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Spivey's Bell Number Formula Revisited

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Abstract

This paper introduces an alternative form of the derivation of Spivey's Bell number formula, which involves the q-Boson operators a and $a^{\dagger}$. Furthermore, a similar formula for the case of the (q,r)-Dowling polynomials is obtained, and is shown to produce a generalization of the latter.

1 Introduction

Consider the Stirling numbers of the second kind, denoted by $\left\{ {m \atop j} \right\}$, which appear as coefficients in the expansion of

$$t^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (t)_k, \tag{1}$$

where $(t)_k = t(t-1)(t-2)\cdots(t-k+1)$.The Bell numbers, denoted by B_n, are defined by

$$B_n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} \tag{2}$$

and are known to satisfy the recurrence relation

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \tag{3}$$

In 2008, Spivey [13] obtained a remarkable formula which unifies the defining relation in (2) and the identity (3). The said formula is given by

$$B_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} j^{n-k} \left\{ {m \atop j} \right\} \binom{n}{k} B_k \tag{4}$$

and is popularly known as "Spivey's Bell number formula". Equation (4) was proved in [13] using a combinatorial approach involving partition of sets. Different proofs and extensions of (4) were later on studied by several authors. For instance, a proof which made use of generating functions was done by Gould and Quaintance [5] which was then generalized by Xu [14] using Hsu and Shuie's [6] generalized Stirling numbers. Belbachir and Mihoubi [2] presented a proof that involves decomposition of the Bell polynomials into a certain polynomial basis. Mező [12] obtained a generalization of the Spivey's formula in terms of the r-Bell polynomials via combinatorial approach. The notion of dual of (4) was also presented in the same paper. On the other hand, the work of Katriel [7] involved the use of the operator X satisfying

$$DX - qXD = 1, \tag{5}$$

where D is the q-derivative defined by

$$Df(x) = \frac{f(qx) - f(x)}{x(q-1)}. \tag{6}$$

For the sake of clarity and brevity, this method will be referred to as "Katriel's proof".

Now, aside from being implicitly implied in Katriel's proof, none of the previously-mentioned studies considered establishing q-analogues. It is, henceforth, the main purpose of this paper to obtain a generalized q-analogue of Spivey's Bell number formula.

2 Alternative form of "Katriel's proof"

We direct our attention to the q-Boson operators a and $a^\dagger$ satisfying the commutation relation

$$[a, a^\dagger]_q = aa^\dagger - qa^\dagger a = 1 \tag{7}$$

(see [1]). We define the Fock space (or Fock states) by the basis $\{|s\rangle\,; s = 0, 1, 2, \ldots\}$ so that the relations $a\,|s\rangle = \sqrt{[s]_q}\,|s-1\rangle$ and $a^\dagger\,|s\rangle = \sqrt{[s+1]_q}\,|s+1\rangle$ form a representation that satisfies (7). The operators $a^\dagger a$ and $(a^\dagger)^k a^k$, when acting on $|s\rangle$, yield

$$a^\dagger a\,|s\rangle = [s]_q\,|s\rangle \tag{8}$$

and

$$(a^\dagger)^k a^k\,|s\rangle = [s]_{q,k}\,|s\rangle\,, \tag{9}$$

respectively, where $[s]_q = \frac{q^s-1}{q-1}$ and $[s]_{q,k} = [s]_q[s-1]_q[s-2]_q\cdots[s-k+1]_q$. Hence, the q-Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_q$ [3] can be defined alternatively as

$$(a^\dagger a)^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q (a^\dagger)^k a^k. \tag{10}$$

From (7), it is clear that

$$[a, (a^\dagger)^k]_{q^k} = [a, (a^\dagger)^{k-1}]_{q^{k-1}} a^\dagger + q^{k-1}(a^\dagger)^{k-1}[a, a^\dagger]_q, \tag{11}$$

and by induction on k, we have

$$[a, (a^\dagger)^k]_{q^k} = [k]_q (a^\dagger)^{k-1}. \tag{12}$$

Since $a\,|0\rangle = 0$, then by (12),

$$\begin{aligned} a(a^\dagger)^\ell\,|0\rangle &= [a, (a^\dagger)^\ell]_{q^\ell}\,|0\rangle \\ &= [\ell]_q (a^\dagger)^{\ell-1}\,|0\rangle\,. \end{aligned}$$

Moreover,

$$a^k (a^\dagger)^\ell\,|0\rangle = \frac{[\ell]_q!}{[\ell-k]_q!}(a^\dagger)^{\ell-k}\,|0\rangle\,, \tag{13}$$

for $k \le \ell$ and

$$a^k (a^\dagger)^\ell\,|0\rangle = 0, \tag{14}$$

for $k > \ell$. Finally,

$$a^k e_q(x a^\dagger)\,|0\rangle = x^k e_q(x a^\dagger)\,|0\rangle\,, \tag{15}$$

where $e_q(xa^\dagger)$ is the q-exponential function defined by

$$e_q(t) = \sum_{\ell=0}^{\infty} \frac{t^\ell}{[\ell]_q!}. \tag{16}$$

Applying (15) to (10) yields

$$(a^\dagger a)^n e_q(t a^\dagger)\,|0\rangle = B_{n,q}(t a^\dagger) e_q(t a^\dagger)\,|0\rangle\,, \tag{17}$$

where $B_{n,q}(ta^\dagger)$ denotes the q-Bell polynomials defined by

$$B_{n,q}(t) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q t^k. \tag{18}$$

Let $x = ta^\dagger$ so that

$$(a^\dagger a)^n e_q(x)\,|0\rangle = B_{n,q}(x) e_q(x)\,|0\rangle\,. \tag{19}$$

Before proceeding, note that by definition,

$$[a, (a^\dagger)^k]_{q^k} = a(a^\dagger)^k - q^k (a^\dagger)^k a. \tag{20}$$

By (12),

$$\begin{aligned} a(a^\dagger)^k - q^k (a^\dagger)^k a &= [k]_q (a^\dagger)^{k-1} \\ a(a^\dagger)^k &= q^k (a^\dagger)^k a + [k]_q (a^\dagger)^{k-1}. \end{aligned}$$

This can be further expressed as

$$(a^\dagger a)(a^\dagger)^k = (a^\dagger)^k \big([k]_q + q^k (a^\dagger a)\big). \tag{21}$$

Now, we have

$$\begin{aligned} (a^\dagger a)^{n+m} &= (a^\dagger a)^n \sum_{j=0}^{m} \left\{ {m \atop j} \right\}_q (a^\dagger)^j a^j \\ &= \sum_{j=0}^{m} \left\{ {m \atop j} \right\}_q (a^\dagger)^j \big([j]_q + q^j (a^\dagger a)\big)^n a^j \\ &= \sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ {m \atop j} \right\}_q \binom{n}{k} [j]_q^{n-k} q^{jk} (a^\dagger)^j (a^\dagger a)^k a^j. \end{aligned}$$

Multiplying both sides with $e_q(x) \left|0\right\rangle$ makes the left-hand side

$$(a^\dagger a)^{n+m} e_q(x) \left|0\right\rangle = B_{n+m,q}(x) e_q(x) \left|0\right\rangle, \tag{22}$$

while the right-hand side becomes

$$\begin{aligned} \sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ {m \atop j} \right\}_q \binom{n}{k} [j]_q^{n-k} q^{jk} (a^\dagger)^j (a^\dagger a)^k e_q(x) \left|0\right\rangle a^j &= \sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ {m \atop j} \right\}_q \binom{n}{k} [j]_q^{n-k} q^{jk} \\ & \quad B_{k,q}(x) e_q(x) \left|0\right\rangle (a^\dagger)^j a^j. \end{aligned}$$

Dividing both sides by $e_q(x) \left|0\right\rangle$ and using (9) gives

$$B_{n+m,q}(x) = \sum_{j=0}^{m} \sum_{k=0}^{n} \left\{ {m \atop j} \right\}_q \binom{n}{k} [j]_q^{n-k} q^{jk} B_{k,q}(x) [x]_{q,j}. \tag{23}$$

As $q \to 1$, we obtain a polynomial version of Spivey's Bell number formula which, in return, reduces to (4) when we set $x = 1$.

It is important to emphasize that this is not a new proof, but an alternative form of Katriel's proof, since the operators a, $a^\dagger$ and the operators X, D generate isomorphic algebras.

3 A generalization of Spivey's Bell number formula

The main result of this paper is the following identity:

$$D_{m,r,q}(n+\ell,x)=\sum_{j=0}^{\ell}\sum_{k=0}^{n}m^jW_{m,r,q}(\ell,j)\binom{n}{k}(m[j]_q+r)^{n-k}q^{jk}D_{m,0,q}(k,x)[x]_{q,j}. \tag{24}$$

Here, $D_{m,r,q}(n,x)$ is a (q,r)-Dowling polynomial defined previously by the author and Katriel [9] as

$$D_{m,r,q}(n,x)=\sum_{k=0}^{n}W_{m,r,q}(n,k)x^k, \tag{25}$$

where $W_{m,r,q}(n,k)$ is the (q,r)-Whitney numbers of the second kind. Several properties of $D_{m,r,q}(n,x)$ can be seen in [8, 9].

To derive (24), we first multiply both sides of (21) by m and then add $r(a^\dagger)^k$ to yield

$$(ma^\dagger a+r)(a^\dagger)^k=(a^\dagger)^k(m[k]_q+r+mq^ka^\dagger a). \tag{26}$$

Also, multiplying both sides of the defining relation in [9, Equation 16] by $e_q(ta^\dagger)\left|0\right\rangle$ and applying (15) yields

$$\begin{aligned}(ma^\dagger a+r)^ne_q(ta^\dagger)\left|0\right\rangle &= \sum_{k=0}^{n}m^kW_{m,r,q}(n,k)(a^\dagger)^ka^ke_q(ta^\dagger)\left|0\right\rangle\\ &= \sum_{k=0}^{n}m^kW_{m,r,q}(n,k)(a^\dagger)^kt^ke_q(ta^\dagger)\left|0\right\rangle\\ &= D_{m,r,q}(n,mta^\dagger)e_q(ta^\dagger)\left|0\right\rangle.\end{aligned}$$

Now, by (26),

$$\begin{aligned}(ma^\dagger a+r)^{n+\ell} &= \sum_{j=0}^{\ell}m^jW_{m,r,q}(\ell,j)(ma^\dagger a+r)^n(a^\dagger)^ja^j\\ &= \sum_{j=0}^{\ell}m^jW_{m,r,q}(\ell,j)(a^\dagger)^j(m[j]_q+r+mq^ja^\dagger a)^na^j\\ &= \sum_{j=0}^{\ell}\sum_{k=0}^{n}m^{j+k}W_{m,r,q}(\ell,j)\binom{n}{k}(a^\dagger)^j(m[j]_q+r)^{n-k}q^{kj}(a^\dagger a)^ka^j.\end{aligned}$$

Applying this expression to the operator identity $e_q(ta^\dagger)\left|0\right\rangle$, combining with the previous equation, using (9), (19) and $W_{m,0,q}(k,i)=m^{k-i}\left\{{k \atop i}\right\}_q$ (see [9, Equation 18]), and then dividing both sides of the resulting identity by $e_q(ta^\dagger)\left|0\right\rangle$ completes the derivation.

4 Remarks

Since $W_{1,0,q}(\ell, j) = \left\{ {\ell \atop j} \right\}_q$, then by setting $x = 1$, $m = 1$ and $r = 0$, we have

$$D_{1,0,q}(n+\ell, 1) = \sum_{j=0}^{\ell} \sum_{k=0}^{n} \left\{ {\ell \atop j} \right\}_q \binom{n}{k} [j]_q^{n-k} q^{jk} B_{k,q}, \tag{27}$$

where $B_{k,q} := B_{k,q}(1)$. This is a q-analogue of (4) which was first obtained by Katriel [7]. On the other hand, setting $x = 1$ and then taking the limit of (24) as $q \to 1$ provides a generalization of Spivey's Bell number formula in terms of the r-Whitney numbers of the second kind, denoted by $W_{m,r}(\ell, j)$, and the r-Dowling numbers, denoted by $D_{m,r}(n)$, (see [4, 11]), given by

$$D_{m,r}(n+\ell) = \sum_{j=0}^{\ell} \sum_{k=0}^{n} m^j W_{m,r}(\ell, j) \binom{n}{k} (mj+r)^{n-k} D_{m,0}(k). \tag{28}$$

In a recent paper, Mansour et al. [10] obtained the following generalization of Spivey's Bell number formula:

$$D_{p,q}(a+b; x) = \sum_{i=0}^{a} \sum_{j=0}^{b} \sum_{\ell=0}^{j} (mq^i)^{j-\ell} x^{i+\ell} \binom{b}{j} \left([r]_p + m[i]_q\right)^{b-j} W_{p,q}(a, i) S_q(j, \ell). \tag{29}$$

Here, $D_{p,q}(n; x)$ and $W_{p,q}(n, k)$ denote the (p, q)-analogues of the r-Dowling polynomials and the r-Whitney numbers of the second kind, respectively. The (p, q)-analogues are natural generalizations of q-analogues. However, since the manner by which the numbers $W_{m,r,q}(n, k)$ were defined in [9] differs from the work of Mansour et al. [10], the main result of this paper is not generalized by (29).

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