Journal of Integer Sequences, Vol. 21 (2018),
Article 18.9.4

About Some Relatives of the Taxicab Number

Viorel Niţică
Department of Mathematics
West Chester University of Pennsylvania
West Chester, PA 19383
USA
vnitica@wcupa.edu

Abstract

The taxicab number, 1729, is the smallest number that can be written as a sum of two cubes in two different ways. It also has the following property: if we add its digits, we obtain 19. The number obtained from 19 by reversing the order of its digits is 91. If we multiply 19 by 91 we again obtain 1729. In the paper we study various generalizations of this property.

1 Introduction

The *taxicab number,* 1729, became well known due to a discussion between Hardy and Ramanujan [9]. It is the smallest positive integer that can be written in two ways as a sum of two cubes: $1^3 + 12^3$ and $9^3 + 10^3$. The number 1729 also has a less well known property: if we add its digits we obtain 19; multiplying 19 by 91, the number obtained from 19 reversing the order of its digits, we again obtain 1729. It is not hard to show that the set of integers with this property is finite and equal to $\{1, 81, 1458, 1729\}$.

In a conversation that the author had with his colleague, Professor Shiv Gupta, Shiv asked if the second property can be generalized. One replaces the sum of the digits of an integer by the sum of the digits times an integer multiplier and then multiplies the product by the number obtained reversing the order of the digits in the product. The taxicab number becomes a particular example with multiplier 1. A computer search produced a large number of examples with larger multiplier. There are 23 integers less than 10000 having this property; see sequence A305131 in the OEIS [10]. For example, 2268 has multiplier 2. The sum of the digits is 18, and we have $18 \times 2 = 36$, and $36 \times 63 = 2268$.

One may replace the last product in the above procedure by a sum. A computer search showed that there are numbers that have the property for sums. There are 264 integers less than 10000 having the property; see sequence A305130 in the OEIS [10]. For example, 121212 has multiplier 6734. The sum of the digits is 9, and we have $9 \times 6732 = 60606$, and $60606 + 60606 = 121212$.

The paper is dedicated to the study of these properties. After the paper was submitted for publication we learned from the editor that our work may be related to the study of Niven (or Harshad) numbers. These are numbers divisible by the sum of their decimal digits. Niven numbers have been extensively studied as one can see for instance from Cai [3], Cooper and Kennedy [4], De Koninck and Doyon [6], Grundman [7]. One of the classes of integers we study, that of multiplicative Ramanujan-Hardy numbers, is a subclass of the class of Niven numbers. Of interest are also q-Niven numbers, which are numbers divisible by the sum of their base q digits. See, for example, Fredricksen, Ionaşcu, Luca, and Stănică [8]. Some other variants of Niven numbers can be found in Boscaro [1] and Bloem [2].

2 Statements of the main results

In what follows let $b \geq 2$ be an arbitrary numeration base.

Definition 1. If N is a positive integer written in base b, we call *reversal* of N and let N^R denote the integer obtained from N by writing its digits in reverse order.

We observe that addition and multiplication are independent of the numeration base. The operation of taking the reversal is not.

Let $s_b(N)$ denote the sum, done in base 10, of the base b digits of an integer N.

Definition 2. A positive integer N written in base b is called *b-additive Ramanujan-Hardy number,* or simply b-ARH number, if there exists a positive integer M, called an *additive multiplier*, such that

$$N = Ms_b(N) + (Ms_b(N))^R, \tag{1}$$

where $(Ms_b(N))^R$ is the reversal of base b-representation of $Ms_b(N)$.

Definition 3. A positive integer N written in base b is called *b-multiplicative Ramanujan-Hardy number,* or simply b-MRH number, if there exists a positive integer M, called a *multiplicative multiplier*, such that

$$N = Ms_b(N) \cdot (Ms_b(n))^R, \tag{2}$$

where $(Ms_b(N))^R$ is the reversal of base b-representation of $Ms_b(N)$.

To simplify the notation, let $s(N)$, ARH, MRH denote $s_{10}(N)$, 10-ARH, 10-MRH.

While b-MRH numbers are b-Niven numbers, b-Niven numbers are not necessarily b-MRH numbers.

Example 4. The number $[144]_7$ is a 7-Niven number but not a 7-MRH number.

We observe that the notions of b-ARH and b-MRH numbers are dependent on the base.

Example 5. The number $[12]_{10}$ is an ARH number, but $[12]_9$ is not a 9-ARH number. The number $[81]_{10}$ is an MRH number, but $[81]_9$ is not a 9-MRH number.

Once these notions are introduced, several natural questions arise.

Question 6. Do there exist infinitely many b-ARH numbers?

Question 7. Do there exist infinitely many b-MRH numbers?

Question 8. Do there exist infinitely many additive multipliers?

Question 9. Do there exist infinitely many multiplicative multipliers?

In what follows, if x is a string of digits, we let $(x)^{\wedge k}$ denote the base 10 integer obtained by repeating x k-times. We also let $[x]_b$ denote the value of the string x in base b.

The following example gives an explicit positive answer to Question 6 if $b = 10$.

Example 10. Consider the numbers

$$N_k = (12)^{\wedge 3^k}, \tag{3}$$

where k is a positive integer. All numbers N_k are ARH numbers and Niven numbers. In particular, there exist infinitly many Niven numbers with no digit equal to zero.

Example 11. If we allow zero digits, then infinitely many b-MRH numbers are given by $\{[1(0)^{\wedge k}]_b | k \in \mathbb{N}\}$. Last example has the unpleasant feature that the apparent multiplicative multiplier of each b-MRH numbers is the number itself and the search for other multipliers is dependent on the base. In order to avoid trivial considerations, we consider from now on only examples of b-ARH and b-MRH numbers that have many digits different from zero.

It follows from the proof of Example 10 that $Ms(N_k) = (Ms(N_k))^R$. The following theorem gives an example in which it is clear from the proof that $Ms_b(N_k) \neq (Ms_b(N_k))^R$ for an arbitrary even base b. One can read from the proof the explicit base b expansion of the multipliers. Counting the multipliers shows that the set of multipliers of a b-ARH number N can grow exponentially in terms of the number of digits of N.

Theorem 12. *Consider the numbers*

$$N_k = [(1)^{\wedge k}]_b, \tag{4}$$

where b is even, $k = [1(0)^{\wedge p}]_b, p \geq 1$, p an arbitrary natural number. All numbers N_k are b-ARH numbers and not b-Niven numbers.

Each N_k has a subset of additive multipliers of cardinality $2^{\frac{k-2p}{2}}$ consisting of all integers $[(1)^{\wedge p} I]_b$, where I is a sequence of 0 and 1 of length $k - 2p$ in which no two digits symmetric about the center of the sequence are identical.

Example 13. We show an example that illustrates the results in Theorem 12. Assume that $b = 2$, $k = 16 = [10000]_2$, and $p = 4$. Then $N_{16} = [(1)^{\wedge 16}]_2$ and $s_2(N_{16}) = 2^4 = [10000]_2$. The following $16 = 2^{\frac{16-2\cdot 4}{2}}$ numbers are additive multipliers of N_{16}:

$$\begin{aligned}
&[111100001111]_2,\ [111100010111]_2,\ [111100101011]_2,\ [111100111100]_2,\\
&[111101001101]_2,\ [111101010101]_2,\ [111101101001]_2,\ [111101110001]_2,\\
&[111110001110]_2,\ [111110010110]_2,\ [111110101010]_2,\ [111110110010]_2,\\
&[111111001100]_2,\ [111111010100]_2,\ [111111101000]_2,\ [111111110000]_2.
\end{aligned}$$

Remark 14. The numbers N_k may have other multipliers, besides those listed in Theorem 12. The growth of the set of multipliers can be larger than that shown in Theorem 12 and depends on the numeration base; see Theorem 15. Nevertheless, for $b = 2$ there are no other multipliers of N_k besides those listed in Theorem 12. We observe that the numbers N_k from Theorem 12 have an even number of digits and the numbers N_k from Theorem 15 have an odd number of digits.

Theorem 15. *Consider the numbers*

$$N_k = [(1)^{\wedge p}(10)^{\wedge k-2p}0(1)^{\wedge p}]_b, \tag{5}$$

where b is even and $k = [1(0)^{\wedge p}]_b, p \geq 1$, p arbitrary natural number. All numbers N_k are b-ARH numbers and not b-Niven numbers.

For each N_k the set of additive multipliers has cardinality $(b-1)^{\frac{k-2p}{2}}$ and consists of all integers $[(1)^{\wedge p}I0]_b$, where I is a concatenation of $k - 2p$ two digits strings of type $0\alpha, \alpha \neq 0$, in which any pair of nonzero digits symmetric about the center of $I0$ have their sum equal to b.

Example 16. We show an example that illustrates the results in Theorem 15. Assume that $b = 4$, $k = 4 = [10]_4$, and $p = 1$. Then $N_4 = [1101001]_4$ and $s_4(N_4) = 4 = [10]_4$. The following $3 = 3^{\frac{4-2\cdot 1}{2}}$ numbers are additive multipliers of N_4:

$$[102020]_4,\ [101030]_4,\ [103010]_4.$$

The following corollary of Theorem 35 gives a partial answer to Question 8.

Corollary 17. *If b is even, there exist infinitely many additive multipliers. Moreover, there exist infinitely many b-ARH numbers that have at least two additive multipliers.*

The numbers N_k from Theorems 12 and 15 are not b-MRH numbers.

Question 18. Do there exist infinitely many b-MRH numbers that have at least two multiplicative multipliers?

Corollary 19. *If b is even, there exist infinitely many b-ARH numbers that are not b-MRH.*

Motivated by the results in Theorems 12 and 15 and by the examples of ARH and MRH numbers shown in Sections 13 and 14, we introduce the following notions.

Definition 20. If N is a b-ARH number, let the *multiplicity* of N be the cardinality of the corresponding set of additive multipliers.

Definition 21. If N is a b-MRH number, let the *multiplicity* of N be the cardinality of the corresponding set of multiplicative multipliers.

Theorem 12 has the following corollary.

Corollary 22. *The multiplicity of b-ARH numbers is unbounded for any even base.*

Question 23. Is the multiplicity of b-MRH numbers bounded?

Remark 24. For Questions 7 and 9 we do not have an answer with b-MRH numbers having all digits different from zero. See Theorem 27 for infinitely many b-MRH numbers with half of the digits different from zero. No prime number can be an MRH number. Note that no integer with two prime factors in the prime factorization can be an MRH number. Such an MRH number has the multiplier equal to 1 and among the MRH numbers with multiplier 1 none has two factors in the prime factorization.

The following theorem shows infinitely many b-Niven numbers that are not b-MRH numbers.

Theorem 25. *Let $b \geq 2$ be a numeration base. For n not divisible by $b-1$ define*

$$R_n = \frac{b^n - 1}{b - 1} = [(1)^{\wedge n}]_b, n \geq 1.$$

Then $(b-1)nR_n$ is a b-Niven number that is not a b-MRH number.

For b-ARH numbers we have the following result.

Theorem 26. *There exist infinitely many integers that are not b-ARH numbers.*

The following Theorem gives a partial answer to Question 7.

Theorem 27. *Let b odd and $k \geq 2$. Then the numbers*

$$N_k = [(b-1)^{\wedge 2^{k-1}-1}(b-2)(0)^{\wedge 2^{k-1}-1}1]_b \tag{6}$$

are b-MRH numbers and $s_b(\sqrt{N_k}) = s_b(N_k)$.

Moreover, if $b \equiv 3 \pmod 4$ then $\sqrt{N_k}$ is itself a b-Niven number.

Example 28. We illustrate the result in Theorem 27.

- For $b = 3, k = 2$ we get $N_2 = [2101]_3$ which is a 3-MRH number. Then $\sqrt{[2101]_3} = [22]_3$, $s_3([2101]_3) = s_3([22]_3) = 4$ and $[22]_3$ is a 3-Niven number.

- For $b = 5, k = 2$ we get $N_2 = [4301]_5$ which is a 5-MRH number. Then $\sqrt{[4301]_5} = [44]_5$, $s_5([4301]_5) = s([44]_5) = 8$ and $[44]_5$ is a 5-Niven number.
- For $b = 17, k = 5$, N_5 is a 17-MRH number, but $\sqrt{N_5}$ is not a 17-Niven number.
- For $b = 7, k = 2$ we get $N_2 = [6501]_7$ which is a 7-MRH number. Then $\sqrt{[6501]_7} = [66]_7$, $s_7([6501]_7) = s_7([66]_7) = 12$ and $[66]_7$ is a 7-Niven number.

Third item shows that the congruence condition in Theorem 27 is necessary. Second item shows that $\sqrt{N_k}$ may be a b-Niven number even without this condition.

The following corollary of Theorem 27 gives a partial positive answer to Question 9.

Corollary 29. *If b is odd there exist infinitely many multiplicative multipliers.*

We show two unexpected corollaries of the proof of Theorem 27.

Corollary 30. *If b is odd there exist infinitely many b-MRH numbers that are perfect squares.*

Corollary 31. *If $b \equiv 3 \pmod 4$, there exist infinitely many b-MRH numbers N for which $\sqrt{N}$ is a b-Niven number and for which $s_b(N) = s_b(\sqrt{N})$.*

The following notions of high degree b-Niven numbers are motivated by Corollary 31, which provides plenty of examples.

Definition 32. An integer N is called *quadratic b-Niven number* if N and N^2 are b-Niven numbers. If in addition $s_b(N) = s_b(N^2)$ then N is called *strongly quadratic b-Niven number*.

The study of high degree b-Niven numbers is continued in Niţică [11]. We show that for each degree there exist infinitely many bases in which b-Niven numbers of that degree appear.

We show in Sections 13 that 6 is not an additive multiplier for base 10 and ARH numbers without zero digits, and that 9 is not an additive multiplier for base 10. We show in Section 14 that 3 is not a multiplicative multiplier for base 10. We do not know how to answer the following questions for any base.

Question 33. Do there exist infinitely many integers that are not additive multipliers?

Question 34. Do there exist infinitely many integers that are not multiplicative multipliers?

In what follows let $\lfloor x \rfloor$ denote the integer part, let $\ln x$ denote the natural logarithm and let $\log_b x$ denote base b logarithm of the positive real number x.

The following theorems give bounds for the number of digits in a b-ARH number in terms of the multiplier.

Theorem 35. *Let N be a b-ARH number with k digits and additive multiplier M. Then*

$$k \leq \begin{cases} M+2, & \text{if } b \geq 4; \\ M+3, & \text{if } b = 2 \text{ or } b = 3. \end{cases}$$

Corollary 36. *For fixed additive multiplier M and base b, the set of b-ARH numbers with multiplier M is finite.*

Theorem 37. *Let N be a b-ARH number with k digits and additive multiplier M. Under any of the following assumptions:*

- $b \geq 10$ *and* $M \geq b^6$;
- $3 \leq b \leq 9$ *and* $M \geq b^7$;
- $b = 2$ *and* $M \geq b^8$,

we have

$$k \leq 2\lfloor \log_b M \rfloor. \tag{7}$$

The following theorems give bounds for the number of digits in a b-MRH number in terms of the multiplier.

Theorem 38. *Let N be a b-MRH number with k digits and multiplicative multiplier M. Then*

$$k \leq \begin{cases} M+4, & \text{if } b \geq 6; \\ M+5, & \text{if } b = 5; \\ M+7, & \text{if } 2 \leq b \leq 4. \end{cases}$$

Theorem 38 shows that a MRH number with multiplicity 1 can have at most 5 digits. A computer search shows that the set of all such numbers is indeed $\{1, 81, 1458, 1729\}$.

Corollary 39. *For fixed multiplicative multiplier M and base b, the set of b-MRH numbers with multiplier M is finite.*

Theorem 40. *Let N be a b-MRH number with k digits and multiplicative multiplier M. Under any of the following assumptions:*

- $b \geq 9$ *and* $M \geq b^9$;
- $5 \leq b \leq 8$ *and* $M \geq b^{10}$;
- $b = 4$ *and* $M \geq b^{11}$;
- $b = 3$ *and* $M \geq b^{12}$;
- $b = 2$ *and* $M \geq b^{16}$;

we have

$$k \leq 3\lfloor \log_b M \rfloor. \tag{8}$$

We summarize the rest of the paper. Example 10 is proved in Section 3, Theorem 12 is proved in Section 4, Theorem 5 is proved in Section 5, Theorem 26 is proved in Section 7, Theorem 25 is proved in Section 6, Theorem 27 is proved in Section 8, Theorem 35 is proved in Section 9, Theorem 37 is proved in Section 10, Theorem 38 is proved in Section 11, and Theorem 40 is proved in Section 12. In Section 13 we show examples of ARH numbers and ask additional questions and in Section 14 we show examples of MRH numbers and ask additional questions. In Section 15 we describe an approach to Question 7 if $b = 10$.

3 Proof of Example 10

Proof. One obtains a formula for N_k by adding two geometric series.

$$\begin{aligned} N_k &= 10^{2\cdot 3^k-1} + 10^{2\cdot 3^k-3} + \ldots + 10 \\ &+2(10^{2\cdot 3^k-2} + 10^{2\cdot 3^k-4} + \ldots + 1) \\ &= 12 \cdot \frac{10^{2\cdot 3^k}-1}{99} = 4 \cdot \frac{10^{2\cdot 3^k}-1}{33}. \end{aligned} \tag{9}$$

Note that $s(N_k) = 3^{k+1}$. We show by induction that $s(N_k)$ divides N_k. The case $k = 0$ gives $s(N_0) = 3$ which divides $N_0 = 12$. Assume that for fixed k, $s(N_k)$ divides N_k.

$$\begin{aligned} N_{k+1} &= 4 \cdot \frac{10^{2\cdot 3^{k+1}}-1}{33} = 4 \cdot \frac{(10^{2\cdot 3^k})^3 - 1^3}{33} \\ = 4 \cdot \frac{10^{2\cdot 3^k}-1}{33} &\cdot (10^{4\cdot 3^k} + 10^{2\cdot 3^k} + 1) = N_k \cdot (10^{4\cdot 3^k} + 10^{2\cdot 3^k} + 1), \end{aligned} \tag{10}$$

which is clearly divisible by $s(N_{k+1}) = 3^{k+2}$ due to N_k divisible by $s(N_k) = 3^{k+1}$ and $10^{4\cdot 3^k} + 10^{2\cdot 3^k} + 1$ divisible by 3. Therefore $s(N_k)$ divides N_k and N_k is a Niven number.

Observe now that $N_k/2 = (N_k/2)^R$. It follows from (3) and the fact that N_k is divisible by $s(N_k) = 3^{k+1}$ that $N_k/2$ is divisible by $s(N_k)$. We conclude that N_k is an ARH number with additive multiplier $M = N_k/(2s(N_k))$. □

4 Proof of Theorem 12

Proof. Let $N_k = [(1)^{\wedge k}]_b$ where k is even and $k = [1(0)^{\wedge p}]_b, p \geq 1$, p arbitrary natural number. Then $s_b(N_k) = [1(0)^{\wedge p}]_b$. Let $M = [(1)^{\wedge p} I]_b$, where I is a string of 0 and 1 of length $k - 2p$ in which no two digits symmetric about the center of the sequence are identical. Note that $M^R = [(I)^R (1)^{\wedge p}]_b$. The following calculation shows that N_k is a b-ARH number. Note that $I + (I)^R = [(1)^{\wedge k-2p}]_b$.

$$
\begin{aligned}
&s_b(N_k)\cdot M+(s_b(N_k)\cdot M)^R\\
&=[1(0)^{\wedge p}]_b\cdot[(1)^{\wedge p}I]_b+([1(0)^{\wedge p}]_b\cdot[(1)^{\wedge p}I]_b)^R\\
&=[(1)^{\wedge p}I(0)^{\wedge p}]_b+([(1)^{\wedge p}I(0)^{\wedge p}]_b)^R\\
&=[(1)^{\wedge p}I(0)^{\wedge p}]_b+[(0)^{\wedge p}(I)^R(1)^{\wedge p}]_b=[(1)^{\wedge k}]_b=N_k.
\end{aligned}
$$

In order to count the multipliers, observe that the length of the string I is $k-2p$. If we know half of its digits we can find the other half using that no two digits symmetric about the center of the string are identical. The number of strings of 0 and 1 of length $\frac{k-2p}{2}$ is $2^{\frac{k-2p}{2}}$. Finally, observe that N_k is not divisible by $s_b(N_k)$, so N_k is not a b-Niven number.. □

5 Proof of Theorem 15

Proof. Let $N_k=[(1)^{\wedge p}(10)^{\wedge k-2p}0(1)^{\wedge p}]_b$ where b is even and $k=[1(0)^{\wedge p}]_b, p\geq 1$. Then $s_b(N_k)=[1(0)^{\wedge p}]_b$. Let $M=[(1)^{\wedge p}I0]_b$. Note that $M^R=[0(I)^R(1)^{\wedge p}]_b$. The following calculation shows that N_k is a b-ARH number. Note that $I0+0(I)^R=[(10)^{\wedge k-2p}0]_b$.

$$
\begin{aligned}
&s_b(N_k)\cdot M+(s_b(N_k)\cdot M)^R\\
&=[1(0)^{\wedge p}]_b\cdot[(1)^{\wedge p}I0]_b+([1(0)^{\wedge p}]_b\cdot[(1)^{\wedge p}I0]_b)^R\\
&=[(1)^{\wedge p}I0(0)^{\wedge p}]_b+([(1)^{\wedge p}I0(0)^{\wedge p}]_b)^R\\
&=[(1)^{\wedge p}I0(0)^{\wedge p}]_b+[(0)^{\wedge p}0(I)^R(1)^{\wedge p}]_b=[(1)^{\wedge p}(10)^{\wedge k-2p}0(1)^{\wedge p}]_b=N_k.
\end{aligned}
$$

In order to count the multipliers, observe that the number of nonzero digits in the string $I0$ is $k-2p$. If we know half of the nonzero digits we can find the other half using that no two digits symmetric about the center of the string $I0$ are identical. There are $\frac{k-2p}{2}$ positions to be filled and each one can be filled in $b-1$ ways. To show that there are no other multiplier it is enough to prove, using induction on length, that the string $[(10)^{\wedge k-2p}0]_b$ cannot be written as a sum of a string J and its reversal except if $J=I0$, where I is as above. Finally, observe that N_k is not divisible by $s_b(N_k)$, so N_k is not a b-Niven number. □

6 Proof of Theorem 25

Proof. McDaniels proved [5, Theorem 2] that if $b=10$ and $m\leq 9R_n$ then $s(9mR_n)=9n$. The proof is valid in any base b and follows readily upon writing m as:

$$
m=\sum_{i=0}^{k}a_ib^i, k<n. \tag{11}
$$

It gives that if $m\leq(b-1)R_n$ then $s_b((b-1)mR_n)=(b-1)n$. If $m=n$ we have $s_b((b-1)nR_n)=(b-1)n$, which shows that $(b-1)nR_n$ is a b-Niven number. By contradiction, assume that $(b-1)nR_n$ is a b-MRH number with multiplier M. It follows that

$$(b-1)nM((b-1)nM)^R = (b-1)nR_n. \tag{12}$$

We recall that a base b number is divisible by $b-1$ if the sum of its base b digits is divisible by $b-1$. The divisibility test and $b-1 \not| n$ imply that $b-1 \not| R_n$, but $b-1 | ((b-1)nM)^R$. As $b-1 \not| n$, there are at least two factors of $b-1$ in the factorization of the left hand side of (12) and only one factor of $b-1$ in the right hand side of (12). This gives a contradiction. $\square$

7 Proof of Theorem 26

Proof. A b-ARH number is a sum of an integer and its reversal. In order to prove the theorem it is enough to show that there exist infinitely many integers that are not a sum of an integer and its reversal. There are $b^k - b^{k-1} = b^{k-1}(b-1)$ base b k-digit numbers. Those of type $N+N^R$, either have $N = [a_k a_{k-1} \cdots a_2 a_1]_b$ with $a_k + a_1 \leq b-1$, or have N with $k-1$ digits. There are $\frac{b(b-1)}{2} \cdot b^{k-2}$ k-digit numbers with $a_k + a_1 \leq b-1$ and there are $b^{k-1} - b^{k-2}$ $(k-1)$-digit numbers. Overall, we have

$$\frac{b(b-1)}{2} \cdot b^{k-2} + (b^{k-1} - b^{k-2}) = b^{k-1}\left(\frac{b+1}{2}\right) - b^{k-2}$$

k-digit numbers of type $N + N^R$. Hence there are

$$b^k - b^{k-1} - \left(b^{k-1}\left(\frac{b+1}{2}\right) - b^{k-2}\right) = b^{k-1}\left(\frac{b-3}{2}\right) + b^{k-2} \tag{13}$$

k-digit numbers that are not of type $N+N^R$. The right hand side of equation (13) has limit ∞ as $k \to \infty$ for $b \geq 3$ and this ends the proof if $b \geq 3$. If $b = 2$, consider the numbers $[(1)^{\wedge k}]_2$. These are not ARH-numbers if k is odd. $\square$

8 Proof of Theorem 27

Proof. As $\gcd(b,2) = 1$ Euler's Theorem implies that 2^k divides $b^{\phi(2^k)} - 1$. Clearly $b-1$ also divides $b^{\phi(2^k)} - 1$. Assume that $\gcd(2^k, b-1) = 2^\ell$. Then $2^{k-\ell}(b-1)$ divides $b^{\phi(2^k)} - 1 = b^{2^{k-1}} - 1$. Consider the product

$$(b^{2^{k-1}} - 1)^2 = b^{2 \cdot 2^{k-1}} - 2b^{2^{k-1}} + 1.$$

The product is divisible by $2^{k-1}(b-1)$, written in base b equals N_k, and $s_b(N_k) = 2^{k-1}(b-1)$. We conclude that N_k is a b-MRH number.

To finish the proof of the theorem observe that if $b \equiv 3 \pmod 4$ then $\gcd(2^k, b-1) = 2$. Therefore $2^{k-1}(b-1)$ divides $b^{2^k} - 1 = [(b-1)^{2^{k-1}}]_b$. Finally

$$s_b(\sqrt{N_k}) = s_b([(b-1)^{2^{k-1}}]_b) = 2^{k-1}(b-1) | b^{2^k} - 1 = \sqrt{N_k}.$$

$\square$

9 Proof of Theorem 35

Proof. As N has k digits, we have

$$N \geq b^{k-1}. \tag{14}$$

The largest possible value for $s_b(N)$ is $(b-1)k$. We observe that reversing the order of the digits in an integer increases its value by at most b times. One has

$$Ms_b(N) + (Ms_b(N))^R \leq (b^2-1)kM. \tag{15}$$

Combining equations (1), (14), (15), we have

$$b^{k-1} \leq (b^2-1)kM. \tag{16}$$

We prove by induction on the variable k that

$$b^{k-1} > (b^2-1)kM, \text{ for } k \geq M+3, M \geq 1, b \geq 4, \tag{17}$$

which combined with (16) gives a contradiction and ends the proof of Theorem 35 for $b \geq 4$.

In the first step $k = M+3$. The statement in (17) becomes

$$b^{M+2} > (b^2-1)(M^2+3M), \text{ for } M \geq 1, b \geq 4. \tag{18}$$

We prove (18) by induction on the variable M. In the initial step $M = 1$ and we have

$$b^3 > 4(b^2-1) \Leftrightarrow b^2(b-4)+4 > 0,$$

which is clearly true for $b \geq 4$.

Now assume that (18) is true for M and prove it for $M+1$. Using the induction hypothesis we have

$$b^{M+3} = b \cdot b^{M+2} > b \cdot (b^2-1)(M^2+3M). \tag{19}$$

In order to finish the proof by induction, we still need to check that

$$b \cdot (b^2-1)(M^2+3M) \geq (b^2-1)\left((M+1)^2+3(M+1)\right). \tag{20}$$

After simplifications, (20) becomes

$$(b-1)M^2 + (3b-5)M - 4 \geq 0. \tag{21}$$

As the left hand side of (21) is larger than M^2+4M-4, which is positive if $M \geq 2$, we conclude that (21) is true for all $M \geq 1$ and finish the proof of (18).

We continue with the general step in the proof of (17). By induction:

$$b^k = b \cdot b^{k-1} > b(b^2-1)kM. \tag{22}$$

We still need to check that

$$b(b^2-1)kM \geq (b^2-1)k(M+1), \tag{23}$$

which is obvious and finishes the proof of (18) and that of Theorem 35 for base $b \geq 4$.

Now assume $b=3$. Equation (16) is still valid.

We prove by induction on the variable k that

$$b^{k-1} > (b^2-1)kM, \text{ for } k \geq M+4, M \geq 1. \tag{24}$$

Equations (16) and (24) give a contradiction that finishes the proof of the theorem.

If $k = M+4$ we have

$$b^{M+3} > (b^2-1)(M^2+4M), \text{ for } M \geq 1, \tag{25}$$

which we prove by induction on M.

The case $M=1$ is true. We assume (25) true for M and prove it for $M+1$. By induction we have

$$b^{M+4} = b \cdot b^{M+3} > b(b^2-1)(M^2+4M).$$

To finish the proof of (25) we still need to check that

$$b(b^2-1)(M^2+4M) \geq (b^2-1)\left((M+1)^2+4(M+1)\right), \tag{26}$$

which simplifies to $(b-1)M^2+(4b-6)M-5 \geq 0$ and is true for $M \geq 1, b=3$.

The rest of the proof of (24) follows from (22) and (23).

Assume $b=2$. Equation (16) is still valid.

We prove by induction on the variable k that

$$b^{k-1} > (b^2-1)kM, \text{ for } k \geq M+4, M \geq 3. \tag{27}$$

Equations (16) and (27) give a contradiction that ends the proof of the theorem for $b = 2, M \geq 3$.

If $k = M+4$ we have

$$2^{M+3} > 3(M^2+4M), \text{ for } M \geq 3, \tag{28}$$

which we prove by induction on M.

The case $M=3$ is true. Assume now that (28) is true for M and prove it for $M+1$.

$$2^{M+4} = 2 \cdot 2^{M+3} > 6(M^2+4M).$$

To finish the proof we still need to check that

$$6(M^2+4M) \geq 3\big((M+1)^2+4M\big).$$

The equation simplifies to $M^2 + 2M - 1 \geq 0$ and it is true for $M \geq 3$.

To finish the proof of the theorem if $b = 2$, it remains to discuss the cases $M = 1, M = 2$.

Let $M = 1$. If $k \leq 4$ the theorem is trivially true, so assume $k \geq 5$. Let N be a 2-ARH number with k digits and $M = 1$. Then $s_2(N) \leq k$ and $N \geq 2^{k-1}$. This implies

$$2^{k-1} \leq 3k. \tag{29}$$

One shows that $3k < 2^{k-1}$ for $k \geq 5$ and gets a contradiction.

Let $M = 2$. If $k \leq 4$ the theorem is trivially true, so assume $k \geq 5$. Let N be a 2-ARH number with k digits and $M = 2$. Then $s_2(N) \leq k$ and $N \geq 2^{k-1}$. This implies

$$2^{k-1} \leq 6k. \tag{30}$$

One shows that $6k < 2^{k-1}$ for $k \geq 5$ and gets a contradiction. □

10 Proof of Theorem 37

Proof. It follows from formula (16) in the proof of Theorem 35 that

$$b^{k-1} \leq (b^2 - 1)kM. \tag{31}$$

We show by induction on the variable k that

$$b^{k-1} > (b^2 - 1)kM \text{ if } M \geq b^6, k \geq 2\lfloor \log_b M \rfloor + 1, b \geq 10 \tag{32}$$

which together with (31) ends the proof of Theorem 37 for base $b \geq 10$.

First we show by induction on the variable M that

$$M > 2b^2(b^2 - 1)\log_b M + b^2(b^2 - 1) \text{ if } M \geq b^5, b \geq 10. \tag{33}$$

If $M = b^6$ (33) is equivalent to

$$b^3 + 13b(1 - b^2) > 0, \tag{34}$$

which is true if $b \geq 10$.

Now assume that (33) is true for a fixed M. One has

$$M + 1 > 2b^2(b^2 - 1)\log_b M + b^(b^2 - 1) + 1.$$

To finish the proof of (33) we still need to check that

$$2b^2(b^2 - 1)\log_b M + b^2(b^2 - 1) + 1 \geq 2b^2(b^2 - 1)\log_b(M + 1) + b^2(b^2 - 1),$$

which after simplifications becomes

$$1 \geq 2b^2(b^2 - 1)\left(\log_b(M + 1) - \log_b M\right),$$

which is true due to $M \geq b^5$ and the mean value theorem.

We start the proof of (32). In the first step $k = 2\lfloor \log_b M \rfloor + 1$ and (32) becomes

$$b^{2\lfloor \log_b M \rfloor} > (b^2 - 1)M(2\lfloor \log_b M \rfloor + 1). \tag{35}$$

Due to $\log_b M - 1 \leq \lfloor \log_b M \rfloor \leq \log_b M$ we have

$$\begin{gathered} b^{2\lfloor \log_b M \rfloor} \geq b^{2(\log_b M - 1)} \\ (b^2 - 1)M(2\lfloor \log_b M \rfloor + 1) \leq (b^2 - 1)M(2\log_b M + 1). \end{gathered} \tag{36}$$

In order to prove (35) it is enough to show that

$$b^{2(\log_b M - 1)} > (b^2 - 1)M(2\log_b M + 1),$$

which is equivalent to (33). This ends the proof of the first induction step.

Now assume that (32) is true for fixed k and show that it is true for $k+1$. Due to the induction hypothesis we have

$$b^k \geq b \cdot (b^2 - 1)kM.$$

To finish the proof of (32) we still need to check that

$$b \cdot (b^2 - 1)kM > b(b^2 - 1)(k + 1)M,$$

which is obviously true.

The proofs of the other cases are similar. The only significant difference appears in (34). If $3 \leq b \leq 9$, (34) becomes $b^4 - 15(b^2 - 1) \geq 0$, which is true. If $b = 2$ (34) becomes $b^6 > 17(b^2 - 1)$, which is true. □

11 Proof of Theorem 38

Proof. As N has k digits we have

$$N \geq b^{k-1}. \tag{37}$$

The largest possible value for $s_b(N)$ is $(b-1)k$. Reversing the order of the digits in an integer increases its value by at most b times. One has

$$Ms_b(N) \cdot (Ms_b(N))^R \leq b(b-1)^2k^2M^2. \tag{38}$$

Combining equations (2), (37), (38) we have

$$b^{k-1} \leq b(b-1)^2k^2M^2. \tag{39}$$

Now we prove by induction on the variable k that

$$b^{k-1} > b(b-1)^2k^2M^2, \text{ for } k \geq M + 5, M \geq 1, b \geq 6 \tag{40}$$

which combined with (39) ends the proof of Theorem 38 for $b \geq 6$.

In the initial induction step $k = M + 5$. The statement in (40) becomes

$$b^{M+4} > b(b-1)^2(M+5)^2M^2, \text{ for } M \geq 1, b \geq 6. \tag{41}$$

We prove (41) by induction on the variable M. If $M = 1$ (41) becomes $b^5 > 36b(b-1)^2$, which is true if $b \geq 6$.

Now we assume that (41) is true for M and prove it for $M + 1$. From the induction hypothesis we have

$$b^{M+5} = b \cdot b^{M+4} > b \cdot b(b-1)^2(M+5)^2M^2. \tag{42}$$

In order to finish the proof, we still need to check that

$$b \cdot b(b-1)^2(M+5)^2M^2 \geq b(b-1)^2(M+6)^2(M+1)^2 \tag{43}$$

for $M \geq 1$.

After simplifications, (43) becomes

$$(b-1)M^4 + (10b-14)M^3 + (25b-61)M^2 - 84M - 36 \geq 0, \tag{44}$$

which is true for $M \geq 1$ and $b \geq 6$.

This finishes the proof of (41).

We continue with the general step in the proof of (40). By induction

$$b^k = b \cdot b^{k-1} > b \cdot b(b-1)^2k^2M^2.$$

To finish the proof of (40) we still need to check that

$$b \cdot b(b-1)^2k^2M^2 \geq b(b-1)^2(k+1)^2M^2,$$

which after simplifications becomes $(b-1)k^2 - 2k - 1 \geq 0$. This is true if $k \geq 1$ and $b \geq 6$.

This finishes the proof of Theorem 38 for $b \geq 6$.

The proof of the case $b = 5$ is similar. The only significant changes appear in (41) and in (44). Equation (41) simplifies to $b^5 > 49(b-1)^2$, which is true for $b = 5$, and (44) becomes

$$(b-1)M^4 + (12b-16)M^3 + (36b-78)M^2 - 112M - 49 \geq 0,$$

which is true if $b = 5$.

If $2 \leq b \leq 4$, using $M \geq 1$, the statement in the theorem is true if $k \leq 8$. We can assume $k \geq 9$.

If $b = 4$ (39) is still true and gives $4^{k-2} \leq 9k^2M^2$. It is easy to show by induction that for $k \geq 9$ we have $4^{k-2} > 9k^2(k-8)^2$. If $k \geq M + 8$ this implies $4^{k-2} > 9k^2M^2$, a contradiction.

If $b = 3$, (39) is still true and gives $3^{k-2} \leq 4k^2M^2$. It is easy to show by induction that for $k \geq 9$ we have $3^{k-2} > 4k^2(k-8)^2$. If $k \geq M + 8$ this implies $3^{k-2} > 8k^2M^2$, a contradiction.

If $b = 2$, (39) is still true and gives $2^{k-2} \leq k^2M^2$. It is easy to show by induction that for $k \geq 9$ we have $2^{k-2} > k^2(k-8)^2$. If $k \geq M + 8$ this implies $2^{k-2} > 8k^2M^2$, a contradiction. $\square$

12 Proof of Theorem 40

Proof. It follows from formula (39) in the proof of Theorem 38 that

$$b^{k-1} \le b(b-1)^2 k^2 M^2. \tag{45}$$

We prove by induction on the variable k that

$$b^{k-1} > b(b-1)^2 k^2 M^2 \text{ for } M \ge b^9, k \ge 3\lfloor \log_b M \rfloor + 1, b \ge 9, \tag{46}$$

which combined with (45) finishes the proof of Theorem 40.

We start showing by induction on M that

$$M > (b-1)^2 b^4 (3\log_b M + 1)^2 \text{for } M \ge b^9, b \ge 9. \tag{47}$$

If $M = b^9$ (47) becomes, after cancellations,

$$b^5 > 28^2 (b-1)^2 \tag{48}$$

which is true for $b \ge 9$.

We assume now that (47) is true for fixed M. We show that it is true for $M+1$. From the induction hypothesis we have

$$M + 1 > (b-1)^2 b^4 (3\log_b M + 1)^2 + 1.$$

To finish the proof of (47), one still needs to check that

$$(b-1)^2 b^4 (3\log_b M + 1)^2 + 1 \ge (b-1)^2 b^4 \left(3\log_b (M+1) + 1\right)^2,$$

which after algebraic manipulations becomes

$$1 \ge b^4 (b-1)^2 (3\log_b (M+1) - 3\log_b M)(3\log_b (M+1)M + 2). \tag{49}$$

Using the mean value theorem, (49) will follow if we show that

$$1 \ge 3b^4 (b-1)^2 \cdot \frac{1}{M} \left(3\log_b (M^2 + M) + 2\right). \tag{50}$$

Consider the function $g(M) = \frac{1}{M}[3\log_b (M^2 + M) + 2]$, with derivative:

$$g'(M) = \frac{\frac{1}{\ln b} \cdot \frac{3}{M^2+M} \cdot (2M+1)M - (3\log_b (M^2+M) + 2)}{M^2}.$$

For $M \ge b^9$ the first term in the numerator of $g'(M)$ is ≤ 6 and the absolute value of the second term is ≥ 30. We conclude that $g'(M)$ is negative and $g(M)$ is decreasing on the interval $[b^9, +\infty)$. The value of

$$3 \cdot b^4 (b-1)^2 \cdot g(M) \tag{51}$$

for $M = b^9$ is larger than $\frac{168(b-1)^2}{b^5}$, which shows that (50) is true if $b \geq 9$. Consequently (49) and (47) are true.

We start the proof of (46). In the first step $k = 3\lfloor \log_b M \rfloor + 1$. Equation (46) becomes

$$b^{3\lfloor \log_b M \rfloor} > (b-1)^2 b(3\lfloor \log_b M \rfloor + 1)^2 M^2. \tag{52}$$

Due to $\log_b M - 1 \leq \lfloor \log_b M \rfloor \leq \log_b M$, (52) follows if we prove that

$$b^{3(\log_b M - 1)} \geq (b-1)^2 b M^2 (3 \log_b M + 1)^2 \text{ for } M \geq b^9. \tag{53}$$

After algebraic manipulations (53) is exactly (47), so it is true. Now we show the general induction step for (46). Assume (46) valid for fixed M. Then we have

$$b^k \geq b \cdot b^{k-1} \geq b \cdot (b-1)^2 b k^2 M^2.$$

To finish we still need to check that

$$b \cdot (b-1)^2 b k^2 M^2 \geq (b-1)^2 b (k+1)^2 M^2,$$

which after simplifications becomes $bk^2 \geq k^2 + 2k + 1$ which is true for $k \geq 1$. This ends the proof of the case $b \geq 9$.

The proofs of the other cases are similar and the only significant changes appear in checking the equation (48) and checking that expression (51) is less than 1. Due to our assumptions the equation remains valid and the expression is less than 1. $\square$

13 Examples of ARH numbers

M	N
1	18, 99
2	12, 33, 66, 99
3	99
4	99
5	11,22,33,44,55,66,77,88,99
6	
7	747

Table 1: ARH numbers with multipliers 1, 2, 3, 4, 5, 7 and without zero digits.

We list in Table 1 small additive multipliers M and the corresponding ARH numbers N without zero digits. Theorem 35 shows that an ARH number with multiplier 6 has at most 8 digits. A computer search through all integers with at most 8 digits and all digits different from zero, shows that 6 is not an additive multiplier for numbers with all digits different from zero. If we allow for zero digits one finds that 909 is an ARH number with multiplier

6. A computer search through all integers with at most 11 digits shows that 9 is not an additive multiplier. These observations motivate Question 33.

We observe that certain ARH numbers, for example 99, have several additive multipliers, respectively $1, 2, 3, 4, 5$. We also observe that certain multipliers, for example 5, have associated several ARH numbers, respectively $11, 22, 33, 44, 55, 66, 77, 88, 99$. The last observation motivates the following definition and questions.

Definition 41. If M is an additive multiplier in a base b, let the *multiplicity* of M be the cardinality of the corresponding set of b-ARH numbers.

Question 42. If we fix the multiplicity and the base, is the set of additive multipliers infinite?

Question 43. If we fix the base, is the multiplicity of additive multipliers bounded?

14 Examples of MRH numbers

We list in Table 2 small multiplicative multipliers M and the corresponding MRH numbers N. Theorem 38 shows that a MRH number with multiplier 3 has at most 7 digits. A computer search through all integers with at most 7 digits shows that 3 is not a multiplicative multiplier. This motivates Question 34.

M	N
1	1, 18, 1458, 1729
2	2268, 736
3	
4	1944, 7744
5	71685

Table 2: MRH numbers with multipliers $1, 2, 3, 4, 5$ and without zero digits.

One can also arrange the data as in Table 3, where, for small values of k, we list multiplicative multipliers M and the corresponding MRH numbers N with k digits.

We observe from Table 3 that certain MRH numbers, for example, 332424, 132192, and 3252312, have several multipliers (respectively $\{27, 38\}$, $\{12, 34\}$, $\{72, 82\}$). We also observe from Table 2 that certain multipliers, for example 4, have associated several MRH numbers, respectively $1944, 7744$. The last observation motivates the following definition and questions.

Definition 44. If M is a multiplicative multiplier in base b, let the *multiplicity* of M be the cardinality of the corresponding set of b-MRH numbers.

Question 45. If we fix the multiplicity and the base, is the set of multiplicative multipliers infinite?

Question 46. If we fix the base, is the multiplicity of multiplicative multipliers bounded?

k	M	N	k	M	N
1	1	1	7	22	9379678
2	1	81		28	6527836
3	2	736		29	9253987
4	1	1458, 1729		32	2892672
	2	2268		33	8673885
	4	1944, 7744		34	7526716
5	5	71685		38	3773932, 6362226
	7	23632		39	5673564
	8	94528		41	2187391
	9	42282		49	4274613, 8239644
	14	51142		63	1821771
	23	78246		72	7651584
6	12	132192		73	2895472
	14	188356, 247324		82	7651584
	19	161595		84	3252312
	21	433755, 496692	8	37	13184839
	22	234256		46	11361448
	23	685584		48	14292288
	26	258778		53	15437628
	27	332424		61	15178752
	29	679354		66	15995232
	31	122512		89	7331464
	33	176418		66	15995232
	34	132192, 751842		68	11715516
	36	271188		71	16746912
	37	215821		74	12419568, 15478432
	38	332424		75	19348875
	39	145314		76	17433792
	44	235224		77	19552995
				78	12661272, 22694256
				79	11437225
				86	21371688
				89	12918439

Table 3: MRH numbers with $1, 2, 3, 4, 5, 6, 7, 8$ digits and no zero digits.

15 Conclusion

In this paper for any numeration base b we introduce two new classes of integers, b-ARH numbers and b-MRH numbers. They have properties that generalize a property of the taxicab number 1729. The second class is a subclass of the class of b-Niven numbers. We ask several natural questions about these classes and partially answer some of them. In particular, we show that the class of b-ARH numbers is infinite if b is even and that the class of b-MRH numbers is infinite if b is odd.

Among the questions left open, the most intriguing is if the set of MRH numbers with all digits different from zero is infinite. One way to attack it is to find infinitely many integers N such that $N = N^R$, N is divisible by $s(N^2)$, and N^2 has no digit equal to zero. Then the squares are infinitely many MRH numbers with nonzero digits. Our data shows some examples of such integers.

- $N^2 = 188356 = 434^2, s(N^2) = 31|434$,
- $N^2 = 234256 = 484^2, s(N^2) = 22|484$,
- $N^2 = 685584 = 828^2, s(N^2) = 36|828$.

16 Acknowledgments

The author would like to thank the editor and the referee for valuable comments that helped him write a better paper. He also thanks the OEIS Wiki community for help with posting the sequences A305130 and A305131 on OEIS.

References

[1] S. Boscaro, Nivenmorphic integers, *J. Rec. Math.*, **28** (1996-1997), 201–205.

[2] E. Bloem, Harshad numbers, *J. Rec. Math.*, **34** (2005), 128.

[3] T. Cai, On 2-Niven numbers and 3-Niven numbers, *Fibonacci Quart.*, **34** (1996), 118–120.

[4] C. N. Cooper and R. E. Kennedy, On consecutive Niven numbers, *Fibonacci Quart.*, **21** (1993), 146–151.

[5] W. L. McDaniel, The existence of infinitely many k-Smith numbers, *Fibonacci Quart.*, **25** (1987) 76–80.

[6] J. M. De Koninck and N. Doyon, Large and small gaps between consecutive Niven numbers, *J. Integer Sequences*, **6** (2003), Article 03.2.5.

[7] H. G. Grundman, Sequences of consecutive Niven numbers, *Fibonacci Quart.*, **32** (1994), 174–175.

[8] H. Fredricksen, E. J. Ionaşcu, F. Luca, and P. Stănică, Remarks on a sequence of minimal Niven numbers, in *Sequences, Subsequences, and Consequences*, Lecture Notes in Comp. Sci., Vol. 4893, Springer, 2007, pp. 162–168.

[9] G. H. Hardy, *Ramanujan: Three Lectures on Subjects Suggested by his Life and Work*, Chelsea, 1999.

[10] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available at `http://oeis.org`.

[11] V. Niţică, High degree b-Niven numbers, preprint, 2018. Available at `http://arxiv.org/abs/1807.02573`.

2010 *Mathematics Subject Classification*: Primary 11B83; Secondary 11B99.
Keywords: base, b-Niven number, reversal, additive b-Ramanujan-Hardy number, multiplicative b-Ramanujan-Hardy number, high degree b-Niven number.

(Concerned with sequences A005349, A067030, A305130, and A305131.)

Received May 6 2018; revised versions received May 7 2018; May 13 2018; May 18 2018; May 19 2018; May 20 2018; July 10 2018; July 13 2018; October 15 2018. Published in *Journal of Integer Sequences*, December 9 2018.

Return to Journal of Integer Sequences home page.