Journal of Integer Sequences, Vol. 22 (2019),
Article 19.3.2

Polynomial Analogues of Restricted b-ary Partition Functions

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Abstract

Given an integer $b \geq 2$, a well studied concept of a b-ary partition function of a positive integer n counts the number of representations of n as sums of powers of b, with each power occurring up to λ times, for a fixed $\lambda \geq 1$. In this paper we introduce and study a multivariable polynomial sequence that reduces to the restricted b-ary partition function when all variables are taken to be 1. In particular, we show that this polynomial sequence characterizes all restricted b-ary partitions for each n, generalizing previous results on hyperbinary and hyper b-ary representations. All this follows from considering more general concepts of restricted b-ary partition functions.

[1]Research supported in part by the Natural Sciences and Engineering Research Council of Canada, Grant # 145628481.

1 Introduction

The (unrestricted) binary partition function, which we shall denote by $S_2(n)$, counts the number of representations of a positive integer n as a sum of powers of 2. This function was apparently first considered by Euler [14, pp. 162ff.] and later by other authors, including Churchhouse [4] who studied congruence properties of $S_2(n)$. Reznick [20] later investigated the following restricted binary partition function. Given an integer $\lambda \geq 1$, let $S_2^\lambda(n)$ denote the number of representations

$$n = \sum_{j\geq 0} c_j 2^j, \qquad c_j \in \{0, 1, \dots, \lambda\}. \tag{1}$$

In particular, we have $S_2^1(n) = 1$ since $\lambda = 1$ corresponds to the unique binary representation of n, and $S_2^2(n) = s(n+1)$, where $\{s(n)\}$ is the well-known Stern sequence; see [20, p. 470].

As an easy example, we consider $n = 6$. Then the binary partitions are

$$4+2,\ 4+1+1,\ 2+2+2,\ 2+2+1+1,\ 2+1+1+1+1,\ 1+1+1+1+1+1.$$

Hence $S_2(6) = 6$, and $S_2^2(6) = 3 = s(7)$, this latter number counting the hyperbinary representations $4+2, 4+1+1$, and $2+2+1+1$.

We can generalize (1) to an arbitrary integer base $b \geq 2$ and consider the number of representations

$$n = \sum_{j\geq 0} c_j b^j, \qquad c_j \in \{0, 1, \dots, \lambda\}, \tag{2}$$

which we denote by $S_b^\lambda(n)$. Analogously, $S_b(n)$ denotes the number of unrestricted b-ary partitions, and we can write $S_b(n) = \lim_{\lambda\to\infty} S_b^\lambda(n)$.

It appears that the function $S_b(n)$ was first studied by Mahler [15] who derived an asymptotic formula, then later by other authors, including Churchhouse [4], Rödseth [21], and Andrews [1], who were mainly interested in congruence properties of the function $S_b(n)$. The restricted b-ary partition function $S_b^\lambda(n)$ was apparently first studied by Dumont et al. [13, Section 6].

As in the case $b = 2$, the function $S_b^{b-1}(n)$ counts the unique b-adic representation of n, and is therefore always equal to 1, and $S_b^b(n)$ counts the number of hyper b-ary expansions of n, also known as base b over-expansions of n. Defant [6] showed that $S_b^b(n) = s_b(n+1)$, where $\{s_b(n)\}$ is the base-b generalized Stern sequence; this is therefore a direct generalization of Reznick's result cited after (1). It should also be noted that for $b \geq 3$ and $1 \leq \lambda < b-1$, we have $S_b^\lambda(n) = 0$ for infinitely many n; see Section 6 for some examples.

In a different direction, the present authors studied the special case $\lambda = b$, first for $b = 2$, i.e., the case of hyperbinary representations. Results of Bates and Mansour [2] and of Stanley and Wilf [22], which are refinements of the Stern identity $S_2^2(n) = s(n+1)$, were further refined in [9] by the introduction of a two-variable (generalized Stern) polynomial sequence that characterizes all hyperbinary representations of n counted by $S_2^2(n)$. This was subsequently extended in [10] to hyper b-ary expansions, i.e., those expansions of the type

(1) that are counted by $S_b^b(n)$ for an arbitrary integer base $b \geq 2$. Further properties of these b-variable polynomials were then derived in [11].

It is the purpose of the present paper to use the methods developed in [9]–[11] to define and investigate polynomial analogues of the restricted b-ary partition function $S_b^\lambda(n)$ for any integer $b \geq 2$ and primarily for $\lambda \geq b-1$. In the process we investigate a more generalized restricted b-ary partition function and its polynomial analogue.

There is no established notation in the literature for binary and b-ary restricted or unrestricted partition functions. We chose the notation $S_b^\lambda(n)$, resp. $S_b(n)$, in order to avoid clashes with notations of other objects. It should also be noted that many authors used the letter m to denote a general integer base, while we are using b and "b-ary".

This paper is structured as follows. In Section 2 we define our principal objects of study, namely the polynomial analogues to the restricted b-ary partition function, and state the main representation theorem. In Section 3 we prove recurrence relations for the polynomials in question and consider a number of special cases. The proof of our main theorem is then given in Section 4, and in Section 5 we derive explicit formulas for our polynomials. We conclude this paper with some further remarks in Section 6.

2 Generating functions and main theorem

It is a well-known fact that partitions, including binary and b-ary partitions, are closely related to generating functions which are usually expressed in the form of infinite products. Indeed, it follows from (2) and the definition of $S_b^\lambda(n)$ that

$$\sum_{n=0}^{\infty} S_b^\lambda(n)\zeta^n = \prod_{j=0}^{\infty}\left(1+\zeta^{b^j}+\zeta^{2\cdot b^j}+\cdots+\zeta^{\lambda\cdot b^j}\right). \tag{3}$$

This can be seen by expanding the right-hand side of (3) and equating coefficients of ζ^n; see also [13, p. 381], or [20, p. 454] for $b=2$. A special case of (3) is the case $\lambda = b$, where $S_b^b(n)$ counts the number of hyper b-ary expansions; see [5]. Since $S_b^b(n) = s_b(n+1)$, as noted in the Introduction, the generating function (3) immediately gives

$$\sum_{n=0}^{\infty} s_b(n)\zeta^n = \zeta\prod_{j=0}^{\infty}\left(1+\zeta^{b^j}+\zeta^{2\cdot b^j}+\cdots+\zeta^{b\cdot b^j}\right), \tag{4}$$

where $s_b(n)$ is the b-ary Stern sequence; see also [6]. More recently the current authors introduced a b-variable polynomial analogue of the sequence $s_b(n)$ in [10] and showed that it characterizes all $s_b(n+1)$ hyper b-ary representations of n. These polynomials can be seen as b-ary generalized Stern polynomials; they were defined by way of recurrence relations generalizing those in [5] and [6], and the corresponding generating function was obtained in the subsequent paper [11]. These polynomials, written here in the slightly different notation

$\omega_{b,T}(n,Z)$, have integer parameters $T=(t_1,\dots,t_b)$ and variables $Z=(z_1,\dots,z_b)$, and satisfy

$$\sum_{n=1}^{\infty}\omega_{b,T}(n;Z)\zeta^n=\zeta\prod_{j=0}^{\infty}\left(1+z_1^{t_1^j}\zeta^{b^j}+z_2^{t_2^j}\zeta^{2\cdot b^j}+\cdots+z_b^{t_b^j}\zeta^{b\cdot b^j}\right). \tag{5}$$

When we compare (4) and (5), it is clear that $\omega_{b,T}(n;1,\dots,1)=s_b(n)$, independent of the b-tuple T.

We now use (5) as the basis for the definitions of the main objects of study in this paper.

Definition 1. Let $b\geq 2$ and $\lambda\geq 1$ be integers, $T=(t_1,\dots,t_\lambda)$ be a λ-tuple of positive integer parameters, and $Z=(z_1,\dots,z_\lambda)$ a λ-tuple of variables. Then we define the sequence of λ-variable polynomials $\omega_{b,T}^{\lambda}(n;Z)$ by the generating function

$$\sum_{n=1}^{\infty}\omega_{b,T}^{\lambda}(n;Z)\zeta^n=\zeta\prod_{j=0}^{\infty}\left(1+z_1^{t_1^j}\zeta^{b^j}+z_2^{t_2^j}\zeta^{2\cdot b^j}+\cdots+z_\lambda^{t_\lambda^j}\zeta^{\lambda\cdot b^j}\right). \tag{6}$$

We immediately see that for $\lambda=b$ this last identity reduces to (5), and for $Z=(1,\dots,1)$ it reduces to (4). Therefore we have for any $b\geq 2$, $\lambda\geq 1$ and for any T and indices $n\geq 0$ that

$$\omega_{b,T}^{b}(n;Z)=\omega_{b,T}(n;Z), \tag{7}$$
$$\omega_{b,T}^{\lambda}(n;1,\dots,1)=S_b^{\lambda}(n-1). \tag{8}$$

By expanding the right-hand side of (6) for a given $b\geq 2$ and $\lambda\geq 1$, one can easily write down the first few polynomials $\omega_{b,T}^{\lambda}(n;Z)$. For instance, those for $b=3$ and $\lambda=4$ are shown in Table 1, where for greater ease of writing we have set $Z=(w,x,y,z)$ and $T=(q,r,s,t)$.

We recall that the partition function $S_b^{\lambda}(n)$ counts the number of b-ary partitions with each power of b occurring at most λ times, and we will see that $\omega_{b,T}^{\lambda}(n;Z)$ characterizes all these partitions. It turns out that without too much additional effort we can do the same with b-ary partitions whose powers of b, if they occur, can only have multiplicities v from an ordered set of λ integers, $V=(v_1,v_2,\dots,v_\lambda)$, where $\lambda\geq 1$ and $1\leq v_1<v_2<\cdots<v_\lambda$. As a consequence we can give a more general definition.

n	$\omega^4_{3,T}(n;w,x,y,z)$	n	$\omega^4_{3,T}(n;w,x,y,z)$
0	0	10	$w^{q^2}+x^ry+y^s$
1	1	11	$w^{1+q^2}+wy^s+x^rz$
2	w	12	$w^{q^2}x+xy^s$
3	x	13	$w^{q+q^2}+w^{q^2}y+y^{1+s}+z^t$
4	w^q+y	14	$w^{1+q+q^2}+w^{q^2}z+y^sz+wz^t$
5	$w^{1+q}+z$	15	$w^{q+q^2}x+xz^t$
6	w^qx	16	$w^{q^2}x^r+w^{q+q^2}y+yz^t$
7	x^r+w^qy	17	$w^{1+q^2}x^r+w^{q+q^2}z+z^{1+t}$
8	wx^r+w^qz	18	$w^{q^2}x^{1+r}$
9	x^{1+r}	19	$x^{r^2}+w^{q^2}x^ry+w^{q^2}y^s$

Table 1: $\omega^4_{3,T}(n;w,x,y,z)$, $T=(q,r,s,t)$, $0\le n\le 19$.

Definition 2. Let b, λ, T and Z be as in Definition 1, and let $V=(v_1,\dots,v_\lambda)$ be a strictly increasing finite sequence of positive integers. Then we define the sequence of λ-variable polynomials $\omega^V_{b,T}(n;Z)$ by the generating function

$$\sum_{n=1}^{\infty}\omega^V_{b,T}(n;Z)\zeta^n=\zeta\prod_{j=0}^{\infty}\left(1+z_1^{t_1^j}\zeta^{v_1b^j}+z_2^{t_2^j}\zeta^{v_2b^j}+\cdots+z_\lambda^{t_\lambda^j}\zeta^{v_\lambda b^j}\right). \tag{9}$$

Obviously, when $V=(1,2,\dots,\lambda)$, then $\omega^V_{b,T}(n;Z)=\omega^\lambda_{b,T}(n;Z)$. Also, in analogy to $S^\lambda_b(m)$, we define $S^V_b(m)$ below. As before, let $V=(v_1,v_2,\dots,v_\lambda)$ be an ordered set of integers with $1\le v_1<\cdots<v_\lambda$, and define

$$\mathbb{H}^V_b(m):=\left\{m=\sum_{j\ge 0}c_jb^j\,|c_j\in\{0,v_1,\dots,v_\lambda\}=\{0\}\cup V\right\}, \tag{10}$$

i.e., the set of restricted b-ary partitions of m, where a power of b either does not occur, or occurs v_1 times, or v_2 times, ..., or v_λ times. If we then define its cardinality as

$$S^V_b(m):=\#\mathbb{H}^V_b(m), \tag{11}$$

then we have, in analogy to (8),

$$\omega^V_{b,T}(n;1,\dots,1)=S^V_b(n-1),\qquad n=1,2,\dots \tag{12}$$

We illustrate the objects defined in (9) and (10) with an example.

Example 3. Let $b=2$, $V=(1,3,4)$, and $n=10$. Then the representations of $m=n-1=9$ given by (10) are

$$8+1,\quad 4+2+1+1+1,\quad 2+2+2+1+1+1,\quad 2+2+2+2+1. \tag{13}$$

Meanwhile, expansion of the right-hand side of (9) gives, with $T=(r,s,t)$ and $Z=(x,y,z)$, the $\lambda=3$ variable polynomial

$$\omega_{2,T}^{V}(10;Z)=x^{1+r^3}+x^{r+r^2}y+y^{1+s}+xz^t. \tag{14}$$

We juxtapose the $S_2^V(9)=4$ representations in (13) and the four monomials in (14) as follows:

$$\begin{array}{lcl} x^{1+r^3} & \longleftrightarrow & 1\cdot 2^0+1\cdot 2^3, \\ x^{r+r^2}y & \longleftrightarrow & 1\cdot 2^1+1\cdot 2^2+3\cdot 2^0, \\ y^{1+s} & \longleftrightarrow & 3\cdot 2^0+3\cdot 2^1, \\ xz^t & \longleftrightarrow & 1\cdot 2^0+4\cdot 2^1. \end{array}$$

This one-to-one correspondence is a special case of the following theorem, which is the main result of this paper.

Theorem 4. *Let $b\geq 2$ and $\lambda\geq 1$ be integers, $T=(t_1,\ldots,t_\lambda)$ a λ-tuple of positive integer parameters, and $V=(v_1,\ldots,v_\lambda)$ an ordered set of strictly increasing positive integers. Then for any integer $m\geq 1$ we have*

$$\omega_{b,T}^{V}(m+1;z_1,\ldots,z_\lambda)=\sum_{h\in\mathbb{H}_b^V(m)} z_1^{p_{h,1}(t_1)}\cdots z_\lambda^{p_{h,\lambda}(t_\lambda)}, \tag{15}$$

where, for each $h\in\mathbb{H}_b^V(m)$, the exponents $p_{h,1}(t_1),\ldots,p_{h,\lambda}(t_\lambda)$ are polynomials with only 0 and 1 as coefficients. Furthermore, if for $1\leq i\leq\lambda$ we write

$$p_{h,i}(t_i)=t_i^{\tau_i(1)}+t_i^{\tau_i(2)}+\cdots+t_i^{\tau_i(\mu_i)},\quad 0\leq\tau_i(1)<\cdots<\tau_i(\mu_i),\quad \mu_i\geq 0, \tag{16}$$

then the powers that are used exactly v_i times in the b-ary partition h are

$$b^{\tau_i(1)},b^{\tau_i(2)},\ldots,b^{\tau_i(\mu_i)}. \tag{17}$$

If $\mu_i=0$ in (16), we set $p_{h,i}(t_i)=0$ by convention, and accordingly (17) is the empty set.

In the special case $\lambda=b$ and $V=(1,2,\ldots,b)$, Theorem 4 reduces to Theorem 11 in [10], which characterizes all hyper b-ary representations of m. In the case $b=2$, $\lambda=3$, and $V=(1,3,4)$, Theorem 4 explains the correspondence given at the end of Example 3.

For the proof of Theorem 4 we need recurrence relations for the polynomials $\omega_{b,T}^{V}(n;Z)$, which are established in the next section.

3 Recurrence relations

As we saw in Section 2, the restricted b-ary partition functions $S_b^\lambda(n-1)$ and $S_b^V(n-1)$ can be seen as generalizations of the b-ary Stern sequence $s_b(n)$ in (4), and the polynomials $\omega_{b,T}^{\lambda}(n;Z)$ and $\omega_{b,T}^{V}(n;Z)$ can be seen as further generalizations of the b-ary generalized Stern polynomials $\omega_{b,T}(n;Z)$. In order to put the results in this section into perspective, we quote the recurrence relation $s_b(0)=0$, $s_b(1)=1$, and for $n\geq 1$,

$$s_b(bn-j) = s_b(n) \qquad (j=0,1,\dots,b-2), \tag{18}$$
$$s_b(bn+1) = s_b(n)+s_b(n+1); \tag{19}$$

see [10] (or [5], where the sequence is shifted by 1). The case $b=2$ in (18), (19) gives the recurrence for the well-known Stern (diatomic) sequence.

The polynomials $\omega_{b,T}(n;Z)$, where $T=(t_1,\dots,t_b)$ and $Z=(z_1,\dots,z_b)$, were shown in [10] to satisfy the recurrence relations $\omega_{b,T}(0;Z)=0$, $\omega_{b,T}(1;Z)=1$, and for $n\geq 1$,

$$\omega_{b,T}(b(n-1)+j+1;z_1,\dots,z_b) = z_j\,\omega_{b,T}(n;z_1^{t_1},\dots,z_b^{t_b}) \quad (1\leq j\leq b-1), \tag{20}$$
$$\omega_{b,T}(bn+1;z_1,\dots,z_b) = z_b\,\omega_{b,T}(n;z_1^{t_1},\dots,z_b^{t_b})+\omega_{b,T}(n+1;z_1^{t_1},\dots,z_b^{t_b}). \tag{21}$$

For $z_1=\cdots=z_b=1$, the identities (20), (21) obviously reduce to (18) and (19).

In the case of general λ, especially when $\lambda>b$, the right-hand sides of (20) and (21) usually have more than just one or two terms. To state and prove the next result in a concise way, we use the following notation. If

$$Z=(z_1,\dots,z_\lambda), \qquad T=(t_1,\dots,t_\lambda) \tag{22}$$

are λ-tuples of variables, resp. positive integers, then we denote

$$Z^T := (z_1^{t_1},z_2^{t_2},\dots,z_\lambda^{t_\lambda}). \tag{23}$$

We can now state and prove the main result of this section.

Theorem 5. *Let $b\geq 2$ and $\lambda\geq 1$ be integers, denote $\ell := \lfloor \lambda/b\rfloor$, and let Z and T be given by (22). Then we have the recurrence relation $\omega_{b,T}^{\lambda}(0;Z)=0$, $\omega_{b,T}^{\lambda}(1;Z)=1$, and for each integer $n\geq 0$,*

$$\omega_{b,T}^{\lambda}(bn+j+1;Z) = \sum_{k=0}^{\ell} z_{kb+j}\omega_{b,T}^{\lambda}(n-k+1;Z^T) \qquad (j=0,1,\dots,b-1), \tag{24}$$

with the conventions that $z_0=1$, $z_\mu=0$ for $\mu>\lambda$, and $\omega_{b,T}^{\lambda}(m;Z)=0$ if $m\leq 0$.

Proof. For ease of notation we write $\omega(n;Z)$ for $\omega_{b,T}^{\lambda}(n;Z)$. Dividing both sides of (6) by ζ and then manipulating the infinite product, we get

$$\begin{aligned}
\sum_{m=0}^{\infty}\omega(m+1;Z)\zeta^m &= \prod_{j=0}^{\infty}\left(1+z_1^{t_1^j}\zeta^{b^j}+z_2^{t_2^j}\zeta^{2\cdot b^j}+\cdots+z_\lambda^{t_\lambda^j}\zeta^{\lambda\cdot b^j}\right)\\
&= (1+z_1\zeta+\cdots+z_\lambda\zeta^\lambda)\prod_{j=0}^{\infty}\left(1+z_1^{t_1^{j+1}}\zeta^{b^{j+1}}+\cdots+z_\lambda^{t_\lambda^{j+1}}\zeta^{\lambda\cdot b^{j+1}}\right)\\
&= (1+z_1\zeta+\cdots+z_\lambda\zeta^\lambda)\prod_{j=0}^{\infty}\left(1+(z_1^{t_1})^{t_1^j}(\zeta^b)^{b^j}+\cdots+(z_\lambda^{t_\lambda})^{t_\lambda^j}(\zeta^b)^{\lambda\cdot b^j}\right).
\end{aligned}$$

Using the notation (23) and again the generating function (6), we then obtain

$$\sum_{m=0}^{\infty}\omega(m+1;Z)\zeta^{m}=(1+z_1\zeta+\cdots+z_\lambda\zeta^\lambda)\sum_{\nu=0}^{\infty}\omega(\nu+1;Z^T)\zeta^{\nu b}. \tag{25}$$

With the conventions $z_0=1$ and $z_\mu=0$ for $\mu>\lambda$, and with $\lambda=\ell b+r$, $0\le r\le b-1$, we get

$$(1+z_1\zeta+\cdots+z_\lambda\zeta^\lambda)=\sum_{k=0}^{\ell}\sum_{j=0}^{b-1}z_{kb+j}\zeta^{kb+j}=\sum_{k=0}^{\ell}\zeta^{kb}\sum_{j=0}^{b-1}z_{kb+j}\zeta^{j}. \tag{26}$$

Next we write $m=bn+j$, $0\le j\le b-1$, in (25) and substitute (26) into (25). Then (25) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{j=0}^{b-1}\omega(bn+j+1;Z)\zeta^{bn+j} \qquad (27)\\
&\qquad=\left(\sum_{k=0}^{\ell}\zeta^{kb}\sum_{j=0}^{b-1}z_{bk+j}\zeta^{j}\right)\cdot\sum_{\nu=0}^{\infty}\omega(\nu+1;Z^T)\zeta^{\nu b}\\
&\qquad=\sum_{n=0}^{\infty}\sum_{j=0}^{b-1}\left(\sum_{k=0}^{\ell}z_{bk+j}\omega(n-k+1;Z^T)\right)\zeta^{nb+j},
\end{aligned}$$

where we have used a Cauchy product, with $k+\nu=n$. Finally, by equating coefficients of ζ^{nb+j} in the first and the third line of (27), we immediately get (24). $\square$

Before considering other consequences and special cases of Theorem 5, we illustrate this result with $n=0$ and $n=1$.

Corollary 6. *Let $b\ge 2$ and $\lambda\ge 2b$ be integers, and Z and T as in (22). Then*

$$\omega_{b,T}^{\lambda}(j;Z)=z_{j-1}\qquad(j=2,3,\dots,b), \tag{28}$$

$$\omega_{b,T}^{\lambda}(b+1;Z)=z_1^{t_1}+z_b, \tag{29}$$

$$\omega_{b,T}^{\lambda}(b+j;Z)=z_1^{t_1}z_{j-1}+z_{b+j-1}\qquad(j=2,3,\dots,b). \tag{30}$$

Proof. As in the proof of Theorem 5 we suppress the subscripts and superscript of ω. The identity (24) with $n=0$ gives, for $j=0,1,\dots,b-1$,

$$\omega(j+1;Z)=z_j\omega(1;Z^T)=z_j.$$

For $j=0$ this is consistent with the convention $z_0=1$, while the cases $j=1,2,\dots,b-1$ give (28) when shifted by 1. Next we take (24) with $n=1$, and again for $j=0,1,\dots,b-1$ we obtain

$$\omega(b+j+1;Z)=z_j\omega(2;Z^T)+z_{b+j}\omega(1;Z^T)=z_jz_1^{t_1}+z_{b+j}, \tag{31}$$

where we have used (23) and (28) for $j=2$. Now (24) with $j=0$ gives (29), and for $j=1,2,\dots,b-1$ we get (30) when j is again shifted by 1. $\square$

As a specific (partial) example of Corollary 6 we consider the case $b = 3$ and $\lambda = 4$ in Table 1. Then (28) and (29) correspond to the entries for $n = 2, 3, 4$. The identity (30) corresponds only to $n = 5$ since we don't have $\lambda \geq 2b$. However, we see that Corollary 6 could easily be extended to give further explicit partial sequences of polynomials when λ is large enough compared with b.

We now consider several special cases of Theorem 5. As a first such case, we let $\lambda = b$ and use the identity (7). Then $\ell = 1$, and (24) reduces to

$$\omega_{b,T}(bn + j + 1; Z) = z_j \omega_{b,T}(n + 1; Z^T) + z_{b+j} \omega_{b,T}(n; Z^T) \tag{32}$$

for $j = 0, 1, \ldots, b-1$. But we have $z_0 = 1$ and $z_{b+j} = 0$ for $j \geq 1$, which leads to the following result.

Corollary 7. *Let $b \geq 2$, $Z = (z_1, \ldots, z_b)$, and $T = (t_1, \ldots, t_b)$. Then we have $\omega_{b,T}(0; Z) = 0$, $\omega_{b,T}(1; Z) = 1$, and for $n \geq 0$,*

$$\begin{aligned} \omega_{b,T}(bn + 1; Z) &= \omega_{b,T}(n + 1; Z^T) + z_b \omega_{b,T}(n; Z^T), \\ \omega_{b,T}(bn + j + 1; Z) &= z_j \omega_{b,T}(n + 1; Z^T) \quad (j = 1, 2, \ldots, b-1). \end{aligned}$$

We have thus recovered (20) and (21). As our next special case we consider $b = 2$, with general $\lambda \geq 1$. We separate the two cases $j = 0$ and $j = b - 1 = 1$ in (24) and note again that $z_0 = 1$. Then we get the following corollary.

Corollary 8. *Let $\lambda \geq 1$ and $Z = (z_1, \ldots, z_\lambda)$, $T = (t_1, \ldots, t_\lambda)$. Then we have $\omega_{2,T}^\lambda(0; Z) = 0$, $\omega_{2,T}^\lambda(1; Z) = 1$, and for $n \geq 0$,*

$$\omega_{2,T}^\lambda(2n + 1; Z) = \omega_{2,T}^\lambda(n + 1; Z^T) + \sum_{k=1}^{\lfloor \lambda/2 \rfloor} z_{2k} \omega_{2,T}^\lambda(n - k + 1; Z^T), \tag{33}$$

$$\omega_{2,T}^\lambda(2n + 2; Z) = \sum_{k=0}^{\lfloor \lambda/2 \rfloor} z_{2k+1} \omega_{2,T}^\lambda(n - k + 1; Z^T), \tag{34}$$

where $z_{\lambda+1} = 0$ and $\omega_{2,T}^\lambda(m; Z) = 0$ when $m \leq 0$.

When $Z = (1, \ldots, 1)$, then T is irrelevant and by (8) we have $\omega_{2,T}^\lambda(n+1; Z) = S_2^\lambda(n)$, the restricted binary partition function defined by (1). Since $z_{\lambda+1} = 0$, the identity (34) needs to be split into two cases according to the parity of λ, as indicated in the following corollary.

Corollary 9. *Let $\lambda \geq 1$ be an integer. Then we have the recurrence relation $S_2^\lambda(0) = 1$, and* *for* $n \geq 0$,

$$S_2^{\lambda}(2n) = \sum_{k=0}^{\lfloor \lambda/2 \rfloor} S_2^{\lambda}(n-k), \tag{35}$$

$$S_2^{\lambda}(2n+1) = \sum_{k=0}^{\lambda/2-1} S_2^{\lambda}(n-k) \qquad (\lambda \text{ even}), \tag{36}$$

$$S_2^{\lambda}(2n+1) = \sum_{k=0}^{(\lambda-1)/2} S_2^{\lambda}(n-k) \qquad (\lambda \text{ odd}), \tag{37}$$

where $S_2^{\lambda}(m) = 0$ *when* $m < 0$.

The three identities in Corollary 9 can be found in Theorem 2.4 of [20], with different notation.

More generally, we can consider arbitrary bases $b \geq 2$, while still setting $Z = (1, 1, \ldots, 1)$ and using (8), namely $\omega_{b,T}^{\lambda}(n+1; Z) = S_b^{\lambda}(n)$. Since $z_\mu = 0$ for $\mu > \lambda$, we also have to split (24) into two cases.

Corollary 10. *Let* $b \geq 2$ *and* $\lambda \geq 1$ *be integers with* $\lambda = \ell b + r$, $0 \leq r \leq b-1$. *Then we have the recurrence relation* $S_b^{\lambda}(0) = 1$, *and for* $n \geq 0$,

$$S_b^{\lambda}(bn+j) = \sum_{k=0}^{\ell} S_b^{\lambda}(n-k) \qquad (0 \leq j \leq r), \tag{38}$$

$$S_b^{\lambda}(bn+j) = \sum_{k=0}^{\ell-1} S_b^{\lambda}(n-k) \qquad (r+1 \leq j \leq b-1), \tag{39}$$

where $S_b^{\lambda}(m) = 0$ *for* $m < 0$.

The identities (38) and (39) were previously obtained by Dumont et al. as Equation (6.2) in [13].

The last corollary of Theorem 5 in this section is a special case that is somewhat similar to Corollary 7 and is related to the unique b-ary representation of a positive integer. If we take $\lambda = b-1$ for any $b \geq 2$, then we have $\ell = 0$, and with (24) and the fact that $z_0 = 1$ we get the following recurrence relation.

Corollary 11. *Let* $b \geq 2$ *be an integer, and* $Z = (z_1, \ldots, z_{b-1})$, $T = (t_1, \ldots, t_{b-1})$ *be ordered sets of variables, resp. positive integers. Then we have* $\omega_{b,T}^{b-1}(0; Z) = 0$, $\omega_{b,T}^{b-1}(1; Z) = 1$, *and for* $n \geq 0$,

$$\omega_{b,T}^{b-1}(bn+1; Z) = \omega_{b,T}^{b-1}(n+1; Z^T), \tag{40}$$

$$\omega_{b,T}^{b-1}(bn+j+1; Z) = z_j \omega_{b,T}^{b-1}(n+1; Z^T) \qquad (1 \leq j \leq b-1). \tag{41}$$

Example 12. We use Corollary 11 with $b = 3$, $Z = (y, z)$, and $T = (s, t)$. Then (40) and (41) can be written as

$$\begin{aligned}
\omega^2_{3,T}(3n+1; Z) &= \omega^2_{3,T}(n+1; y^s, z^t),\\
\omega^2_{3,T}(3n+2; Z) &= y \cdot \omega^2_{3,T}(n+1; y^s, z^t),\\
\omega^2_{3,T}(3n+3; Z) &= z \cdot \omega^2_{3,T}(n+1; y^s, z^t).
\end{aligned}$$

The first 14 nontrivial polynomials, evaluated using these recurrences, are listed in Table 2. They are obviously all monomials.

m	ternary	$\omega^2_{3,T}(m+1; Z)$	m	ternary	$\omega^2_{3,T}(m+1; Z)$
1	(1)	y	8	(22)	z^{1+t}
2	(2)	z	9	(100)	y^{s^2}
3	(10)	y^s	10	(101)	y^{1+s^2}
4	(11)	y^{1+s}	11	(102)	$y^{s^2}z$
5	(12)	$y^s z$	12	(110)	y^{s+s^2}
6	(20)	z^t	13	(111)	y^{1+s+s^2}
7	(21)	yz^t	14	(112)	$y^{s+s^2}z$

Table 2: $\omega^2_{3,T}(m+1; y, z)$, $T = (s, t)$, $1 \le m \le 14$.

Table 2 can also be seen as another illustration of Theorem 4 for $V = (1, 2)$. Consider, for example, the last entry in Table 2. The polynomial $y^{s+s^2}z$ indicates that the ternary representation of $m = 14$ is unique, and the powers 3^1 and 3^2 each occur once, while 3^0 occurs twice. This is consistent with $14 = (112)$ in ternary representation.

To conclude this section, we state and prove a recurrence relation for the more general polynomials $\omega^V_{b,T}(n; Z)$ that were introduced in Definition 2.

Theorem 13. *Let $b \ge 2$ and $\lambda \ge 1$ be integers, and $V = (v_1, \dots, v_\lambda)$ a strictly increasing sequence of positive integers. Furthermore, let Z and T be as specified in (22), and define the v_λ-tuple $\widetilde{Z} = (\widetilde{z}_1, \widetilde{z}_2, \dots, \widetilde{z}_{v_\lambda})$ by*

$$\widetilde{z}_j = \begin{cases} z_k, & \text{when} \quad j = v_k,\ k = 1, 2, \dots, \lambda; \\ 0, & \text{when} \quad j \notin V. \end{cases} \tag{42}$$

Then we have the recurrence relation $\omega^V_{b,T}(0; Z) = 0$, $\omega^V_{b,T}(1; Z) = 1$, and for each integer $n \ge 0$,

$$\omega^V_{b,T}(bn + j + 1; Z) = \sum_{k=0}^{\lfloor v_\lambda/b \rfloor} \widetilde{z}_{kb+j}\,\omega^V_{b,T}(n - k + 1; Z^T) \qquad (j = 0, 1, \dots, b-1), \tag{43}$$

with the conventions that $\widetilde{z}_0 = 1$, $\widetilde{z}_\mu = 0$ for $\mu > \lambda$, and $\omega^V_{b,T}(m; Z) = 0$ when $m \le 0$.

Proof. In analogy to (42) we set

$$\widetilde{t}_j = \begin{cases} t_k, & \text{when} \quad j = v_k,\ k = 1, 2, \ldots, \lambda; \\ 1, & \text{when} \quad j \notin V. \end{cases}$$

Then with this definition and with (42) we see that we can write

$$\omega_{b,T}^{V}(n; Z) = \omega_{b,\widetilde{T}}^{v_\lambda}(n; \widetilde{Z}), \tag{44}$$

where the right-hand side of (44) is a polynomial of the type in Definition 1, with v_λ used in place of λ. We can therefore apply Theorem 5, which immediately gives Theorem 13, with $\lfloor v_\lambda/b \rfloor$ in place of ℓ. $\square$

Example 14. Let $b = 2$ and $V = (1, 3, 4)$. Then $\lfloor v_\lambda/b \rfloor = 2$, $Z = (x, y, z)$, $T = (r, s, t)$, and $\widetilde{Z} = (x, 0, y, z)$. Therefore

$$\widetilde{z}_0 = 1, \quad \widetilde{z}_1 = x, \quad \widetilde{z}_2 = 0, \quad \widetilde{z}_3 = y, \quad \widetilde{z}_4 = z, \quad \widetilde{z}_5 = 0,$$

and thus (43), written separately for $j = 0$ and $j = 1$, gives

$$\omega_{2,T}^{V}(2n+1; x, y, z) = \omega_{2,T}^{V}(n+1; x^r, y^s, z^t) + z \cdot \omega_{2,T}^{V}(n-1; x^r, y^s, z^t),$$
$$\omega_{2,T}^{V}(2n+2; x, y, z) = x \cdot \omega_{2,T}^{V}(n+1; x^r, y^s, z^t) + y \cdot \omega_{2,T}^{V}(n; x^r, y^s, z^t).$$

These last two identities, together with the initial conditions $\omega_{2,T}^{V}(0; Z) = 0$ and $\omega_{2,T}^{V}(1; Z) = 1$, can be used to easily compute the polynomials in question; see Table 3. We note that the entry for $m = 10$ agrees with (14) in Example 3.

m	$\omega_{2,T}^{V}(m; x, y, z)$	m	$\omega_{2,T}^{V}(m; x, y, z)$
1	1	6	$x^{1+r^2} + x^r y$
2	x	7	$x^{r+r^2} + y^s + x^r z$
3	x^r	8	$x^{1+r+r^2} + xy^s + x^{r^2} y$
4	$x^{1+r} + y$	9	$x^{r^3} + z^t + x^{r^2} z$
5	$x^{r^2} + z$	10	$x^{1+r^3} + xz^t + x^{r+r^2} y + y^{1+s}$

Table 3: $\omega_{2,T}^{V}(m; x, y, z)$, $T = (r, s, t)$, $V = (1, 3, 4)$, $1 \le m \le 10$.

4 Proof of Theorem 4

In this relatively brief section we use the general recurrence relation, Theorem 13, to prove our main representation theorem, namely Theorem 4. In what follows, in order to simplify terminology, we refer to a restricted b-ary partition of m in the sense of (10) simply as a "representation of m".

We fix the integers $b \geq 2$, $\lambda \geq 1$, and the λ-tuples T, V, and Z as in Theorems 4 and 13. As we did in earlier proofs, we suppress the subscripts and superscript of ω. We now proceed by induction on m.

1. For the induction beginning we consider (43) with $n = 0$. Since $\omega(0; Z) = 0$ and $\omega(1; Z) = 1$, we find for $j = 0, 1, \ldots, b-1$ that

$$\omega(j+1; Z) = \widetilde{z}_j = \begin{cases} z_k, & \text{when } j = v_k,\ k = 1, 2, \ldots, \lambda; \\ 0, & \text{when } j \notin V,\ j \geq 1; \\ 1, & \text{when } j = 0. \end{cases} \tag{45}$$

Here we have used (42) and the convention $\widetilde{z}_0 = 1$.

On the other hand, the only representation of $j = 1, 2, \ldots, b-1$ is $v_k = v_k b^0$ when $j = v_k$, while j has no representation when $j \notin V$. This is consistent with (15) and (16), namely

$$\omega(j+1; Z) = z_k^1, \qquad p_{h,k}(t_k) = 1 = t_k^0,$$

as the power b^0 is used exactly v_k times in the representation of j. Hence Theorem 4 holds for $m = j$, $1 \leq j \leq b-1$. This concludes the induction beginning.

2. Now we assume that Theorem 4 is true for all $m \leq nb - 1$, for some $n \geq 1$. We wish to show that it is then also true for $m = nb + j$, for all $j = 0, 1, \ldots, b-1$. To do so, we fix j, $0 \leq j \leq b-1$, and consider all representations of $nb + j$. They can be obtained recursively as follows. We fix an integer $k \geq 0$ and

(a) take all representations of $n - k$ and multiply them by b,

(b) add to each such representation $kb + j$ times the part b^0,

(c) ignore (a) and (b) when $kb + j \notin V$,

(d) do (a) and (b) for all $k \geq 0$ that satisfy $kb + j \leq v_\lambda$.

This procedure gives all representations of $nb + j$ since $(n-k)b + (kb + j) = nb + j$. Also note that the maximal k given by (d) is $\lfloor v_\lambda / b \rfloor$.

3. Using the induction hypothesis and (15), we have

$$\omega(n-k+1; Z) = \sum_{h \in \mathbb{H}_b^V(n-k)} z_1^{p_{h,1}(t_1)} \cdots z_\lambda^{p_{h,\lambda}(t_\lambda)}, \tag{46}$$

with exponents $p_{h,1}(t_1), \ldots, p_{h,\lambda}(t_\lambda)$ as in (16). In order to lift the representations of $n-k$ to those of $(n-k)b$, which corresponds to step (a), all powers of t_i, $1 \leq i \leq \lambda$, are augmented by 1. In addition, for each k with $0 \leq k \leq \lfloor v_\lambda / b \rfloor$ and satisfying $kb + j \in V$ (say $kb + j = v_r$, $1 \leq r \leq \lambda$) we add t_r^0, corresponding to b^0 being used exactly v_r times; this, in turn, corresponds to step (b). If $kb + j \notin V$, there is no contribution to the representations of $nb + j$, which corresponds to step (c).

Altogether, then, carrying out the procedure of the previous paragraph for each k for $0 \le k \le \lfloor v_\lambda/b \rfloor$ (step (d)), we get the polynomials

$$P_k := \sum_{h' \in \mathbb{H}_b^V(nb+j)} z_1^{p_{h',1}(t_1)} \cdots z_\lambda^{p_{h',\lambda}(t_\lambda)}, \tag{47}$$

where h' ranges over all those representations of $nb + j$ that have exactly $v_r = kb + j$ times the part b^0, and

$$p_{h',r}(t_r) = 1 + t_r p_{h,r}(t_r), \qquad p_{h',i}(t_i) = t_i p_{h,i}(t_i) \quad (1 \le i \le \lambda, i \ne r). \tag{48}$$

Now with (46) and (48), the sum in (47) becomes

$$\begin{aligned} P_k &= z_r \sum_{h \in \mathbb{H}_b^V(n-k)} (z_1^{t_1})^{p_{h,1}(t_1)} \cdots (z_\lambda^{t_\lambda})^{p_{h,\lambda}(t_\lambda)} \\ &= z_r \omega(n-k+1; Z^T), \end{aligned} \tag{49}$$

where we have used the notation (23).

Finally we sum (49) over all k, $0 \le k \le \lfloor v_\lambda/b \rfloor$, and use the fact that by (42) we have $\widetilde{z}_{kb+j} = z_r$ when $kb+j \in V$, and $\widetilde{z}_{kb+j} = 0$ otherwise. Then the representations of $nb+j$ are characterized by

$$\sum_{k=0}^{\lfloor v_\lambda/b \rfloor} \widetilde{z}_{kb+j} \omega(n-k+1; Z^T) = \omega(bn+j+1; Z), \tag{50}$$

where the right-hand side of (50) comes from (43). Therefore all the representations of $nb+j$ are characterized by $\omega(bn+j+1; Z)$; and since this holds for any j, $0 \le j \le b-1$, the proof by induction of the main statement of Theorem 4 is now complete.

It only remains to show that for each $h \in \mathbb{H}_b^V(m)$ the polynomials $p_{h,i}(t_i)$, $i = 1, \dots, \lambda$, have only 0 and 1 as coefficients. But this follows from (48), again by induction.

5 Explicit Formulas

In [3] Carlitz proved, in a different notation, that the number of odd binomial coefficients $\binom{n-k}{k}$ is given by $s(n+1)$, where $\{s(n)\}$ is the Stern sequence mentioned in the Introduction. If we set $\binom{n}{k}^* = \binom{n}{k} \pmod 2$ with $\binom{n}{k}^* \in \{0,1\}$, then Carlitz's result can be written as

$$s(n+1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}^* \qquad (n \ge 0). \tag{51}$$

In [9] we extended the identity (51) to 2-variable Stern polynomials, again in base $b = 2$. To state this and further results, we need the following definition from [9].

For an integer $k \geq 0$, let $k = \sum_{j\geq 0} c_j 2^j$, $c_j \in \{0, 1\}$, be the binary expansion of k. Then for an integer base $t \geq 1$ we define

$$d_t(k) := \sum_{j\geq 0} c_j t^j. \tag{52}$$

Various small values of $d_t(k)$ can be found in Table 2 of [8], with references to the OEIS [19].

With the definition (52) and notation from (6), the polynomial extension of (51) in [9] can be stated as

$$\omega_{2,T}^2(n+1; y, z) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}^* y^{d_s(n-2k)} z^{d_t(k)}, \tag{53}$$

where $T = (s, t)$. As indicated in [18], the identity (51) can be rewritten as

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}^* = \sum_{j+2k=n} \binom{j+k}{k}^*, \tag{54}$$

and similarly, we can rewrite (53) as

$$\omega_{2,T}^2(n+1; y, z) = \sum_{k_1+2k_2=n} \binom{k_1+k_2}{k_2}^* y^{d_s(k_1)} z^{d_t(k_2)}. \tag{55}$$

This last identity may serve as motivation for the following main result of this section. In analogy to $\binom{n}{k}^*$ we define $\binom{k_1+\cdots+k_\lambda}{k_1,\ldots,k_\lambda}^*$ to be the least nonnegative residue of the multinomial coefficient $\binom{k_1+\cdots+k_\lambda}{k_1,\ldots,k_\lambda}$ modulo 2.

Theorem 15. *Let $\lambda \geq 1$ be an integer, $V = (v_1, \ldots, v_\lambda)$ a strictly increasing sequence of positive integers, and Z and T as in (22). Then for $n \geq 0$ we have*

$$\omega_{2,T}^V(n+1; Z) = \sum_{v_1k_1+v_2k_2+\cdots+v_\lambda k_\lambda=n} \binom{k_1+\cdots+k_\lambda}{k_1,\ldots,k_\lambda}^* z_1^{d_{t_1}(k_1)} \cdots z_\lambda^{d_{t_\lambda}(k_\lambda)}. \tag{56}$$

Example 16. With $\lambda = 3$, $V = (1, 3, 4)$, $Z = (x, y, z)$, and $T = (r, s, t)$, the identity (56) becomes

$$\omega_{2,T}^V(n+1; Z) = \sum_{k_1+3k_2+4k_3=n} \binom{k_1+k_2+k_3}{k_1,k_2,k_3}^* x^{d_r(k_1)} y^{d_s(k_2)} z^{d_t(k_3)}. \tag{57}$$

For $n = 9$, the conditions $k_1 + 3k_2 + 4k_3 = n$ and $\binom{k_1+k_2+k_3}{k_1,k_2,k_3}^* = 1$ are satisfied by the triples $(k_1, k_2, k_3) = (9, 0, 0), (6, 1, 0), (0, 3, 0), (1, 0, 2)$, so that (57) becomes

$$\begin{aligned}\omega_{2,T}^V(10; Z) &= x^{d_r(9)} + x^{d_r(6)} y^{d_s(1)} + y^{d_s(3)} + x^{d_r(1)} z^{d_t(2)} \\ &= x^{1+r^3} + x^{r+r^2} y + y^{1+s} + xz^t.\end{aligned}$$

This is consistent with (14) in Example 3; see also Table 3.

We note that the sum in (56) is taken over all partitions of n with parts in V. In the special case $V = (1, 2, \dots, \lambda)$, we get the following immediate consequence of Theorem 15.

Corollary 17. *Let $\lambda \geq 1$ be an integer, and Z and T as in (22). Then for $n \geq 0$ we have*

$$\omega_{2,T}^{\lambda}(n+1; Z) = \sum_{k_1+2k_2+\cdots+\lambda k_\lambda = n} \binom{k_1+\cdots+k_\lambda}{k_1, \dots, k_\lambda}^{*} z_1^{d_{t_1}(k_1)} \cdots z_\lambda^{d_{t_\lambda}(k_\lambda)}. \tag{58}$$

We now see that the identity (55) is the case $\lambda = 2$ of Corollary 17. If we set $Z = (1, \dots, 1)$ in (56) and use (12), we get the following extension of (51) and (54).

Corollary 18. *Let $\lambda \geq 1$ be an integer and $V = (v_1, \dots, v_\lambda)$ a strictly increasing sequence of positive integers. Then for $n \geq 0$ we have*

$$S_2^V(n) = \sum_{v_1k_1+v_2k_2+\cdots+v_\lambda k_\lambda = n} \binom{k_1+\cdots+k_\lambda}{k_1, \dots, k_\lambda}^{*}, \tag{59}$$

where $S_2^V(n)$ is the number of restricted binary partitions of n, as defined in (11).

Theorem 15 is a special case of a more general result, which in turn is a consequence of the following lemma.

Lemma 19. *Let $b \geq 2$ and $\lambda \geq 1$ be integers, $V = (v_1, \dots, v_\lambda)$ a strictly increasing sequence of positive integers, and Z and T as in (22). Then for $n \geq 0$ we have*

$$\omega_{b,T}^{V}(n+1; Z) = \sum \prod_{j \geq 0} (z_{i_j})^{t_{i_j}^{j}}, \tag{60}$$

where the sum is taken over all representations $v_{i_0} + v_{i_1}b + v_{i_2}b^2 + \cdots = n$ in $\mathbb{H}_b^V(n)$ as defined in (10), and where we set $z_0 = 1$ and $v_0 = 0$ by convention.

Example 20. Let $b = 3$, $V = \{1, 2, 3, 4, 5\}$, $Z = (v, w, x, y, z)$, and $T = (p, q, r, s, t)$. The representations of $n = 15$ and the corresponding monomials, according to Lemma 19, are then given as follows:

$$\begin{array}{rclcl}
2 \cdot 3 + 1 \cdot 3^2 & \longleftrightarrow & z_2^{t_2} z_1^{t_1^2} & = & w^q v^{p^2}, \\
3 \cdot 1 + 1 \cdot 3 + 1 \cdot 3^2 & \longleftrightarrow & z_3^{t_3^0} z_1^{t_1} z_1^{t_1^2} & = & x v^{p+p^2}, \\
5 \cdot 3 & \longleftrightarrow & z_5^{t_5} & = & z^t, \\
3 \cdot 1 + 4 \cdot 3 & \longleftrightarrow & z_3^{t_3^0} z_4^{t_4} & = & x y^s.
\end{array}$$

In accordance with (60) these monomials are the terms of the polynomial

$$\omega_{3,T}^{5}(16; Z) = w^q v^{p^2} + x v^{p+p^2} + z^t + x y^s.$$

Proof of Lemma 19. If we divide both sides of (9) by ζ, we see that the infinite product on the right-hand side of (9) is the generating function of the polynomial sequence $\omega_{b,T}^{V}(n+1;Z)$ for $n \geq 0$.

On the other hand, for a fixed n the product in (9) shows that all powers of ζ^n are of the form

$$\zeta^n = \prod_{j\geq 0} \zeta^{v_{i_j} b^j},$$

or, in terms of the exponents,

$$n = \sum_{j\geq 0} v_{i_j} b^j, \qquad 0 \leq i_j \leq \lambda, \qquad v_0 = 0,\ v_i \in V\ (1 \leq i \leq \lambda). \tag{61}$$

The structure of the factors on the right-hand side of (9) also shows that the monomial $z_1, \ldots, z_\lambda$ corresponding to the exponent n in (61) is

$$\prod_{j\geq 0} (z_{i_j})^{t_{i_j}^j}. \tag{62}$$

Summing over all representations of n as in (61), we obtain the identity (60). $\square$

In order to state and prove our next result, we need some additional definitions with corresponding notations. For the remainder of this section, let $b \geq 2$ be a fixed integer.

First we define the set

$$M_b := \left\{ \sum_{j\geq 0} c_j b^j \mid c_j \in \{0,1\} \right\}, \tag{63}$$

i.e., the set of all nonnegative integers whose b-ary digits are only 0 or 1. Clearly we have $M_2 = \mathbb{N} \cup \{0\}$.

Next we extend (52) as follows. Let $k \in M_b$ with $k = \sum_{j\geq 0} c_j b^j$, $c_j \in \{0,1\}$. Then for an integer base $t \geq 1$ we define

$$d_t^b(k) := \sum_{j\geq 0} c_j t^j. \tag{64}$$

It is clear that $d_t^2(k) = d_t(k)$ and $d_b^b(k) = k$ for all integers $b \geq 2$ and $k \in M_b$.

Finally, we extend the multinomial coefficient modulo 2, as used in Theorem 15. If $k_1, \ldots, k_\lambda \in M_b$, we set

$$\binom{k_1 + \cdots + k_\lambda}{k_1, \ldots, k_\lambda}_b^* := \binom{d_2^b(k_1) + \cdots + d_2^b(k_\lambda)}{d_2^b(k_1), \ldots, d_2^b(k_\lambda)}^*, \tag{65}$$

with the right-hand side of (65) as defined earlier, just before Theorem 15. We are now ready to state a result that generalizes Theorem 15.

Theorem 21. *Let $b \geq 2$ and $\lambda \geq 1$ be integers, $V = (v_1, \dots, v_\lambda)$ a strictly increasing sequence of positive integers, and Z and T as in (22). Then for $n \geq 0$ we have*

$$\omega_{b,T}^{V}(n+1;Z) = \sum_{\substack{v_1k_1+v_2k_2+\cdots+v_\lambda k_\lambda=n \\ k_1,\dots,k_\lambda\in M_b}} \binom{k_1+\cdots+k_\lambda}{k_1,\dots,k_\lambda}_b^{*} z_1^{d_{t_1}^b(k_1)} \cdots z_\lambda^{d_{t_\lambda}^b(k_\lambda)}. \tag{66}$$

Before proving this result, we note that it immediately implies Theorem 15. Indeed, when $b = 2$, the multinomial coefficient defined in (65) reduces to the one in (56) since $d_2^2(k_i) = k_i$. Then in the $b = 2$ case we also have $d_{t_i}^b(k_i) = d_{t_i}(k_i)$, and the conditions concerning M_b become irrelevant.

For the proof of Theorem 21 we require the following result due to Dickson [7], which we state as a lemma. See also the more recent paper [17].

Lemma 22 (Dickson)**.** *Let p be a prime and $\lambda \geq 2$ an integer, and suppose that the positive integers $k_1, \dots, k_\lambda$ and $k := k_1 + \cdots + k_\lambda$ are given in base-p representation as*

$$\begin{aligned} k_i &= a_0^i + a_1^i p + \cdots + a_m^i p^m, \qquad i = 1, \dots, \lambda, \\ k &= a_0 + a_1 p + \cdots + a_m p^m. \end{aligned}$$

Then we have

$$\binom{k}{k_1,\dots,k_\lambda} \not\equiv 0 \pmod{p}, \tag{67}$$

if and only if

$$a_j = a_j^1 + \cdots + a_j^\lambda \tag{68}$$

holds for all $j = 1, \dots, m$.

Proof of Theorem 21. We begin with Lemma 19 and its proof. The representation of n in (61) can be rewritten as

$$n = v_1k_1 + v_2k_2 + \cdots + v_\lambda k_\lambda, \tag{69}$$

where for each $r = 1, 2, \dots, \lambda$ we have

$$k_r := \sum_{i_j=r} b^j,$$

for integers i_j with $0 \leq i_j \leq \lambda$. This means that each power $b^0, b^1, b^2, \dots$ occurs in the base-b representation of at most one of the integers $k_1, k_2, \dots, k_\lambda$, which in turn implies that

(i) $k_i \in M_b$ for all $i = 1, 2, \dots, \lambda$, and

(ii) the powers $b^0, b^1, b^2, \dots$ do not "overlap" in the sum $k_1 + \cdots + k_\lambda$.

Furthermore, as n is rewritten in the form (69), the corresponding monomial in (62) can be rewritten as

$$\prod_{i=1}^{\lambda} z_i^{d_{t_i}^b(k_i)}, \tag{70}$$

where we have used the definition (64).

We are now going to characterize condition (ii), noting that condition (i) is taken care of by the summation in (66). If for each $i = 1, 2, \ldots, \lambda$ we replace

$$k_i = c_0^i + c_1^i b + \cdots + c_{j_i} b^{j_i} \in M_b$$

by

$$d_2^b(k_i) = c_0^i + c_1^i 2 + \cdots + c_{j_i} 2^{j_i},$$

then condition (ii) means that there is no "carry" as we take the sum $S := d_2^b(k_1) + \cdots + d_2^b(k_\lambda)$. This, in turn, means that for each power of 2, say 2^ν, the sum of the coefficients of 2^ν in $d_2^b(k_1), \ldots, d_2^b(k_\lambda)$, namely 0 or 1, is exactly the coefficient of 2^ν in S. Hence the condition (68) in Lemma 22 with $p = 2$ is satisfied, and by (67) and (65) the modified multinomial coefficient in (66) is 1. Therefore the product (70) is included, as required.

On the other hand, if condition (ii) is not satisfied, then there is a carry in the sum S. This means that there is at least one index $\nu \geq 0$ such that

$$c_\nu^1 + c_\nu^2 + \cdots + c_\nu^\lambda \geq 2,$$

while the coefficient of 2^ν in S is only 0 or 1. Therefore the condition (68) in Lemma 22 (again with $p = 2$) is not satisfied, and by (67) and (65) the modified multinomial coefficient in (66) is 0. This completes the proof of Theorem 21. □

Example 23. As in Example 20, we choose again $b = 3$, $V = \{1, 2, 3, 4, 5\}$, and $n = 15$. Of the 84 partitions of $n = 15$ with parts in V, 25 satisfy the second summation condition in (66), namely $k_i \in M_3 = \{0, 1, 3, 4, 9, 10, 12, 27, \ldots\}$, $1 \leq i \leq 5$. For only four of these partitions the modified multinomial coefficient in (66) is 1, namely for

$$(k_1, \ldots, k_5) = (9, 3, 0, 0, 0), \quad (12, 0, 1, 0, 0), \quad (0, 0, 1, 3, 0), \quad (0, 0, 0, 0, 3). \tag{71}$$

Hence (66) with $Z = (v, w, x, y, z)$ and $T = (p, q, r, s, t)$ gives

$$\begin{aligned}\omega_{3,T}^5(16; Z) &= v^{d_p^3(9)} w^{d_q^3(3)} + v^{d_p^3(12)} x^{d_r^3(1)} + x^{d_r^3(1)} y^{d_s^3(3)} + z^{d_t^3(3)} \\ &= v^{p^2} w^q + v^{p+p^2} x + x y^s + z^t,\end{aligned}$$

which is consistent with the result of Example 20.

Remark. A necessary condition for the product (70) to be included in (66) is $k_1 + \cdots + k_\lambda \in M_b$, since otherwise condition (ii) in the proof of Theorem 21 is obviously not satisfied. However, this condition is not sufficient, as the following example shows.

On the other hand, we conjecture that a necessary *and* sufficient condition for the product (70) to be included in (66) is that all possible partial sums of elements of $\{k_1, \dots, k_\lambda\}$ be in M_b. A further analysis would go beyond the scope of this paper; we leave this to the interested reader.

Example 24. Continuing with Example 23, we note that of the 25 partitions that satisfy $k_i \in M_3$, $1 \le i \le 5$, there are 19 such that $k_1 + \cdots + k_5 \notin M_3$. The remaining six partitions obviously include the four in (71), while $(k_1, \dots, k_5) = (10, 1, 1, 0, 0)$ and $(4, 4, 1, 0, 0)$ are excluded by way of the modified multinomial coefficients as in (66), or by the fact that, for instance, $1 + 1 \notin M_3$ and $4 + 1 \notin M_3$.

6 Further Remarks

We recall that Definition 1 and most of the results that follow are valid for all $\lambda \ge 1$, regardless of the base $b \ge 2$. While in most of this paper we were interested in the case $\lambda \ge b - 1$ (or even $\lambda \ge b$), we are now going to consider the case $\lambda \le b - 2$ when $b \ge 3$. We begin with a consequence of Theorem 5.

Corollary 25. *Let $b \ge 3$ and $1 \le \lambda \le b - 2$ be integers, and let Z and T be given by (22). Then we have $\omega_{b,T}^{\lambda}(0; Z) = 0$, $\omega_{b,T}^{\lambda}(1; Z) = 1$, and for each $n \ge 0$ we have*

$$\omega_{b,T}^{\lambda}(bn + j; Z) = \begin{cases} \omega_{b,T}^{\lambda}(n + 1; Z^T), & \text{when } \lambda = 1; \\ z_j \omega_{b,T}^{\lambda}(n + 1; Z^T), & \text{when } 2 \le j \le \lambda; \\ 0, & \text{when } \lambda + 1 \le j \le b - 1. \end{cases} \tag{72}$$

This follows immediately from Theorem 5 since $\ell = \lfloor \lambda/b \rfloor = 0$, $z_0 = 1$, and $z_{\lambda+1} = \cdots = z_{b-1} = 0$. Corollary 25 can also be seen as an extension of Corollary 11.

By iterating (72), we immediately get the following result which characterizes the cases in which the polynomials are zero when $\lambda \le b - 2$.

Corollary 26. *Let $b \ge 3$ and $1 \le \lambda \le b - 2$ be integers, and Z and T as in (22). If the integer $m \ge 0$ has the base-b representation $m = \sum_{i=0}^{r} c_i b^i$, $0 \le c_i \le b - 1$ for $0 \le i \le r$, then*

$$\omega_{b,T}^{\lambda}(m + 1; Z) = 0 \quad \text{if and only if} \quad \lambda \le c_i \le b - 2$$

for at least one index i, $0 \le i \le r$.

Example 27. Let $b = 4$ and $\lambda = 2$, with $Z = (y, z)$ and $T = (s, t)$. Then Corollary 25 leads to the entries of Table 4, which also illustrates Corollary 26.

m	$\omega(m+1;Z)$	m	$\omega(m+1;Z)$	m	$\omega(m+1;Z)$	m	$\omega(m+1;Z)$
0	1	5	y^{1+s}	10	z^{1+t}	15	0
1	y	6	$y^s z$	11	0	16	y^{s^2}
2	z	7	0	12	0	17	y^{1+s^2}
3	0	8	z^t	13	0	18	$y^{s^2}z$
4	y^s	9	yz^t	14	0	19	0

Table 4: $\omega(m+1;Z)=\omega_{4,T}^2(m+1;Z)$, $T=(s,t)$, $Z=(y,z)$, $0\le m\le 19$.

Using (8), we get the following result for the integers $S_b^\lambda(m)$ defined by (2).

Corollary 28. *Let $b\ge 3$ and $1\le\lambda\le b-2$ be integers, and suppose that $m\ge 0$ has the base-b representation $m=\sum_{i=0}^r c_ib^i$, $0\le c_i\le b-1$ for $0\le i\le r$. Then*

$$S_b^\lambda(m)=\begin{cases}1, & \text{when } 0\le c_i\le\lambda-1 \text{ for all } 0\le i\le r;\\ 0, & \text{otherwise.}\end{cases}$$

This last result is not difficult to obtain directly. The special case where b is a prime in Corollary 28 was mentioned in [12] in connection with a study of products of cyclotomic polynomials.

As a second remark, we note that Mansour and Shattuck [16] defined the following concept.

Fix $m\ge 2$ and $0\le c\le m-1$. By a c-hyper m-expansion of a positive integer n, we mean a partition of n into powers of m in which a given power can appear exactly j times, where $j\in\{0,1,\dots,m-1,m+c\}$. See [16, Def. 1.1].

If we compare this definition with (10), we see that the set of c-hyper m-expansions is, in our notation,

$$\mathbb{H}_m^V(n),\quad\text{where}\quad V=(1,2,\dots,m-1,m+c).$$

Furthermore, again for $m\ge 2$ and $0\le c\le m-1$, Mansour and Shattuck define the sequence of polynomials $f_{m,c}(d;q)$ for $d\ge 0$ by

$$f_{m,c}(mn+j;q)=f_{m,c}(n;q),\qquad 0\le j\le m-1,\quad j\ne c, \tag{73}$$
$$f_{m,c}(mn+c;q)=f_{m,c}(n;q)+q\,f_{m,c}(n-1;q), \tag{74}$$

with $f_{m,c}(0;q)=1$ and $f_{m,c}(d;q)=0$ for $d<0$. See [16, Def. 1.2].

If we compare (73), (74) with Theorem 13, where $b=\lambda=m$ and $V=(1,2,\dots,m-1,m+c)$, then we find

$$f_{m,c}(d;q)=\omega_{m,T}^V(d+1,Z), \tag{75}$$

with the m-tuples $T=(1,\dots,1)$ and $Z=(1,\dots,1,q)$. Also, a special case of our Theorem 4 for the polynomials in (75) was obtained in [16]. Apart from these connections, however, the paper [16] has a different emphasis from ours and deals mainly with generalizations of the well-known Calkin-Wilf tree.

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2010 *Mathematics Subject Classification*: Primary 11P81; Secondary 11B37, 11B83.
Keywords: binary partition, b-ary partition, restricted partition, polynomial analogue, generating function, recurrence relation, multinomial coefficient.

(Concerned with sequences A002487 and A018819.)

Received December 2 2018; revised version received May 8 2019. Published in *Journal of Integer Sequences*, May 17 2019.

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