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A Method For Examining Divisibility Properties Of Some Binomial Sums

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Abstract

We introduce a notion that we call a "D sum" and use it to examine the divisibility properties of some binomial sums. The method of D sums consists of two theorems and their proofs. We present two applications of our method to non-negative sums with absolute values. The third application is for a known alternating binomial sum. In particular, our method of D sums can be used to prove Dixon's formula.

1 Introduction

Let m, n, and k be non-negative integers such that $m \geq 2$. We consider the sum

$$S(n,m) = \sum_{k=0}^{n} \binom{n}{k}^m F(n,k), \tag{1}$$

where $F(n,k)$ is an integer-valued function that depends only on n and k.

The aim is to examine some divisibility properties of sums of the form $S(n,m)$. To do this, we introduce the notion of "D sums".[1]

[1]So-named in honor of professor Duško Jojić.

Definition 1. Let n, j, and t be non-negative integers such that $j \le \lfloor \frac{n}{2} \rfloor$. Then the D sums for $S(n,m)$ are as follows:

$$D_S(n,j,t) = \sum_{l=0}^{n-2j} \binom{n-j}{l}\binom{n-j}{j+l}\binom{n}{j+l}^t F(n,j+l). \tag{2}$$

Obviously, for $m \ge 2$, the equation

$$S(n,m) = D_S(n,0,m-2) \tag{3}$$

holds.

Hence, we can see the sum $D_S(n,j,m-2)$ as a generalization of the sum $S(n,m)$.

We search for new theorems and facts about $D_S(n,j,t)$ sums, which may be useful for studying $S(n,m)$ sums.

Let n, j, and t be as in Definition 1. The method of D sums consists of the following two theorems:

Theorem 2.

$$D_S(n,j,t+1) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n}{j+u}\binom{n-j}{u} D_S(n,j+u,t).$$

Theorem 3.

$$D_S(n,j,0) = \sum_{u=0}^{\lfloor \frac{n-2j}{2} \rfloor} \binom{n-j}{j+u}\binom{n-2j-u}{u} \sum_{v=0}^{n-2j-2u} \binom{n-2j-2u}{v} F(n,j+u+v).$$

Together, Theorems 2 and 3 give a recursive definition of a D_S sum.

2 Background

In 1891, Dixon [11] found the following identity [16, Eq. (6.6), p. 51]:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n \binom{2n}{n}\binom{3n}{2n}, \tag{4}$$

where n is a non-negative integer. The absolute value of the right-hand side is sequence A006480 in the *On-Line Encyclopedia of Integer Sequences*.

Moreover, Dixon [12] established the following generalization of Eq. (4):

$$\sum_{k=-a}^{a} (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a! \cdot b! \cdot c!}, \tag{5}$$

where a, b, and c are non-negative integers. Both identities are known in the literature as Dixon's identities or Dixon's formulas. For $a = b = c = n$, Eq. (5) becomes Eq. (4).

There are many proofs of Eq. (5): for example, see [13, 15, 17, 21]. However, there are not so many direct proofs of Eq. (4). Our goal is to give an elementary proof of Eq. (4) without using Eq. (5).

In order to find the desired proof, we discovered the method of D sums. Unfortunately, the method of D sums does not work on Eq. (5).

In 1998, Calkin [9, Thm. 1] proved that the alternating binomial sum $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. Calkin used arithmetical techniques in his proof.

There are several generalizations of Calkin's result. In 2007, Guo, Jouhet, and Zeng proved, among other things, two generalizations of Calkin's result [18, Thm. 1.2, Thm. 1.3]. In the same paper they proposed several conjectures including Conjecture 5.3 which is a refinement of Calkin's result. In 2010, Cao and Pan [10, Thm. 1.1] proved, among other results, Conjecture 5.3.

The rest of the paper is structured as follows. In Sections 3 and 4, we prove our main Theorems 2 and 3, respectively. In Section 5, we give a brief explanation of our method of D sums.

In Section 6, we consider two non-negative sums with absolute values: $S_1(2n, m) = \sum_{k=0}^{2n}\binom{2n}{k}^m|n-k|$ and $S_2(2n+1, m) = \sum_{k=0}^{2n+1}\binom{2n+1}{k}^m\left|\frac{2n+1}{2} - k\right|$. We assert, among other, that the first sum $S_1(2n, m)$ is divisible by $n\binom{2n}{n}$ for all positive integers n and all positive integers m . Similarly, we assert, among other, that the second sum $S_2(2n+1, m)$ is divisible by $(2n+1)\binom{2n}{n}$ for all non-negative integers n and all positive integers m.

In Section 7, we begin with motivation for studying $S_1(2n, m)$ and $S_2(2n+1, m)$ sums. Then we apply the method of D sums to prove our assertions for the sum $S_1(2n, m)$. Proofs of our assertions for the second sum $S_2(2n+1, m)$ are omitted because they are similar to the proofs for the first sum $S_1(2n, m)$.

In Section 8, we consider the alternating binomial sum $S_3(2n, m) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m$.

We give a sketch of the proof of Calkin's result [9, Thm. 1] by using our method of D sums. Also, we give a sketch of the proof of Eq. (4).

3 Proof of Theorem 2

We use three well-known binomial identities.

The first one is the Chu-Vandermonde convolution formula:

$$\sum_{k=0}^{c}\binom{a}{k}\binom{b}{c-k} = \binom{a+b}{c}, \tag{6}$$

where a, b, and c are non-negative integers.

Let a, b, and c be non-negative integers such that $a \geq b \geq c$. The second identity is

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}. \tag{7}$$

The third identity is symmetry of binomial coefficients.

Proof. Let n, j, and t be as in Definition 1. By Definition 1, we know that

$$D_S(n,j,t+1) = \sum_{l=0}^{n-2j} \binom{n-j}{l}\binom{n-j}{j+l}\binom{n}{j+l}^{t+1} F(n,j+l). \tag{8}$$

According to Eq. (6), we have

$$\binom{n-j}{l} = \sum_{u=0}^{l} \binom{n-2j-l}{u}\binom{j+l}{l-u}. \tag{9}$$

Obviously, the following inequalities

$$u \leq l, \tag{10}$$
$$u \leq n-2j-l. \tag{11}$$

hold.

From Inequalities (10) and (11), we obtain the following inequalities:

$$0 \leq u \leq \left\lfloor \frac{n-2j}{2} \right\rfloor, \tag{12}$$
$$u \leq l \leq n-2j-u. \tag{13}$$

We use Eq. (9), the symmetry $\binom{n-j}{j+l} = \binom{n-j}{n-2j-l}$, and the symmetry $\binom{j+l}{l-u} = \binom{j+l}{j+u}$, and then permute terms in Eq. (8). Thus Eq. (8) becomes as follows:

$$\begin{aligned}
D_S(n,j,t+1) &= \sum_{l=0}^{n-2j}\sum_{u=0}^{l} \binom{n-2j-l}{u}\binom{j+l}{l-u}\binom{n-j}{j+l}\binom{n}{j+l}^{t+1} F(n,j+l) \text{ (Eq. (9))} \\
&= \sum_{l=0}^{n-2j}\sum_{u=0}^{l} \binom{n-2j-l}{u}\binom{j+l}{j+u}\binom{n-j}{n-2j-l}\binom{n}{j+l}\binom{n}{j+l}^{t} F(n,j+l) \text{ (by symmetry)} \\
&= \sum_{l=0}^{n-2j}\sum_{u=0}^{l} \binom{n-j}{n-2j-l}\binom{n-2j-l}{u}\binom{n}{j+l}\binom{j+l}{j+u}\binom{n}{j+l}^{t} F(n,j+l) \text{ (perm.)}
\end{aligned} \tag{14}$$

We apply Eq. (7) twice. It follows that

$$\binom{n-j}{n-2j-l}\binom{n-2j-l}{u} = \binom{n-j}{u}\binom{n-j-u}{n-2j-l-u} \text{ and} \tag{15}$$

$$\binom{n}{j+l}\binom{j+l}{j+u} = \binom{n}{j+u}\binom{n-j-u}{l-u}. \tag{16}$$

If we use Eqns. (15) and (16), Eq. (14) becomes

$$D_S(n,j,t+1) = \sum_{l=0}^{n-2j}\sum_{u=0}^{l}\binom{n-j}{u}\binom{n-j-u}{n-2j-l-u}\binom{n}{j+u}\binom{n-j-u}{l-u}\binom{n}{j+l}^t F(n,j+l). \tag{17}$$

We now exchange the order of summation. If we take Inequalities (12) and (13) into consideration along with the symmetry $\binom{n-j-u}{n-2j-l-u} = \binom{n-j-u}{j+l}$, Eq. (17) becomes

$$\begin{aligned} D_S(n,j,t+1) &= \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\sum_{l=u}^{n-2j-u}\binom{n-j}{u}\binom{n-j-u}{n-2j-l-u}\binom{n}{j+u}\binom{n-j-u}{l-u}\binom{n}{j+l}^t F(n,j+l) \text{ (Ineq.)} \\ &= \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\sum_{l=u}^{n-2j-u}\binom{n-j}{u}\binom{n-j-u}{j+l}\binom{n}{j+u}\binom{n-j-u}{l-u}\binom{n}{j+l}^t F(n,j+l) \text{ (symmetry)} \\ &= \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n}{j+u}\binom{n-j}{u}\sum_{l=u}^{n-2j-u}\binom{n-j-u}{l-u}\binom{n-j-u}{j+l}\binom{n}{j+l}^t F(n,j+l). \end{aligned} \tag{18}$$

We substitute $u+v$ for l. From the Inequality (12), we know that $0 \le j+u \le \lfloor\frac{n}{2}\rfloor$. Then Eq. (18) becomes as follows:

$$\begin{aligned} &\sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n}{j+u}\binom{n-j}{u}\sum_{v=0}^{n-2j-2u}\binom{n-j-u}{v}\binom{n-j-u}{j+u+v}\binom{n}{j+u+v}^t F(n,j+u+v) \\ &= \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n}{j+u}\binom{n-j}{u} D_S(n,j+u,t) \text{ (by Eq. (2))}. \end{aligned} \tag{19}$$

From Eqns. (18) and (19), we obtain

$$D_S(n,j,t+1) = \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor}\binom{n}{j+u}\binom{n-j}{u} D_S(n,j+u,t).$$

This completes the proof of Theorem 2. $\square$

4 Proof of Theorem 3

The proof of Theorem 3 is similar to the proof of Theorem 2. We use the same three binomial identities as in the previous proof.

Proof. By Definition 1 and Eq. (2), we know that

$$D_S(n,j,0) = \sum_{l=0}^{n-2j} \binom{n-j}{l}\binom{n-j}{j+l} F(n,j+l). \tag{20}$$

We use Eq. (9) and the symmetry $\binom{j+l}{l-u} = \binom{j+l}{j+u}$. Again, Inequalities (10), (11), (12), and (13) hold.

Then Eq. (20) becomes as follows:

$$\begin{aligned}
D_S(n,j,0) &= \sum_{l=0}^{n-2j} \left(\sum_{u=0}^{l} \binom{n-2j-l}{u}\binom{j+l}{l-u} \right) \binom{n-j}{j+l} F(n,j+l) \text{ (by Eq. (9))} \\
&= \sum_{l=0}^{n-2j} \sum_{u=0}^{l} \binom{n-j}{j+l}\binom{j+l}{l-u}\binom{n-2j-l}{u} F(n,j+l) \text{ (permutation)} \\
&= \sum_{l=0}^{n-2j} \sum_{u=0}^{l} \binom{n-j}{j+l}\binom{j+l}{j+u}\binom{n-2j-l}{u} F(n,j+l) \text{ (by symmetry).}
\end{aligned} \tag{21}$$

By Eq. (7) and the symmetry $\binom{n-2j-u}{l-u} = \binom{n-2j-u}{n-2j-l}$, it follows that

$$\binom{n-j}{j+l}\binom{j+l}{j+u} = \binom{n-j}{j+u}\binom{n-2j-u}{n-2j-l}. \tag{22}$$

If we use Eq. (22), Eq. (21) becomes

$$D_S(n,j,0) = \sum_{l=0}^{n-2j} \sum_{u=0}^{l} \binom{n-j}{j+u}\binom{n-2j-u}{n-2j-l}\binom{n-2j-l}{u} F(n,j+l). \tag{23}$$

By Eq. (7) and the symmetry $\binom{n-2j-2u}{n-2j-l-u} = \binom{n-2j-2u}{l-u}$, it follows that

$$\binom{n-2j-u}{n-2j-l}\binom{n-2j-l}{u} = \binom{n-2j-u}{u}\binom{n-2j-2u}{l-u}. \tag{24}$$

If we use Eq. (24), Eq. (23) becomes

$$D_S(n,j,0) = \sum_{l=0}^{n-2j} \sum_{u=0}^{l} \binom{n-j}{j+u}\binom{n-2j-u}{u}\binom{n-2j-2u}{l-u} F(n,j+l). \tag{25}$$

We exchange the order of summation. If we take Inequalities (12) and (13) into consideration, Eq. (25) becomes as follows:

$$\begin{aligned} D_S(n,j,0) &= \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor} \sum_{l=u}^{n-2j-u} \binom{n-j}{j+u}\binom{n-2j-u}{u}\binom{n-2j-2u}{l-u} F(n,j+l) \text{ (Ineq.)} \\ &= \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor} \binom{n-j}{j+u}\binom{n-2j-u}{u} \sum_{l=u}^{n-2j-u} \binom{n-2j-2u}{l-u} F(n,j+l). \end{aligned} \tag{26}$$

We substitute l by $u+v$. Then Eq. (26) becomes

$$D_S(n,j,0) = \sum_{u=0}^{\lfloor\frac{n-2j}{2}\rfloor} \binom{n-j}{j+u}\binom{n-2j-u}{u} \sum_{v=0}^{n-2j-2u} \binom{n-2j-2u}{v} F(n,j+u+v). \tag{27}$$

Eq. (27) completes the proof of Theorem 3. □

5 How Does This Method Work?

In one particular situation, Theorem 2 implies a simple consequence which is important for us.

Let n be a fixed non-negative integer. Let t_0 be a non-negative integer, and let j be an arbitrary integer in the range $0 \le j \le \lfloor\frac{n}{2}\rfloor$. Suppose that $q = q(n)$ is a positive integer which divides $D_S(n,j,t_0)$ sums for all j in the given range. We want q to be as large as possible. Then it can be shown, by Theorem 2, that q divides $D_S(n,j,t_0+1)$ for all j in the given range.

By induction, it follows that q divides $D_S(n,j,t)$ for all t such that $t \ge t_0$ and for all j in the given range. By Eq. (3), it follows that q divides $S(n,t+2)$ for all t such that $t \ge t_0$. This is exactly how this method works.

6 Two applications for non-negative sums

We give two applications of our method of D sums for non-negative sums with absolute values.

Let n and j be non-negative integers such that $j \le n$, and let t and m be positive integers. First, we consider the sum $S_1(2n,m) = \sum_{k=0}^{2n} \binom{2n}{k}^m |n-k|$. It is known [4, p. 3] that

$$\sum_{k=0}^{2n} \binom{2n}{k} |n-k| = n\binom{2n}{n}. \tag{28}$$

Therefore, we have

$$S_1(2n,1) = n\binom{2n}{n}.$$

We establish the following two lemmas:

Lemma 4.

$$D_{S_1}(2n,j,0) = n\binom{2n-j}{n}\binom{2n-j-1}{n} \quad \textit{for} \quad 0 \le j \le n.$$

Lemma 5.

$$D_{S_1}(2n,j,1) = n\binom{2n}{n}\sum_{u=0}^{n-1-j}\binom{n}{j+u}\binom{2n-j}{u}\binom{2n-j-u-1}{n} \quad \textit{for} \quad 0 \le j \le n.$$

In particular, Lemmas 4 and 5 imply the following formulas:

$$S_1(2n,2) = n\binom{2n}{n}\binom{2n-1}{n}, \tag{29}$$

$$S_1(2n,3) = n\binom{2n}{n}\sum_{u=0}^{n-1}\binom{n}{u}\binom{2n}{u}\binom{2n-u-1}{n}. \tag{30}$$

If n is a positive integer, Lemma 5 suggests setting $q_1(2n) = n\binom{2n}{n}$.

We establish the following theorem:

Theorem 6. *Let n be a positive integer. The sum $D_{S_1}(2n,j,t)$ is divisible by $n\binom{2n}{n}$ for all positive integers t and all integers j such that $0 \le j \le n$.*

We conclude that

Corollary 7. *Let n be a positive integer. The sum $S_1(2n,m)$ is divisible by $n\binom{2n}{n}$ for all positive integers m .*

Next, we consider the sum $S_2(2n+1,m) = \sum_{k=0}^{2n+1}\binom{2n+1}{k}^m\left|\frac{2n+1}{2}-k\right|$. It is known [4, p. 3] that

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}\left|\frac{2n+1}{2}-k\right| = (2n+1)\binom{2n}{n}. \tag{31}$$

Therefore, we have

$$S_2(2n+1,1) = (2n+1)\binom{2n}{n}.$$

We establish the following two lemmas:

Lemma 8.

$$D_{S_2}(2n+1,j,0) = (2n+1-j)\binom{2n-j}{n}^2 \quad \textit{for} \quad 0 \le j \le n.$$

Lemma 9.

$$D_{S_2}(2n+1,j,1) = (2n+1)\binom{2n}{n}\sum_{u=0}^{n-j}\binom{n}{j+u}\binom{2n+1-j}{u}\binom{2n-j-u}{n} \quad \textit{for} \quad 0 \le j \le n.$$

In particular, Lemmas 8 and 9 imply the following formulas:

$$S_2(2n+1,2) = (2n+1)\binom{2n}{n}^2, \tag{32}$$

$$S_2(2n+1,3) = (2n+1)\binom{2n}{n}\sum_{u=0}^{n}\binom{n}{u}\binom{2n+1}{u}\binom{2n-u}{n}. \tag{33}$$

Lemma 9 suggests setting $q_2(2n+1) = (2n+1)\binom{2n}{n}$. We establish the following theorem:

Theorem 10. *Let n be a non-negative integer. The sum $D_{S_2}(2n+1,j,t)$ is divisible by $(2n+1)\binom{2n}{n}$ for all positive integers t and all integers j such that $0 \le j \le n$.*

We conclude that

Corollary 11. *Let n be a non-negative integer. Then the sum $S_2(2n+1,m)$ is divisible by $(2n+1)\binom{2n}{n}$ for all positive integers m.*

Due to clarity and brevity of this paper, we prove only Lemma 4, Lemma 5, Theorem 6, and Corollary 7. Proofs of Lemma 8, Lemma 9, Theorem 10, and Corollary 11 are similar to proofs of Lemma 4, Lemma 5, Theorem 6, and Corollary 7, respectively. Therefore, proofs of Lemma 8, Lemma 9, Theorem 10, and Corollary 11 are omitted.

7 Details of Theorem 6

7.1 Motivation

The Identity (28) has a long history [6, Introduction]. It was a problem in the 1974 Putnam competition [2, Problem A4]. Best [3, Thm. 3] considered this identity in an application to Hadamard matrices. See also [14, Thm. 15.2], [1, Chapter 2.5], and [7].

Tuenter [22] considered centered binomial sums of the form

$$S_r(n) = \sum_{k=0}^{2n}\binom{2n}{k}|n-k|^r,$$

which are a generalization of the Identity (28).

Brent [4] considered binomial sums

$$U_r(n) = \sum_{k=0}^{n}\binom{n}{k}|n/2-k|^r,$$

which are a generalization of Tuenter's sums. Brent gave recurrence relations for $U_r(n)$, where he used facts $U_1(2n) = n\binom{2n}{n}$ and $U_1(2n+1) = (2n+1)\binom{2n}{n}$. Brent cited a solution by Hillman [20] of the Putnam problem 35-A4. Hillman gave a closed form for $U_1(n)$, i.e., $U_1(n) = n\binom{n-1}{\lfloor \frac{n}{2} \rfloor}$.

Identities (28) and (31) are connected with the following binomial identity [5, 6]:

$$\sum_{i=-n}^{n}\sum_{j=-n}^{n}\binom{2n}{n+i}\binom{2n}{n+j}|i^2-j^2| = 2n^2\binom{2n}{n}^2. \tag{34}$$

The Identity (34) can be used in proofs of lower bounds for the Hadamard maximal determinant problem.

We consider the sum $S_1(2n,m)$ as another generalization of the Identity (28). Similarly, we consider the sum $S_2(2n+1,m)$ as another generalization of the Identity (31).

7.2 Proof of Lemma 4

Let n be a non-negative integer, and let m be a positive integer. We defined the sum $S_1(2n,m)$ as

$$S_1(2n,m) = \sum_{k=0}^{2n}\binom{2n}{k}^m |n-k|. \tag{35}$$

Obviously, the sum $S_1(2n,m)$ is an instance of the sum (1), where

$$F_1(2n,k) = |n-k|. \tag{36}$$

Let j be a non-negative integer such that $j \leq n$.

The proof of Lemma 4 is based on Theorem 3 and Eq. (28).

Proof. By Theorem 3, we have

$$\begin{aligned} D_{S_1}(2n,j,0) &= \sum_{u=0}^{\lfloor \frac{2n-2j}{2} \rfloor}\binom{2n-j}{j+u}\binom{2n-2j-u}{u}\sum_{v=0}^{2n-2j-2u}\binom{2n-2j-2u}{v}F_1(2n,j+u+v) \\ &= \sum_{u=0}^{n-j}\binom{2n-j}{j+u}\binom{2n-2j-u}{u}\sum_{v=0}^{2n-2j-2u}\binom{2n-2j-2u}{v}F_1(2n,j+u+v) \qquad (37) \\ &= \sum_{u=0}^{n-j}\binom{2n-j}{j+u}\binom{2n-2j-u}{u}\sum_{v=0}^{2n-2j-2u}\binom{2n-2j-2u}{v}|n-j-u-v|. \qquad (38) \end{aligned}$$

Note that in Eq. (37), we used Eq. (36).

By Eq. (28), we obtain

$$\sum_{v=0}^{2n-2j-2u}\binom{2n-2j-2u}{v}|n-j-u-v| = (n-j-u)\binom{2(n-j-u)}{n-j-u}. \tag{39}$$

If we use Eq. (39), the symmetry $\binom{2n-j}{j+u} = \binom{2n-j}{2n-2j-u}$, and the symmetry $\binom{2n-2j-u}{u} = \binom{2n-2j-u}{2n-2j-2u}$, Eq. (38) becomes as follows:

$$\sum_{u=0}^{n-j} \binom{2n-j}{j+u}\binom{2n-2j-u}{u}(n-j-u)\binom{2(n-j-u)}{n-j-u} \text{ (by Eq. (39))} \tag{40}$$

$$= \sum_{u=0}^{n-1-j} \binom{2n-j}{2n-2j-u}\binom{2n-2j-u}{2n-2j-2u}\binom{2n-2j-2u}{n-j-u}(n-j-u) \text{ (symmetry).} \tag{41}$$

Note that the last term of the sum in Eq. (40) is equal to zero.

Furthermore, we have

$$\begin{aligned}\binom{2n-2j-u}{2n-2j-2u}\binom{2n-2j-2u}{n-j-u} &= \binom{2n-2j-u}{n-j-u}\binom{n-j}{n-j-u} \text{ (by Eq. (7))}\\ &= \binom{2n-2j-u}{n-j}\binom{n-j}{u} \text{ (by symmetry).}\end{aligned}$$

Therefore, we obtain

$$\binom{2n-2j-u}{2n-2j-2u}\binom{2n-2j-2u}{n-j-u} = \binom{2n-2j-u}{n-j}\binom{n-j}{u}. \tag{42}$$

If we use Eq. (42), Eq. (41) becomes

$$D_{S_1}(2n, j, 0) = \sum_{u=0}^{n-1-j} \binom{2n-j}{2n-2j-u}\binom{2n-2j-u}{n-j}\binom{n-j}{u}(n-j-u). \tag{43}$$

By Eq. (7) and the symmetry $\binom{2n-j}{n-j} = \binom{2n-j}{n}$, it follows that

$$\binom{2n-j}{2n-2j-u}\binom{2n-2j-u}{n-j} = \binom{2n-j}{n}\binom{n}{n-j-u}. \tag{44}$$

If we use Eq. (44), Eq. (43) becomes as follows:

$$\begin{aligned}D_{S_1}(2n, j, 0) &= \sum_{u=0}^{n-1-j} \binom{2n-j}{n}\binom{n}{n-j-u}\binom{n-j}{u}(n-j-u) \text{ (by Eq. (44))}\\ &= \binom{2n-j}{n}\sum_{u=0}^{n-1-j}\binom{n-j}{u}(n-j-u)\binom{n}{n-j-u} \text{ (permutation)}\end{aligned} \tag{45}$$

It is well-known that

$$k\binom{n}{k} = n\binom{n-1}{k-1}. \tag{46}$$

By Eq. (46), we get

$$(n-j-u)\binom{n}{n-j-u} = n\binom{n-1}{n-1-j-u}. \tag{47}$$

If we use Eq. (47), Eq. (45) becomes

$$D_{S_1}(2n,j,0) = n\binom{2n-j}{n}\sum_{u=0}^{n-1-j}\binom{n-j}{u}\binom{n-1}{n-1-j-u}. \tag{48}$$

By Eq. (6) and the symmetry $\binom{2n-1-j}{n-1-j} = \binom{2n-1-j}{n}$, we get

$$\sum_{u=0}^{n-1-j}\binom{n-j}{u}\binom{n-1}{n-1-j-u} = \binom{2n-1-j}{n}. \tag{49}$$

Finally, if we put Eq. (49) in Eq. (48), we obtain

$$D_{S_1}(2n,j,0) = n\binom{2n-j}{n}\binom{2n-1-j}{n}.$$

This completes the proof of Lemma 4. □

7.3 Proof of Lemma 5

This proof is based on Theorem 2 and Lemma 4.

Proof. By Theorem 2, we have

$$\begin{aligned} D_{S_1}(2n,j,1) &= \sum_{u=0}^{\lfloor\frac{2n-2j}{2}\rfloor}\binom{2n}{j+u}\binom{2n-j}{u}D_{S_1}(2n,j+u,0) \quad \text{(by Theorem 2)} \\ &= \sum_{u=0}^{n-j}\binom{2n}{j+u}\binom{2n-j}{u}D_{S_1}(2n,j+u,0). \end{aligned} \tag{50}$$

By Lemma 4, we get

$$D_{S_1}(2n,j+u,0) = n\binom{2n-j-u}{n}\binom{2n-1-j-u}{n}. \tag{51}$$

Then Eq. (50) becomes as follows:

$$\begin{aligned} D_{S_1}(2n,j,1) &= \sum_{u=0}^{n-j}\binom{2n}{j+u}\binom{2n-j}{u}n\binom{2n-j-u}{n}\binom{2n-1-j-u}{n} \quad \text{(by Eq. (51))} \\ &= n\sum_{u=0}^{n-j}\binom{2n}{j+u}\binom{2n-j-u}{n}\binom{2n-j}{u}\binom{2n-1-j-u}{n} \quad \text{(permutation)} \\ &= n\sum_{u=0}^{n-j}\binom{2n}{2n-j-u}\binom{2n-j-u}{n}\binom{2n-j}{u}\binom{2n-1-j-u}{n}. \end{aligned} \tag{52}$$

Note that we used the symmetry $\binom{2n}{j+u} = \binom{2n}{2n-j-u}$ in Eq. (52). Furthermore, the last term in Eq. (52) equals zero.

It is readily verified that

$$\binom{2n}{2n-j-u}\binom{2n-j-u}{n} = \binom{2n}{n}\binom{n}{j+u}. \tag{53}$$

Then Eq. (52) becomes as follows:

$$\begin{aligned} D_{S_1}(2n, j, 1) &= n\sum_{u=0}^{n-j}\binom{2n}{n}\binom{n}{j+u}\binom{2n-j}{u}\binom{2n-1-j-u}{n} \text{ (by Eq. (53))} \\ &= n\binom{2n}{n}\sum_{u=0}^{n-1-j}\binom{n}{j+u}\binom{2n-j}{u}\binom{2n-1-j-u}{n}. \end{aligned} \tag{54}$$

Eq. (54) implies that

$$D_{S_1}(2n, j, 1) = n\binom{2n}{n}\sum_{u=0}^{n-1-j}\binom{n}{j+u}\binom{2n-j}{u}\binom{2n-1-j-u}{n}.$$

This completes the proof of Lemma 5. □

7.4 Proof of Theorem 6

We assume that n is a fixed positive integer and j is a fixed non-negative integer such that $j \le n$. We use induction on t.

Proof. When $t = 1$, $D_{S_1}(2n, j, t)$ is divisible by $n\binom{2n}{n}$. This follows from Lemma 5. The base case of induction is confirmed.

We now assume that $D_{S_1}(2n, k, t)$ is divisible by $n\binom{2n}{n}$ for all non-negative integers k such that $k \le n$.

What happens with $D_{S_1}(2n, j, t+1)$? By Theorem 2, we have

$$D_{S_1}(2n, j, t+1) = \sum_{u=0}^{n-j}\binom{2n}{j+u}\binom{2n-j}{u}D_{S_1}(2n, j+u, t). \tag{55}$$

Obviously, $0 \le u + j \le n$ for $0 \le u \le n - j$. By induction hypothesis, $D_{S_1}(2n, j+u, t)$ is divisible by $n\binom{2n}{n}$. By Eq. (55), it follows that $D_{S_1}(2n, j, t+1)$ is divisible by $n\binom{2n}{n}$. By induction, Theorem 6 follows. □

7.5 Proof of Corollary 7

Proof. By Eq. (28), it follows that $S_1(2n,1)$ is divisible by $n\binom{2n}{n}$.

Setting $j=0$ in Lemma 4, we obtain Eq. (29). By Eq. (29), it follows that $S_1(2n,2)$ is divisible by $n\binom{2n}{n}$.

Let $m \geq 3$. By Eq. (3), we know that $S_1(2n,m) = D_{S_1}(2n,0,m-2)$. Since $m-2 \geq 1$, we can apply Theorem 6. By Theorem 6, $D_{S_1}(2n,0,m-2)$ is divisible by $n\binom{2n}{n}$. By Eq. (3), $S_1(2n,m)$ is divisible by $n\binom{2n}{n}$ for $m \geq 3$. This completes the proof of Corollary 7. □

Remark 12. Note that

$$S_1(2n,2) = \frac{n}{2}\binom{2n}{n}^2. \tag{56}$$

Setting $j=0$ in Lemma 5 and using Eq. (3), we obtain Eq. (30).

8 The third application for the alternating sum

We give a sketch of the proof of Calkin's result [9, Thm. 1] by using our method of D sums.

We use the well-known identity [16, Eq. (1.25), p. 4]

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k} = \begin{cases} 0, & \text{if } n>0; \\ 1, & \text{if } n=0, \end{cases} \tag{57}$$

where n is a non-negative integer. Eq. (57) plays the same role as Eq. (28) in the proof of Theorem 6.

Let n be a non-negative integer and let m be a positive integer. Let $S_3(2n,m)$ denote the sum $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^m$. Obviously, the sum $S_3(2n,m)$ is an instance of the sum (1), where $F_3(2n,k) = (-1)^k$. By Eq. (57), we conclude that $S_3(2n,1)$ is divisible by $\binom{2n}{n}$.

Furthermore, it is known

$$S_3(2n,2) = (-1)^n\binom{2n}{n}, \qquad \text{(Kummer's formula)}$$

$$S_3(2n,3) = (-1)^n\binom{2n}{n}\binom{3n}{2n}. \qquad \text{(Dixon's formula (4))}$$

Therefore, it follows that $S_3(2n,m)$ is divisible by $\binom{2n}{n}$ for $1 \leq m \leq 3$.

By Eq. (57) and Theorem 3, it can be proved that

$$D_{S_3}(2n,j,0) = (-1)^n\binom{2n-j}{n} \quad \text{for} \quad 0 \leq j \leq n. \tag{58}$$

By Theorem 2 and Eq. (58), it can be shown that

$$D_{S_3}(2n,j,1) = (-1)^n\binom{2n}{n}\binom{3n-j}{2n} \quad \text{for} \quad 0 \leq j \leq n. \tag{59}$$

Setting $j = 0$ in Eq. (59) and using Eq. (3), we obtain Eq. (4). Hence, the method of D sums gives a proof of Eq. (4).

Furthermore, Eq. (59) suggests setting $q_3(2n) = \binom{2n}{n}$.

By Theorem 2, it can be shown that $D_{S_3}(2n, j, t)$ is divisible by $\binom{2n}{n}$ for all positive integers t and all integers j such that $0 \leq j \leq n$.

Let $m \geq 4$. By Eq. (3), we know that $S_3(2n, m) = D_{S_3}(2n, 0, m-2)$. Since $m - 2 \geq 2$, the sum $D_{S_3}(2n, 0, m-2)$ is divisible by $\binom{2n}{n}$. By Eq. (3), $S_3(2n, m)$ is divisible by $\binom{2n}{n}$ for $m \geq 4$.

Finally, we can conclude that $S_3(2n, m)$ is divisible by $\binom{2n}{n}$ for all non-negative integers n and all positive integers m. This proves Calkin's result.

Remark 13. By using asymptotic methods, Bruijn [8] has proved that no closed form exists for $S_3(n, m)$ when $m \geq 4$. However, there are formulas for $S_3(2n, 4)$ and $S_3(2n, 5)$ [19, Eq. (5.13), Eq. (5.12)]. Both formulas can be derived by using the method of D sums.

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