \sum_{P \in \widetilde{\mathcal{P}}_n} z^n t^{\mathrm{dist}(P)}.$$

An identical argument to the one we used to deduce Equation (6) yields

$$\widetilde{R}(z,t) = \frac{1 - z(1-t)\widetilde{P}(z,t)}{1 - z\widetilde{P}(z,t)}, \tag{12}$$

so it is enough to compute $\widetilde{P}(z,t)$.

If P is an irreducible multi-marked Dyck path, then there is a unique multi-marked Dyck path P' such that either $P = EP'N$ or $P = EP'N_t^*$ for some $t \geq 2$. Paths P for which P' is empty contribute $zt + \frac{z^2t}{1-z}$ to the generating function $\widetilde{P}(z,t)$, because in this case P is an elbow. On the other hand, paths P for which P' is nonempty contribute $\left(z + \frac{z^2t}{1-z}\right)\left(\widetilde{P}(z,t) - 1\right)$ to $\widetilde{P}(z,t)$. Given that every path in $\widetilde{\mathcal{P}}_n$ has a unique decomposition as a sequence of irreducible multi-marked Dyck paths, it follows that

$$\widetilde{P}(z,t) = \frac{1}{1 - \left[z(t-1) + \left(z + \frac{z^2t}{1-z}\right)\widetilde{P}(z,t)\right]}.$$

Solving for $\widetilde{P}(z,t)$ yields

$$\widetilde{P}(z,t) = \frac{(1-z)\left(1 + z - zt - \sqrt{(1+z-zt)^2 - 4\left(z\frac{z^2t}{1-z}\right)}\right)}{2z(1-z+zt)}.$$

Hence, Equation (12) implies that

$$\widetilde{R}(z,t) = \frac{1 + z(2-t) - z^2(1-t) + (z-1)\sqrt{(1+z-zt)^2 + \frac{4z(1-z+zt)}{z-1}}}{2z(2-z)}. \qquad \square$$

By setting $t = 1$ in Theorem 51, we find that

$$\sum_{n\geq 0} |\mathbf{I}_n(>,<,-)|\, z^n = \frac{1 + z - \sqrt{1 - 6z + 5z^2}}{2z(2-z)}, \tag{13}$$

which appears as A033321 in the OEIS [22]. This is also the generating function for $|S_n(2143, 3142, 4132)|$, that is, permutations simultaneously avoiding the classical patterns 2143, 3142 and 4132 [6, Lem. 3.2]. Burstein and Stromquist [8] point out that $\pi \to \Theta\left(\pi^{RC}\right)$ is a bijection between $S_n(2143, 3142, 4132)$ and $\mathbf{I}_n(>,<,-)$.

In Section 4.9 we provided a bijection between $\mathbf{I}_n(>,\leq,-)$ and $\mathcal{R}'_{n-1}$. To extend this construction to inversion sequences avoiding $(>,<,-)$, let $\widetilde{\mathcal{R}}'_n$ be the set of underdiagonal paths from $(0,0)$ to the line $x = n$ with steps N, E and $D_t = (t-1,t)$ for $t \geq 2$, and note that $\mathcal{R}'_n \subseteq \widetilde{\mathcal{R}}'_n$. We construct a bijection between $\mathbf{I}_n(>,<,-)$ and $\widetilde{\mathcal{R}}'_{n-1}$ as follows. Given $e \in \mathbf{I}_n(>,<,-)$, first let $R \in \widetilde{\mathcal{R}}_n$ be its corresponding multi-marked Dyck path with an unmarked tail. Then replace each marked step N_t^* in R by a step D_t, and delete the last run of N steps as well as the E step immediately preceding it, to obtain a path $R' \in \widetilde{\mathcal{R}}'_{n-1}$ (see Figure 9 for an example). It follows from this bijection and Equation (13) that

$$\sum_{n\geq 0} \left|\widetilde{\mathcal{R}}'_n\right| z^n = \frac{1 - 3z + 2z^2 - \sqrt{1 - 6z + 5z^2}}{2z^2(2-z)}.$$

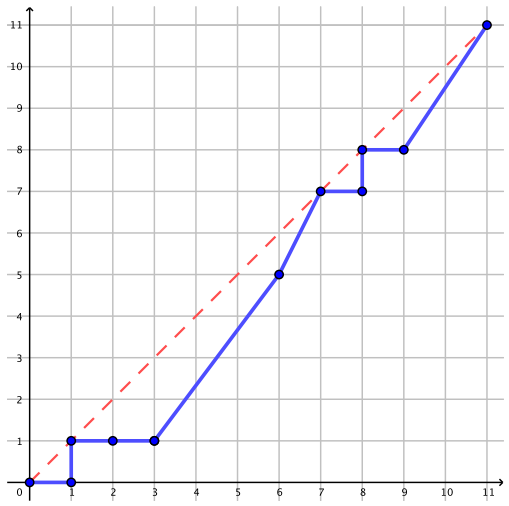


Figure 9: The path $R' = ENEED_4D_2ENED_3 \in \widetilde{\mathcal{R}}'_{11}$ corresponding to $e = 011345442111$ (from Figure 8) under the bijection $\mathbf{I}_{12}(>,<,-) \to \widetilde{\mathcal{R}}'_{11}$.

Let (R_1, R_2, R_3) be one of the patterns listed in Table 3. Then $\mathbf{I}_n(R_1, R_2, R_3) \subseteq \mathbf{I}_n(>,<,-)$. Indeed, Table 3 shows that the unimodality condition characterizing $\mathbf{I}_n(R_1, R_2, R_3)$ is weaker than that characterizing $\mathbf{I}_n(>,<,-)$. Hence, an inversion sequence $e \in \mathbf{I}_n(R_1, R_2, R_3)$ can still be represented as a multi-marked Dyck path $\varphi(e)$ with an unmarked tail, where φ is the bijection in the proof of Theorem 51. For each pattern in Table 3, φ induces a bijection between $\mathbf{I}_n(R_1, R_2, R_3)$ and a subset of $\widetilde{\mathcal{R}}_n$, which can be characterized as shown in Table 4. These descriptions allow us to find the following expressions for $I_{(R_1,R_2,R_3)}(z,t)$ using the symbolic method.

Theorem 52. *For patterns (R_1, R_2, R_3) such that $\mathbf{I}_n(R_1, R_2, R_3)$ is characterized by unimodality conditions, the generating functions $I_{(R_1,R_2,R_3)}(z,t)$ are as follows:*

$$I_{(<,-,<)}(z,t) = \frac{1 - 3z + zt + 3z^2 - 2z^2t + z^2t^2 - z^3 + z^3t}{(1-z)^3}, \tag{14}$$

$$I_{(\neq,<,-)}(z,t) = \frac{1 - 4z + zt + 6z^2 - 4z^2t + z^2t^2 - 4z^3 + 5z^3t - z^3t^2 + z^4 - 2z^4t + z^4t^2}{(1-z)^2(1-2z+z^2-z^2t)},$$

$$I_{(\neq,\leq,-)}(z,t) = \frac{1 - 2z + zt + z^2 - 2z^2t + z^2t^2 + z^3t}{(1-z)(1-z-z^2t)},$$

$$I_{(>,<,-)}(z,t) = \frac{1 + z(2-t) - z^2(1-t) - (1-z)\sqrt{(1+z-zt)^2 - \frac{4z(1-z+zt)}{1-z}}}{2z(2-z)}, \tag{15}$$

$$I_{(>,\leq,-)}(z,t) = \frac{1 + z(3-t) - \sqrt{1 - z\,(2 + 2t - z + 6zt - zt^2)}}{4z}, \tag{16}$$

$$I_{(>,\neq,-)}(z,t) = \frac{1-2z+z^2(1-t)^2+(1-z+zt)\sqrt{(1+z-zt)^2-4z}}{2(1-z)\sqrt{(1+z-zt)^2-4z}},$$

$$I_{(\geq,\neq,-)}(z,t) = \frac{1-z-zt+2z^2t}{(1-z)(1-zt)^2}, \qquad I_{(=,<,-)}(z,t) = \frac{1-z}{1-z-zt},$$

$$I_{(=,\leq,-)}(z,t) = \frac{1}{1-zt-z^2t}, \qquad I_{(\geq,\leq,\neq)}(z,t) = \frac{1-z+z^3t}{(1-z)(1-zt-z^2t)}.$$

Equations (15) and (16) are restatements of Theorems 51 and 40, respectively, included here for the sake of completeness. By setting $t = 1$ in the generating functions $I_{(R_1,R_2,R_3)}(z,t)$ in Theorem 52, we obtain expressions for $\sum_{n\geq 0} |\mathbf{I}_n(R_1,R_2,R_3)|\, z^n$ in each case, from where one can recover the results listed in Table 3.

Pattern (R_1,R_2,R_3)	Description of paths in $\varphi(\mathbf{I}_n(R_1,R_2,R_3)) \subseteq \widetilde{\mathcal{R}}_n$
$(<,-,<)$	empty path; or $E^{r_1}(N_t^*)^{r_2}N^{r_3}EN^{r_4}$, where $r_2 = 0,1$ and $r_4 > 0$
$(\neq,<,-)$	E^rN^r; or $E^{r_1}PE^{r_2}N^{r_3}$, where P is a nonempty path with steps N_t^* and N, and $r_1,r_2,r_3 > 0$
$(\neq,\leq,-)$	E^rN^r; or $E^{r_1}PEN^{r_2}$, where P is a nonempty path with steps N_2^* and N, and $r_1,r_2 > 0$
$(>,<,-)$	any path in $\widetilde{\mathcal{R}}_n$
$(>,\leq,-)$	contains no N_t^* steps for $t \geq 3$
$(>,\neq,-)$	contains at most one marked step N_t^*
$(\geq,\neq,-)$	consists of alternating E and vertical (N_t^* or N) steps, with exactly one marked step; or PE^rN^r, where P consists of alternating E and N steps, starting with E and ending with N
$(=,<,-)$	PE^rN^r, where P consists of alternating E and vertical (N_t^* or N) steps, starting with E and ending with a vertical step, unless it is empty
$(=,\leq,-)$	empty path; or PE^rN^r, where P consists of alternating E and either N_2^* or N steps, starting with E and ending with either N_2^* or N, unless it is empty, and $r = 1,2$
$(\geq,\leq,\neq)$	empty path; or PE^rN^r, where P consists of alternating E and either N_2^* or N steps, starting with E and ending with either N_2^* or N, unless it is empty, and $r > 0$

Table 4: Characterization of the image of the image of $\mathbf{I}_n(R_1,R_2,R_3)$ under φ for each of the patterns (R_1,R_2,R_3) in Table 3. Exponents indicate step repetition, and r and r_i are arbitrary nonnegative integers unless otherwise specified.

For the pattern $(>,\neq,-)$, the generating function in Theorem 52 is obtained from the characterization in Table 4 using a path decomposition like the one for the patterns $(>,<,-)$ and $(>,\leq,-)$. For the other patterns in Table 4, $I_{(R_1,R_2,R_3)}(z,t)$ is obtained directly from this characterization of $\varphi(\mathbf{I}_n(R_1,R_2,R_3))$. For instance, the description of $\varphi(\mathbf{I}_n(<,-,<))$ implies that any R in this set satisfies one of the following:

- R is the empty path, contributing 1 to $I_{(<,-,<)}(z,t)$;
- $R = E^{r_1+1}N^{r_4}$, with $r_1+1=r_4>0$, contributing $\frac{zt}{1-z}$ to $I_{(<,-,<)}(z,t)$;
- $R = E^{r_1}N^{r_3}EN^{r_4}$, with $r_1+1=r_3+r_4$ and $r_1,r_3,r_4>0$, contributing $\frac{z^2t^2}{(1-z)^2}$ to $I_{(<,-,<)}(z,t)$;
- $R = E^{r_1}N_t^*N^{r_3}EN^{r_4}$, with $r_1=r_3+r_4$; $r_1,r_4>0$; and $r_3\geq 0$, contributing $\frac{z^3t^2}{(1-z)^3}$ to $I_{(<,-,<)}(z,t)$.

Adding these four contributions, we obtain Equation (14).

It is natural to ask if an analogue of Corollary 41 holds for any of the patterns in Table 3. In addition to $(>,\leq,-)$, the only other pattern on this list for which the distribution of dist on $\mathbf{I}_n(R_1,R_2,R_3)$ is symmetric is $(=,<,-)$. In this case, $I_{(=,<,-)}(z,t) = 1+\sum_{n\geq 1} t(t+1)^{n-1}z^n$. Computations for small values of n also suggest that the distribution of dist on $\mathbf{I}_n(R_1,R_2,R_3)$ is unimodal for each of the patterns in Table 3.

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