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The Central Coefficients of a Family of Pascal-like Triangles and Colored Lattice Paths

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Abstract

We study the central coefficients of a family of Pascal-like triangles defined by Riordan arrays. These central coefficients count left-factors of colored Schröder paths. We give various forms of the generating function, including continued fraction forms, and we calculate their Hankel transform. By using the A and Z sequences of the defining Riordan arrays, we obtain a matrix whose row sums are equal to the central coefficients under study. We explore the row polynomials of this matrix. We give alternative formulas for the coefficient array of the sequence of central coefficients.

1 Introduction

In the mathematics literature, the designation of "central binomial coefficients" is normally reserved for the binomial coefficients $\binom{2n}{n}$ A000984, as they form the central spine of Pascal's triangle A007318 when it is viewed as a centrally symmetric (or palindromic) triangle.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \boxed{1} & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & \boxed{2} & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & \boxed{6} & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & \boxed{20} & & 15 & & 6 & & 1
\end{array}
$$

When we view Pascal's triangle as a lower-triangular matrix, these numbers skip alternate rows.

$$
\begin{pmatrix}
\boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & \boxed{2} & 1 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 & 0 & 0 \\
1 & 4 & \boxed{6} & 4 & 1 & 0 & 0 \\
1 & 5 & 10 & 10 & 5 & 1 & 0 \\
1 & 6 & 15 & \boxed{20} & 15 & 6 & 1
\end{pmatrix}.
$$

In this note, we shall be principally concerned with generalizations of the central coefficients $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ A001405, which take numbers from each row when we view Pascal's triangle as a lower-triangular matrix. To distinguish $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ from the usual central binomial coefficients, we shall call $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, and their generalizations that we shall soon introduce, *complete central coefficients*.

$$
\begin{pmatrix}
\boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\boxed{1} & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & \boxed{2} & 1 & 0 & 0 & 0 & 0 \\
1 & \boxed{3} & 3 & 1 & 0 & 0 & 0 \\
1 & 4 & \boxed{6} & 4 & 1 & 0 & 0 \\
1 & 5 & \boxed{10} & 10 & 5 & 1 & 0 \\
1 & 6 & 15 & \boxed{20} & 15 & 6 & 1
\end{pmatrix}.
$$

We recall that $\binom{2n}{n}$ has the generating function (g.f.)

$$\sum_{k=0}^{n} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

while $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ has the generating function

$$g(x) = \sum_{k=0}^{n} \binom{n}{\lfloor \frac{n}{2} \rfloor} x^n = \frac{-1+2x+\sqrt{1-4x^2}}{2x-4x^2}.$$

This generating function can be alternatively expressed as

$$g(x)=\frac{1+xc(x^2)}{\sqrt{1-4x^2}}=\frac{1}{1-x-x^2c(x^2)},$$

where

$$c(x)=\frac{1-\sqrt{1-4x}}{2x}$$

is the generating function of the sequence of Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$ A000108. We note that $\binom{n}{\lfloor\frac{n}{2}\rfloor}$ is given by the interlacing of $\binom{2n}{n}$ and $\binom{2n+1}{n+1}$ A001700 so that we have

$$\binom{n}{\lfloor\frac{n}{2}\rfloor}=\binom{n}{n/2}\frac{1+(-1)^n}{2}+\binom{n}{(n+1)/2}\frac{1-(-1)^n}{2}.$$

Now the generating function of $\binom{2n+1}{n+1}$ is $\frac{c(x)}{\sqrt{1-4x}}$, so that we obtain that

$$g(x)=\frac{1}{\sqrt{1-4x^2}}+\frac{1-\sqrt{1-4x^2}}{2x\sqrt{1-4x^2}}=\frac{-1+2x+\sqrt{1-4x^2}}{2x-4x^2}.$$

We have previously [5] defined a family of Pascal-like triangles whose elements are (ordinary) Riordan arrays. In this note, we study the complete central coefficients of this family of triangles.

Integer sequences in this note are referred to by their *Annnnnn* number from the On-Line Encyclopedia of Integer Sequences [14, 15] where such a number is known. All matrices in this note are integer valued, lower triangular, and invertible. In particular the main diagonal will always consist of all 1's. Where examples are given, a suitable truncation is shown.

We recall that an ordinary Riordan array $(g(x), f(x))$ [2, 12, 13] is defined by two generating functions,

$$g(x)=g_0+g_1x+g_2x^2+\ldots,\quad g_0\neq 0,$$

and

$$f(x)=f_1x+f_2x^2+f_3x^3+\ldots,\quad f_1\neq 0,$$

with the (n,k)-th element of the corresponding lower-triangular matrix being given by

$$[x^n]g(x)f(x)^k.$$

In the sequel we shall always assume that $g_0=f_1=1$. The inverse of the Riordan array $(g(x), f(x))$ is given by

$$(g(x),f(x))^{-1}=\left(\frac{1}{g(\bar{f}(x))},\bar{f}\right),$$

where $\bar{f}(x)$ denotes the compositional inverse of $f(x)$. Thus we have $f(\bar{f}(x))=x$ and $\bar{f}(f(x))=x$. We also use the notation $\text{Rev}(f)(x)=\bar{f}(x)$. The product law for Riordan arrays (which coincides with matrix multiplication) is given by

$$(g(x),f(x))\cdot(u(x),v(x))=(g(x)u(f(x)),v(f(x))).$$

In the group of Riordan arrays, a Bell matrix is a Riordan array defined by a pair $(g(x), xg(x))$. The row sums of a Bell matrix have a generating function given by

$$(g(x), xg(x)) \cdot \frac{1}{1-x} = \frac{g(x)}{1-xg(x)}.$$

The Hankel transform [9, 10] of a sequence a_n is the sequence h_n where $h_n = |a_{i+j}|_{0\leq i,j\leq n}$. If the g.f. of a_n is expressible as a continued fraction [4, 16] of the form

$$\cfrac{\mu_0}{1-\alpha_0 x-\cfrac{\beta_1 x^2}{1-\alpha_1 x-\cfrac{\beta_2 x^2}{1-\alpha_2 x-\cfrac{\beta_3 x^2}{1-\alpha_3 x-\cdots}}}},$$

then the Hankel transform of a_n is given by [9] the Heilermann formula

$$h_n = \mu_0^{n+1}\beta_1^n\beta_2^{n-1}\cdots\beta_{n-1}^2\beta_n.$$

If the g.f. of a_n is expressible as the following type of continued fraction:

$$\cfrac{\mu_0}{1+\cfrac{\gamma_1 x}{1+\cfrac{\gamma_2 x}{1+\cdots}}},$$

then we have

$$h_n = \mu_0^{n+1}(\gamma_1\gamma_2)^n(\gamma_3\gamma_4)^{n-1}\cdots(\gamma_{2n-3}\gamma_{2n-2})^2(\gamma_{2n-1}\gamma_{2n}).$$

A Motzkin path of length n is a lattice path from $(0,0)$ to $(n,0)$ with steps northeast, $(1,1)$, east $(1,0)$ and southeast, $(1,-1)$, that does not go below the x-axis. A left-factor of a Motzkin path is the portion of a Motzkin path that goes from $(0,0)$ to the line $x=n$.

A Schröder path of length n is a lattice path from $(0,0)$ to $(2n,0)$ with steps northeast, $(1,1)$, east $(2,0)$ and southeast, $(1,-1)$, that does not go below the x-axis. A left-factor of a Schröder path is the portion of a Schröder path that goes from $(0,0)$ to the line $x=n$.

2 The family of Pascal-like triangles

The Riordan array

$$\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$$

defines a Pascal-like matrix [5] with general term $T_{n,k}$ given by

$$T_{n,k}(r) = \sum_{j=0}^{k} \binom{k}{j} \binom{n-j}{n-k-j} r^j = \sum_{j=0}^{n-k} \binom{k}{j} \binom{n-k}{j} (r+1)^j.$$

These number triangles are "Pascal-like" in the sense that

$$T_{n,k} = T_{n,n-k}, T_{n,0} = 1, T_{n,n} = 1, T_{n,k} = 0 \,\text{for}\, n > k.$$

We have

$$T_{n,k}(0) = \binom{n}{k},$$

hence for $r = 0$, the obtained number triangle is indeed Pascal's triangle A007318.

The central coefficients of these triangles are given by

$$T_{2n,n}(r) = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n-k} r^k = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} r^k = \sum_{k=0}^{n} \binom{n}{k}^2 (r+1)^k.$$

Similarly, we have that

$$T_{2n+1,n+1}(r) = \sum_{j=0}^{n+1} \binom{n+1}{j} \binom{2n+1-j}{n-j} r^j = \sum_{j=0}^{n} \binom{n+1}{j} \binom{n}{j} (r+1)^j.$$

Proposition 1. *The generating function of $T_{2n,n}(r)$ is given by*

$$\frac{1}{\sqrt{1 - 2x(r+2) + r^2x^2}}$$

and the generating function of $T_{2n+1,n+1}(r)$ is given by

$$\frac{1 - rx - \sqrt{1 - 2x(r+2) + r^2x^2}}{2x\sqrt{1 - 2x(r+2) + r^2x^2}}.$$

Proof. Define $G(x)$ and $F(x)$ by

$$\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right) = (G(x), xF(x)).$$

Then

$$\begin{aligned} T_{2n,n} &= [x^{2n}]G(x)(xF(x))^n \\ &= [x^n]G(x)F(x)^n \\ &= [x^n]\frac{G(x)}{F(x)}F(x)^{n+1} \\ &= (n+1)\frac{1}{n+1}[x^n]\frac{G(x)}{F(x)}F(x)^{n+1} \\ &= [x^n]\frac{G(v(x))}{F(v(x))}v'(x) \end{aligned}$$

where

$$v(x) = \text{Rev}\left(\frac{x}{F(x)}\right),$$

and where we have used Lagrange inversion [3, 11]. Now in this case we have

$$v(x) = \text{Rev}\left(\frac{x(1-x)}{1+rx}\right) = \frac{1-rx-\sqrt{1-2x(r+2)+r^2x^2}}{2},$$

whence the result follows. In similar fashion, we have

$$T_{2n+1,n+1} = [x^n]G(v(x))v'(x).$$

$\square$

From the above we have that

$$T_{2n,n}(r) = r^n P_n\left(\frac{r+2}{r}\right),$$

where $P_n(x)$ is the n-th Legendre polynomial. In particular we have

$$T_{2n,n}(r) = \sum_{k=0}^{n}(n-k+1)\tilde{T}_{n,k}r^k,$$

where

$$\tilde{T}_{n,k} = \frac{1}{n-k+1}\binom{2n-k}{n}\binom{n}{k}$$

is the number triangle A060693 that counts the number of Schröder paths of length n with k peaks. This latter triangle is closely related to the Narayana triangle $(N_{n,k})$ that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 6 & 1 & 0 \\ 0 & 1 & 10 & 20 & 10 & 1 \end{pmatrix}.$$

In fact we have

$$(N_{n,k}) \cdot B = (\tilde{T}_{n,k}),$$

where $B = (\binom{n}{k})$ is the binomial matrix.

We now turn our attention to the complete central coefficients $T_{n,\lfloor \frac{n}{2} \rfloor}(r)$ which define the polynomial sequence that begins

$$1, 1, r+2, 2r+3, r^2+6r+6, \ldots.$$

Example 2. The Pascal-like Riordan array $\left(\frac{1}{1-x}, \frac{x(1+x)}{1-x}\right)$ (the Delannoy triangle A008288) begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 \\ 1 & 7 & 13 & 7 & 1 & 0 & 0 \\ 1 & 9 & 25 & 25 & 9 & 1 & 0 \\ 1 & 11 & 41 & 63 & 41 & 11 & 1 \end{array}\right).$$

The complete central coefficients form the sequence A026003 that begins

$$1, 1, 3, 5, 13, 25, 63, \ldots.$$

This sequence counts left-factors of Schröder paths (from $(0,0)$ to the line $x=n$).

Example 3. The Pascal-like triangle $\left(\frac{1}{1-x}, \frac{x(1+2x)}{1-x}\right)$ A081577 begins

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 7 & 1 & 0 & 0 & 0 \\ 1 & 10 & 22 & 10 & 1 & 0 & 0 \\ 1 & 13 & 46 & 46 & 13 & 1 & 0 \\ 1 & 16 & 79 & 136 & 79 & 16 & 1 \end{array}\right).$$

The complete central coefficients form the sequence that begins

$$1, 1, 4, 7, 22, 46, 136, \ldots.$$

This sequence counts left-factors of Schröder paths (from $(0,0)$ to the line $x=n$), where the horizontal steps come in two colors. See Figure 1.

Proposition 4. *The generating function of the complete central coefficients $T_{n,\lfloor\frac{n}{2}\rfloor}(r)$ of the Pascal-like Riordan array $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$ is given by*

$$g(x;r) = \frac{\sqrt{1-2x^2(r+2)+r^2x^4}+rx^2+2x-1}{2x(1-2x-rx^2)}.$$

Proof. By the previous proposition, we have

$$\begin{aligned} g(x;r) &= \frac{1}{\sqrt{1-2x^2(r+2)+r^2x^4}} + x\frac{1-rx^2-\sqrt{1-2x^2(r+2)+r^2x^4}}{2x\sqrt{1-2x^2(r+2)+r^2x^4}} \\ &= \frac{1+2x-rx^2-\sqrt{1-2x^2(r+2)+r^2x^4}}{2x\sqrt{1-2x^2(r+2)+r^2x^4}} \\ &= \frac{\sqrt{1-2x^2(r+2)+r^2x^4}+rx^2+2x-1}{2x(1-2x-rx^2)}. \end{aligned}$$

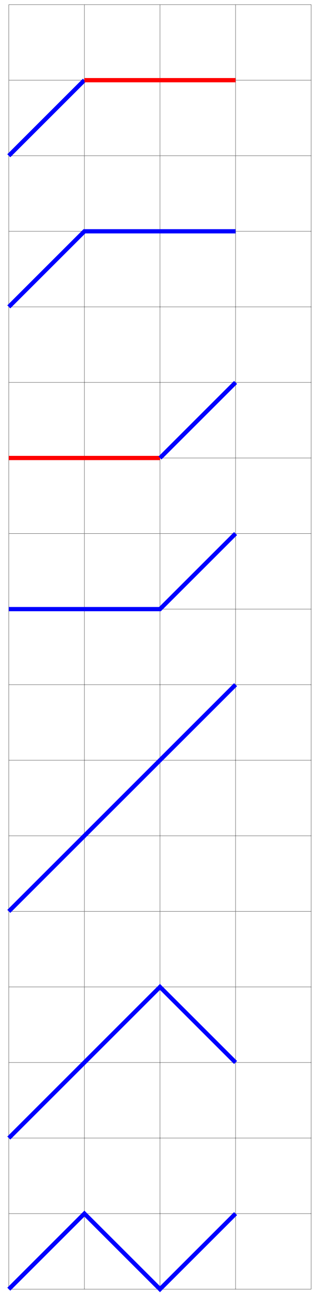

Figure 1: The 7 Schröder left-factors at $x = 3$ for $r = 2$

$\square$

Corollary 5. *We have the following continued fraction form for $g(x;r)$.*

$$g(x;r) = \cfrac{1}{1 - x - \cfrac{(r+1)x^2}{1 - \cfrac{x^2}{1 - \cfrac{(r+1)x^2}{1 - \cfrac{x^2}{1 - \cdots}}}}}.$$

Proof. We solve the equation

$$u = \frac{1}{1 - \frac{x^2}{1-(r+1)x^2 u}}.$$

Then we find that

$$g(x;r) = \frac{1}{1 - x - (r+1)x^2 u}.$$

$\square$

Corollary 6. *We have the following continued fraction form for $g(x;r)$.*

$$g(x;r) = \cfrac{1}{1 - x - rx^2 - \cfrac{x^2}{1 - rx^2 - \cfrac{x^2}{1 - rx^2 - \cdots}}}.$$

Corollary 7. *We have*

$$g(x;r) = \frac{1}{1-2x-rx^2} c\left(\frac{-x}{1-2x-rx^2}\right).$$

Corollary 8. *The Hankel transform $h_n = |T_{i+j,\lfloor \frac{i+j}{2} \rfloor}|_{0 \le i,j \le n}$ of $T_{n,\lfloor \frac{n}{2} \rfloor}(r)$ is given by*

$$h_n = (r+1)^{\lfloor \frac{(n+1)^2}{4} \rfloor}.$$

Proof. We have

$$g(x;r) = \cfrac{1}{1 - x - \cfrac{(r+1)x^2}{1 - \cfrac{x^2}{1 - \cfrac{(r+1)x^2}{1 - \cfrac{x^2}{1 - \cdots}}}}}$$

whose expansion has the same Hankel transform as the expansion of its INVERT(1) transform

$$\cfrac{1}{1-\cfrac{(r+1)x^2}{1-\cfrac{x^2}{1-\cfrac{(r+1)x^2}{1-\cfrac{x^2}{1-\cdots}}}}}.$$

The Hankel transform result follows from Heilermann's formula. ☐

Note that if a sequence has generating function $f(x)$, then its INVERT(r) transform is the sequence with generating function $\frac{f(x)}{1-rxf(x)}$. The Hankel transform of a sequence and that of its INVERT(r) transform are equal [6]. We let

$$\mathfrak{c}(x)=\frac{1+xc(x^2)}{\sqrt{1-4x^2}}=\frac{1}{1-x-x^2c(x^2)}=\frac{\sqrt{1-4x^2}+2x-1}{2x(1-x)}$$

be the generating function of $\binom{n}{\lfloor \frac{n}{2}\rfloor}$. We have

$$\mathfrak{c}(x)=\left(\frac{1}{1-2x},\frac{-x}{1-2x}\right)\cdot c(x).$$

Corollary 9. *The generating function $g(x;r)$ of the complete central coefficients $T_{n,\lfloor \frac{n}{2}\rfloor}$ of the Pascal-like matrix $\left(\frac{1}{1-x},\frac{x(1+rx)}{1-x}\right)$ is given by*

$$g(x;r)=\left(\frac{1}{1-rx^2},\frac{x}{1-rx^2}\right)\cdot\mathfrak{c}(x)=\frac{1}{1-rx^2}\mathfrak{c}\left(\frac{x}{1-rx^2}\right).$$

Proof. This follows since we have the Riordan array factorization

$$\left(\frac{1}{1-2x-rx^2},\frac{-x}{1-2x-rx^2}\right)=\left(\frac{1}{1-rx^2},\frac{x}{1-rx^2}\right)\cdot\left(\frac{1}{1-2x},\frac{-x}{1-2x}\right).$$

☐

Corollary 10. *We have*

$$T_{n,\lfloor \frac{n}{2}\rfloor}=\sum_{k=0}^{n}\binom{\frac{n+k}{2}}{k}r^{\frac{n-k}{2}}\frac{1+(-1)^{n-k}}{2}\binom{k}{\lfloor \frac{k}{2}\rfloor}=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-k}{k}r^{n-2k}\binom{n-2k}{\lfloor \frac{n-2k}{2}\rfloor}.$$

Proof. The general term of the Riordan array $\left(\frac{1}{1-rx^2},\frac{x}{1-rx^2}\right)$ is given by $\binom{\frac{n+k}{2}}{k}r^{\frac{n-k}{2}}\frac{1+(-1)^{n-k}}{2}$. ☐

3 A matrix whose row sums are $T_{n,\lfloor \frac{n}{2} \rfloor}(r)$

In this section, we wish to construct a matrix whose row sums are equal to the complete central coefficients $T_{n,\lfloor \frac{n}{2} \rfloor}(r)$. In order to do this, we will make use of the A and the Z sequences of the Pascal-like triangle $(T_{n,k})$.

Proposition 11. *For the Pascal-like triangle*

$$\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right),$$

we have

$$Z(x) = 1, \quad A(x) = \frac{1+x+\sqrt{1+2x(2r+1)+x^2}}{2}.$$

Proof. We let $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right) = (g(x), f(x))$. Then

$$A(x) = \frac{x}{\bar{f}(x)},$$

where $\bar{f}(x)$ is the solution of the equation $f(u) = x$ for which $u(0) = 0$. Thus we are looking for the solution of the equation

$$\frac{u(1+ru)}{1-u} = x.$$

We find that

$$\bar{f}(x) = \frac{\sqrt{1+2x(2r+1)+x^2}-x-1}{2r},$$

and so

$$A(x) = \frac{1+x+\sqrt{1+2x(2r+1)+x^2}}{2}.$$

That $Z(x) = 1$ follows from the fact that $g(x) = \frac{1}{1-x}$. $\square$

We now wish to construct the Riordan array M whose A-sequence is $A(x^2)$, and whose Z-sequence is $Z(x^2)$. By the theory of Riordan arrays, this matrix will have its inverse given by

$$M^{-1} = \left(1 - \frac{xZ(x^2)}{A(x^2)}, \frac{x}{A(x^2)}\right) = \left(1 - \frac{x}{A(x^2)}, \frac{x}{A(x^2)}\right).$$

Now

$$A(x^2) = \frac{1+x^2+\sqrt{1+2x^2(2r+1)+x^4}}{2}.$$

Thus if $M = (u(x), v(x))$, we have

$$v(x) = \text{Rev}\left(\frac{x}{A(x^2)}\right) = \text{Rev}\left(\frac{\sqrt{1+2x^2(2r+1)+x^4}-x^2-1}{2rx}\right).$$

We obtain that
$$v(x)=\frac{1-rx^2-\sqrt{1-2x^2(r+2)+r^2x^4}}{2x}.$$
It follows that
$$u(x)=\frac{1}{1-x}.$$
We then have the following result.

Proposition 12. *We consider the Pascal-like Riordan array*
$$\left(\frac{1}{1-x},\frac{x(1+rx)}{1-x}\right).$$
Let the Z-sequence, respectively the A-sequence of this matrix be given by $Z(x)$, respectively $A(x)$. Then the matrix whose Z-sequence is $Z(x^2)$, and whose A-sequence is $A(x^2)$, is given by
$$\left(\frac{1}{1-x},\frac{1-rx^2-\sqrt{1-2x^2(r+2)+r^2x^4}}{2x}\right).$$

Corollary 13. *The row sums of the matrix M have generating function given by*
$$\frac{2x}{(1-x)(\sqrt{r^2x^4-2x^2(r+2)+1}+rx^2+2x-1)}.$$

Proof. This follows since the row sums of $M=(u,v)$ have generating function $\frac{u}{1-v}$. □

We now form the Riordan array
$$\tilde{M}=(1-x,x)M=(1,v(x))$$
which will have row sums whose generating function is given by
$$\frac{2x}{\sqrt{r^2x^4-2x^2(r+2)+1}+rx^2+2x-1}=\frac{\sqrt{1-2x^2(r+2)+r^2x^4}-rx^2-2x+1}{2(1-2x-rx^2)}.$$
We then have

Proposition 14. *The generating function $s(x)$ of the row sums of the matrix $\tilde{M}$ satisfies*
$$s(x)=1+xg(x),$$
where $g(x)$ is the generating function of the complete central coefficients of the Pascal-like matrix
$$\left(\frac{1}{1-x},\frac{x(1+rx)}{1-x}\right).$$

Corollary 15. *Let $A(x)$ be the A-sequence of the Pascal-like triangle $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$. The row sums of the Bell matrix*

$$\left(\frac{v(x)}{x}, v(x)\right) = \left(\frac{1}{x} Rev\left(\frac{x}{A(x^2)}\right), Rev\left(\frac{x}{A(x^2)}\right)\right)$$

are the complete central coefficients $T_{n,\lfloor\frac{n}{2}\rfloor}(r)$ of the Pascal-like matrix

$$\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right).$$

Proof. We have

$$v(x) = \frac{1 - rx^2 - \sqrt{1 - 2x^2(r+2) + r^2x^4}}{2x}.$$

The row sums of $\left(\frac{v(x)}{x}, v(x)\right)$ then have generating function

$$\frac{\frac{v(x)}{x}}{1 - v(x)} = g(x; r).$$

□

The matrix $(1, v(x))$ begins

$$\left(\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & r+1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2r+2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & r^2+3r+2 & 0 & 3r+3 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 3r^2+8r+5 & 0 & 4r+4 & 0 & 1 & 0 & 0 \\
0 & r^3+6r^2+10r+5 & 0 & 6r^2+15r+9 & 0 & 5r+5 & 0 & 1 & 0 \\
0 & 0 & 4r^3+20r^2+30r+14 & 0 & 10r^2+24r+14 & 0 & 6r+6 & 0 & 1
\end{array}\right),$$

while the matrix $\left(\frac{v(x)}{x}, v(x)\right)$ begins

$$\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
r+1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2r+2 & 0 & 1 & 0 & 0 & 0 & 0 \\
r^2+3r+2 & 0 & 3r+3 & 0 & 1 & 0 & 0 & 0 \\
0 & 3r^2+8r+5 & 0 & 4r+4 & 0 & 1 & 0 & 0 \\
r^3+6r^2+10r+5 & 0 & 6r^2+15r+9 & 0 & 5r+5 & 0 & 1 & 0 \\
0 & 4r^3+20r^2+30r+14 & 0 & 10r^2+24r+14 & 0 & 6r+6 & 0 & 1
\end{array}\right).$$

The matrix $R = \left(\frac{1}{x}\mathrm{Rev}\left(\frac{x}{A(x^2)}\right), \mathrm{Rev}\left(\frac{x}{A(x^2)}\right)\right)$ can be written as

$$R = \left(\frac{1}{1-rx^2} c\left(\frac{x^2}{(1-rx^2)^2}\right), \frac{x}{1-rx^2} c\left(\frac{x^2}{(1-rx^2)^2}\right)\right).$$

We define the row polynomials of this matrix $R_n(y;r)$ to be the elements of the sequence with generating function

$$\mathcal{R}(x,y)=R\cdot\frac{1}{1-yx}=\left(\frac{1}{1-rx^2}c\left(\frac{x^2}{(1-rx^2)^2}\right),\frac{x}{1-rx^2}c\left(\frac{x^2}{(1-rx^2)^2}\right)\right)\cdot\frac{1}{1-yx}.$$

We find that

$$\mathcal{R}(x,y)=\frac{1-2xy-rx^2-\sqrt{1-2x^2(r+2)+r^2x^4}}{2x(rx^2y+x(y^2+1)-y)}.$$

This expands to give the sequence of polynomials $R_n(y;r)$ that begins

$$1,y,y^2+r+1,y^3+2y(r+1),y^4+3y^2(r+1)+(r+1)(r+2),\ldots.$$

The complete central coefficients $T_{n,\lfloor\frac{n}{2}\rfloor}(r)$ are given by $R_n(1;0)$.

We have the following result.

Proposition 16. *The generating function $\mathcal{R}(x,y;r)$ of the row polynomials $R_n(y;r)$ can be expressed as the following continued fraction.*

$$\mathcal{R}(x,y;r)=\cfrac{1}{1-yx-\cfrac{(r+1)x^2}{1-\cfrac{x^2}{1-\cfrac{(r+1)x^2}{1-\cfrac{x^2}{1-\cdots}}}}}.$$

Corollary 17. *The Hankel transform of the row polynomial sequence $R_n(y)$ is given by*

$$h_n=(r+1)^{\lfloor\frac{(n+1)^2}{4}\rfloor}.$$

Example 18. For $(y,r)=(0,1)$ we have that $R_n(0;1)$ is the sequence of aerated large Schröder numbers

$$1,0,2,0,6,0,22,0,90,0,394,0,1806,\ldots$$

that counts Schröder paths of length n.

Example 19. For $(y,r)=(2,0)$, we have that $R_n(2;0)$ is the sequence A054341 which begins

$$1,2,5,12,30,74,185,460,1150,2868,7170,17904,44760,\ldots$$

and which counts Motzkin paths of length n for which the horizontal steps at level 0 come in two colours, and which have no horizontal steps at higher level (Emeric Deutsch).

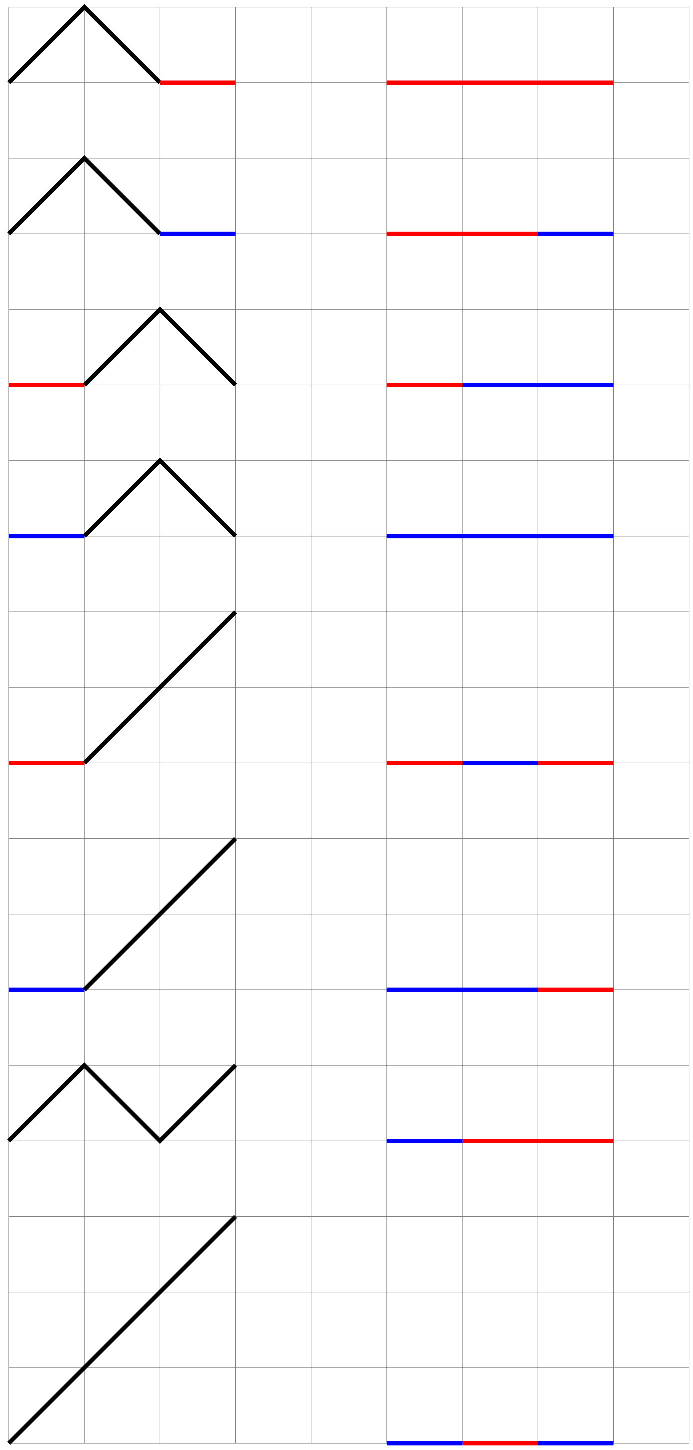

Figure 2: The $16 = R_2(2,1)$ Motzkin left-factors for $y = 2$ and $r = 1$ at $x = 3$.

Example 20. For $(y, r) = (2, 1)$, we have that $R_n(2, 1)$, which begins

$$1, 2, 6, 16, 46, 128, 366, 1032, 2946, 8352, 23826, 67720, 193126, \ldots,$$

counts left-factors of length n of Motzkin paths whose the horizontal steps at level 0 come in two colours, and which have no horizontal steps at higher level. See Figure 2.

We note that the Riordan array A110109 [1]

$$\left(\frac{1-x^2-\sqrt{1-6x^2+x^4}}{2x^2}, \frac{1-x^2-\sqrt{1-6x^2+x^4}}{2x}\right)$$

counts the number of left factors of Schröder paths from $(0, 0)$ to (n, k). Its general term [8] is given by

$$L_{n,k} = \sum_{j=0}^{\frac{n}{2}} \binom{n-j}{j} \left(\binom{n-2j}{\frac{n+k-2j}{2}} - \binom{n-2j}{\frac{n+k-2j+2}{2}} \right),$$

with the convention that a binomial coefficient whose lower parameter is not an integer is 0. We have

$$R_n(2, 1) = \sum_{k=0}^{n} L_{n,k} 2^k,$$

and in general

$$R_n(y, 1) = \sum_{k=0}^{n} L_{n,k} y^k.$$

This corresponds to the fact that the generating function

$$\cfrac{1}{1-yx-\cfrac{2x^2}{1-\cfrac{x^2}{1-\cfrac{2x^2}{1-\cdots}}}}$$

expands to the matrix that begins

$$\left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 6 & 0 & 6 & 0 & 1 & 0 & 0 & 0 \\ 0 & 16 & 0 & 8 & 0 & 1 & 0 & 0 \\ 22 & 0 & 30 & 0 & 10 & 0 & 1 & 0 \\ 0 & 68 & 0 & 48 & 0 & 12 & 0 & 1 \end{array}\right).$$

This is $(L_{n,k})$.

4 Further results

We begin this section by considering the coefficient array of the complete central coefficients $T_{n,\lfloor\frac{n}{2}\rfloor}(r)$. We have

$$T_{n,\lfloor\frac{n}{2}\rfloor}(r)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor}{k}\binom{n-k}{n-\lfloor\frac{n}{2}\rfloor-k}r^k,$$

and hence the coefficient array has coefficients

$$\binom{\lfloor\frac{n}{2}\rfloor}{k}\binom{n-k}{n-\lfloor\frac{n}{2}\rfloor-k}=\binom{\lfloor\frac{n}{2}\rfloor}{k}\binom{n-k}{\lfloor\frac{n}{2}\rfloor}.$$

This coefficient array begins

$$\left(\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
6 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
10 & 12 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
20 & 30 & 12 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
35 & 60 & 30 & 4 & 0 & 0 & 0 & 0 & 0 & 0\\
70 & 140 & 90 & 20 & 1 & 0 & 0 & 0 & 0 & 0\\
126 & 280 & 210 & 60 & 5 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

The (n,k)-th element of this array represents the number of left-factors of Schröder paths (Schröder paths from $(0,0)$ to the line $x=n$) with k peaks. This gives us the interpretation of the complete central coefficients $T_{n,\lfloor\frac{n}{2}\rfloor}(r)$ as counting left-factors of Schröder paths where the horizontal steps can have any of r colors.

Rectifying the above matrix (let $n\mapsto n+k$), we get the array that begins

$$\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0\\
2 & 2 & 1 & 0 & 0 & 0 & 0\\
3 & 6 & 3 & 1 & 0 & 0 & 0\\
6 & 12 & 12 & 4 & 1 & 0 & 0\\
10 & 30 & 30 & 20 & 5 & 1 & 0\\
20 & 60 & 90 & 60 & 30 & 6 & 1
\end{array}\right).$$

We recognise in this the exponential Riordan array A107230

$$[I_0(2x)+I_1(2x),x]\,.$$

Since the Bessel function $I_n(x)$ is given by

$$I_n(x) = \sum_{i=0}^{\infty} \frac{1}{i!(n+i)!} \left(\frac{x}{2}\right)^{n+2i},$$

we can also describe the elements of the coefficient array as

$$\frac{(n-k)!}{k!} \left([n-2k \,\text{is even}] \frac{1}{\frac{n-2k}{2}!^2} + [n-2k-1 \,\text{is even}] \frac{1}{\frac{n-2k-1}{2}!\frac{n-2k+1}{2}!} \right).$$

Here we have used the Iverson bracket notation $[\mathcal{P}]$, where $[\mathcal{P}]$ is 1 if the proposition $\mathcal{P}$ is true, and 0 if not [7].

We can generalize $T_{n,\lfloor \frac{n}{2} \rfloor}(r)$ by defining

$$T_{n,\lfloor \frac{n}{2} \rfloor}(r,s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{k} \binom{n-k}{\lfloor \frac{n}{2} \rfloor} s^{n-k} r^k.$$

We have the following.

Proposition 21. *The sequence* $T_{n,\lfloor \frac{n}{2} \rfloor}(r,s)$ *is the complete central coefficient sequence of the Riordan array*

$$\left(\frac{1}{1-sx}, \frac{sx(1+rx)}{1-sx} \right).$$

.

Proof. We find that

$$T_{n,k}(r,s) = \sum_{j=0}^{k} \binom{k}{j} \binom{n-j}{k} r^j s^{n-j}.$$

Thus

$$T_{n,\lfloor \frac{n}{2} \rfloor}(r,s) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{j} \binom{n-j}{\lfloor \frac{n}{2} \rfloor} r^j s^{n-j}.$$

$\square$

As a consequence of this proposition, we can obtain the following result.

Proposition 22. *The generating function of* $T_{n,\lfloor \frac{n}{2} \rfloor}(r,s)$ *is given by*

$$g(x;r,s) = \frac{1-2sx-rsx^2-\sqrt{1-2sx^2(r+2s)+r^2s^2x^4}}{2sx(rsx^2+2sx-1)}.$$

We can express this as the continued fraction

$$\cfrac{1}{1-sx-\cfrac{s(r+s)x^2}{1-\cfrac{s^2x^2}{1-\cfrac{s(r+s)x^2}{1-\cfrac{s^2x^2}{1-\cdots}}}}}.$$

Corollary 23. *The generating function of $T_{n,\lfloor\frac{n}{2}\rfloor}(r,s)$ is given by*

$$\left(\frac{1}{1-2sx-rsx^2}, \frac{-sx}{1-2sx-rsx^2}\right)\cdot c(x) = \left(\frac{1}{1-rsx^2}, \frac{sx}{1-rsx^2}\right)\cdot \mathfrak{c}(x).$$

Corollary 24. *The Hankel transform of $T_{n,\lfloor\frac{n}{2}\rfloor}(r,s)$ is given by*

$$h_n = s^{2\lfloor\frac{n^2}{4}\rfloor}(s(r+s))^{\lfloor\frac{(n+1)^2}{4}\rfloor}.$$

Example 25. For $(r,s)=(0,2)$, we get the sequence A060899 or $2^n\binom{n}{\lfloor\frac{n}{2}\rfloor}$ that begins

$$1, 2, 8, 24, 96, 320, 1280, 4480, 17920, 64512, 258048, \ldots.$$

This counts walks of length n on a square lattice, starting at the origin, staying on points with $x+y\geq 0$.

For $(r,s)=(1,2)$, we get the sequence that begins

$$1, 2, 10, 32, 148, 536, 2440, 9344, 42256, 167072, 752800, \ldots.$$

This counts left-factors of Schröder paths where each type of step can have a choice of two colors.

For $(r,s)=(2,2)$, we get the sequence that begins

$$1, 2, 12, 40, 208, 800, 4032, 16512, 82176, 348672, 1723392, \ldots.$$

This counts left-factors of Schröder paths where each type of step can have a choice of two colors, and where additionally the horizontal step can be dashed or dotted. See Figure 3.

5 Conclusions

We have studied the sequences that are given by the complete central sequences of the Pascal-like triangles defined by the Riordan arrays $\left(\frac{1}{1-x}, \frac{x(1+rx)}{1-x}\right)$. We have given generating functions and closed from formulas. We have also presented Riordan arrays whose row sums are equal to these sequences. Throughout, we have found sequences that are linked to the statistics of Schröder and Motzkin paths. As the techniques used have been of an algebraic and analytical nature, there is as yet no clear combinatorial view of why this should be so. This therefore merits further study.

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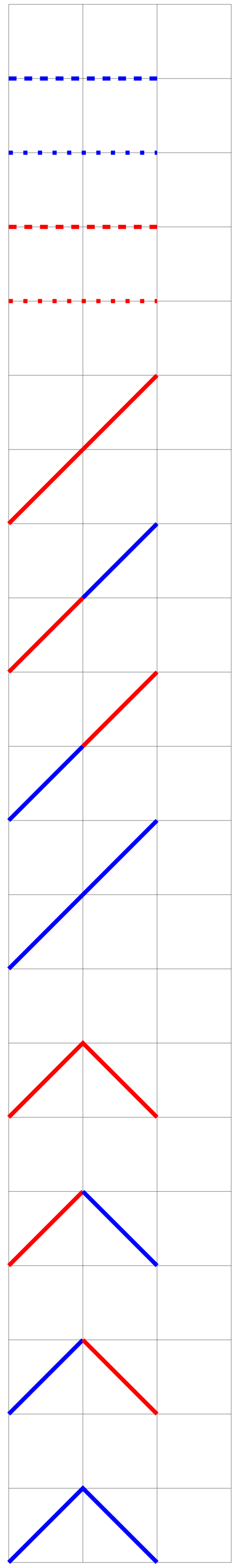

Figure 3: The 12 Schröder left-factors for $x = 2$ and $r = 2$, $s = 2$.

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