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Motzkin Numbers: an Operational Point of View

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Abstract

The Motzkin numbers can be derived as coefficients of hybrid polynomials. Such an identification allows the derivation of new identities for this family of numbers, and offers a tool to investigate previously unnoticed links with the theory of special functions and with the relevant treatment in terms of operational means. The use of umbral methods opens new directions for further developments and generalizations, which leads, e.g., to the identification of new Motzkin-associated forms.

1 Introduction

The telephone numbers (T_n), also called convolution numbers, provide a very well-known example of the link between special numbers and special polynomials. The (T_n) can be expressed in terms of the coefficients of Hermite polynomials (h_s) [1, pp. 85–86]. Two of the present authors (M. A. and G. D.) have recently pointed out [2] that the Padovan and Perrin numbers [3, 4] can be recognized as associated with particular values of two-variable Legendre polynomials [5].

The Motzkin numbers plays a significant role in combinatorics (see, e.g., Bernhart [6]), Donaghey and Shapiro pointed out fourteen different manifestations of these numbers [7]. Blasiak et al. and Dattoli et al. have studied [8, 9] the connection between Motzkin numbers and a family of hybrid polynomials and the relevant properties of Motzkin numbers.

In a recent paper, Zhao and Qi [10] discussed similar results employing the link between Motzkin and Catalan numbers.

The hybrid polynomials are defined as follows [9]:

$$P_n^{(q)}(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2r} y^r}{(n-2r)!r!(r+q)!}, \tag{1}$$

and the relevant generating function is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(q)}(x,y) = \frac{I_q(2\sqrt{y}\ t)}{(\sqrt{y}\ t)^q} e^{xt}, \tag{2}$$

where $I_q(x)$ is the modified Bessel function of the first kind of order q.

Within the present framework, the Motzkin number sequence can be specified as follows [8]:

$$\begin{aligned} m_n &= P_n^{(1)}(1,1) = \sum_{s=0}^{n} m_{n,s}, \\ m_{n,s} &= \binom{n}{s} f_s, \\ f_s &= \frac{s!}{\Gamma\left(\frac{s}{2}+2\right)\Gamma\left(\frac{s}{2}+1\right)} \left|\cos\left(s\frac{\pi}{2}\right)\right|, \end{aligned} \tag{3}$$

where the coefficients $m_{n,s}$ can be represented as the triangle reported in the following table, in which $m_{n,2}$ corresponds, in the On-Line Encyclopedia of Integer Sequences (OEIS), to the sequence A000217, $m_{n,4}$ to A034827, $m_{n,6}$ to A000910, and so on.

Parameter		$m_{n,s}$ coefficients s								m_n Motzkin $\sum_{s=0}^{n} m_{n,s}$
		0	1	2	3	4	5	6	7	
n	0	1								1
	1	1	0							1
	2	1	0	1						2
	3	1	0	3	0					4
	4	1	0	6	0	2				9
	5	1	0	10	0	10	0			21
	6	1	0	15	0	30	0	5		51
	7	1	0	21	0	70	0	35	0	127

Table 1: Motzkin numbers and their coefficients

According to Eq. (2), the Motzkin numbers can also be defined as the coefficients of the following series expansion

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} m_n = \frac{I_1(2t)}{t} e^t. \tag{4}$$

In what follows we show how some progress in the study of the relevant properties can be achieved by the use of the umbral notation.

2 Motzkin numbers and the umbral calculus

In order to simplify most of the algebra associated with the study of the properties of the Motzkin numbers, and to get new relevant identities, we introduce a formalism successfully exploited elsewhere [11] based on of the umbral calculus [12].

To this aim, we note that the function

$$C_q(x) = \frac{I_q(2\sqrt{x})}{(\sqrt{x})^q} = \sum_{r=0}^{\infty} \frac{x^r}{r!(q+r)!} \tag{5}$$

can be cast in the form

$$C_q(x) = \hat{c}^q \circ e^{\hat{c}x}, \tag{6}$$

where $\hat{c}$ is an umbral operator defined according to the equation

$$\hat{c}^\mu = \frac{1}{\Gamma(\mu+1)}, \tag{7}$$

with μ real and not necessarily integer.

We define the following composition rule

$$\hat{c}^{\mu} \circ \hat{c}^{\nu} = \hat{c}^{\mu+\nu}, \tag{8}$$

and we let $\hat{C} = \{\hat{c}^{\alpha}, \alpha \in \mathbb{C}\}$ denote the set of $\hat{c}$-operators. Then the pair $(\hat{C}, \circ)$ satisfies the Abelian group property. The mathematical foundations of the theory of $\hat{c}$-operators can be traced back to those underlying the Borel transform and have been carefully discussed [12].

The use of this formalism allows one to restyle the hybrid polynomials in the form

$$P_n^{(q)}(x, y) = \hat{c}^q \circ H_n(x, \hat{c}\, y), \tag{9}$$

where

$$H_n(x, y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2r} y^r}{(n-2r)! r!} \tag{10}$$

are the two-variable Hermite-Kampé de Fériét polynomials of order 2.

We can accordingly use the wealth of properties of this family of polynomials to derive further and new relations regarding those of the Motzkin numbers family.

Indeed, by recalling the generating function [9]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n+l}(x, y) = H_l(x + 2yt, y) e^{xt+yt^2}, \tag{11}$$

we find

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} m_{n+l} = \hat{c} \circ H_l(1 + 2\hat{c}t, \hat{c}) e^{t+\hat{c}t^2}, \tag{12}$$

which, after using Eqs. (8), (10), and (6), finally yields

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!} m_{n+l} = \mu_l(t)\; e^t, \\ &\mu_l(t) = l! \sum_{r=0}^{\lfloor \frac{l}{2} \rfloor} \frac{1}{r!} \sum_{s=0}^{l-2r} \frac{2^s}{s!(l-2r-s)!} \frac{I_{s+r+1}(2t)}{t^{r+1}}. \end{aligned} \tag{13}$$

Furthermore, the same procedure and the use of the Hermite polynomial duplication formula [13]

$$H_{2n}(x, y) = \sum_{r=0}^{n} \binom{n}{r}^2 r!\; (2y)^r \left(H_{n-r}(x, y)\right)^2, \tag{14}$$

yields the following identity for Motzkin numbers

$$\begin{aligned} m_{2n} &= \hat{c} \circ \sum_{r=0}^{n} r! \binom{n}{r}^2 (2\hat{c})^r \circ H_{n-r}(1, \hat{c}) \circ H_{n-r}(1, \hat{c}) = \\ &= \sum_{r=0}^{n} \binom{n}{r}^2 2^r r! (n-r)! \sum_{s=0}^{\lfloor \frac{n-r}{2} \rfloor} \frac{m_{n-r}^{(r+s+1)}}{(n-r-2s)! s!}, \end{aligned} \tag{15}$$

where

$$m_n^{(q)} = P_n^{(q)}(1,1) = \hat{c}^q \circ H_n(1,\hat{c}) \tag{16}$$

are associated Motzkin numbers [8].

The identification of Motzkin numbers as in Eq. (16), along with the use of the recurrences of Hermite polynomials, yields, e.g., the identities

$$\begin{aligned} m_{n+1}^{(q)} &= m_n^{(q)} + 2\,n\,m_{n-1}^{(q+1)}, \\ m_{n+p} &= \sum_{s=0}^{\min[n,p]} 2^s s! \binom{p}{s}\binom{n}{s} M_{p-s,\,n-s,\,s}, \\ M_{p,\,n,\,t} &= p! \sum_{r=0}^{\lfloor \frac{p}{2} \rfloor} \frac{m_n^{(t+r+1)}}{(p-2r)!r!}, \end{aligned} \tag{17}$$

in which the second identity has been derived from the Nielsen formula for $H_{n+m}(x,y)$ [14].

3 Final comments

In this paper we have shown that a fairly straightforward extension of a formalism put forward in [8] permits non-trivial progress in the theory of Motzkin numbers. Further relations can be easily obtained by applying the method we have envisaged as, e.g.,

$$\sum_{s=0}^{n} m_{n-s}\,m_s = 2\,(n+1)\,m_n^{(2)}, \tag{18}$$

which represents a discrete self-convolution of Motzkin numbers.

The associated Motzkin numbers, originally introduced [8], have been defined here on purely algebraic grounds (see Eq. 16). Strictly speaking they are not integers and therefore they are not amenable to a combinatorial interpretation; however, redefining them as

$$\tilde{m}_n^{(q)} = \frac{(n+q)!}{n!} P_n^{(q)}(1,1), \tag{19}$$

we obtain for $q = 2$ the sequences in OEIS A014531, while for $q = 3$ the sequences A014532, and so on.

A more appropriate interpretation in combinatorial terms can be obtained by following, e.g., the procedures indicated [15] and deserves further investigations, out of the scope of the present paper.

We have mentioned in the introduction the theory of telephone numbers $T(n)$ [16, pp. 65–67], whose importance in chemical graph theory has been recently emphasized [17]. As is well known, they can be expressed in terms of ordinary Hermite polynomials; however, the use of the two-variable extension is more effective. Indeed, they can be expressed as $T(n) = H_n(1, \frac{1}{2})$.

The use of Hermite polynomial properties, such as the index duplication formula, yields

$$T(2n) = \sum_{r=0}^{n} \binom{n}{r}^2 r!\, T(n-r)^2. \tag{20}$$

The use of the Hermite numbers h_s [18] allows the derivation of the following additional expression

$$\begin{aligned} T(n) &= \sum_{s=0}^{n} t_{n,s}, \\ t_{n,s} &= \binom{n}{s} h_s\left(\frac{1}{2}\right), \\ h_s(y) &= y^{\frac{s}{2}}\Gamma\left(\frac{s}{2}+2\right) f_s = \frac{y^{\frac{s}{2}} s!}{\Gamma\left(\frac{s}{2}+1\right)} \left|\cos\left(s\,\frac{\pi}{2}\right)\right|. \end{aligned} \tag{21}$$

The coefficients $t_{n,s}$ of the telephone numbers can be arranged in the following triangle, in which the numbers belonging to the column $s = 4$, $(3, 15, 45, 105, 210, 378, \ldots)$ are identified, in the OEIS, with the sequence A050534 and the column in $s = 6$, $(15, 105, 420, 1260, 3150, \ldots)$, with A240440.

$t_{n,s}$ coefficients										$T(n)$ telephone numbers
Parameter		s								$\sum_{s=0}^{n} t_{n,s}$
		0	1	2	3	4	5	6	7	
	0	1								1
	1	1	0							1
	2	1	0	1						2
	3	1	0	3	0					4
n	4	1	0	6	0	3				10
	5	1	0	10	0	15	0			26
	6	1	0	15	0	45	0	15		76
	7	1	0	21	0	105	0	105	0	232

Table 2: Telephone number coefficients

The use of the identification with two-variable Hermite polynomials opens further perspectives. By exploiting the polynomials (see [1, pp. 85–86] and references therein)

$$H_n^{(m)}(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{x^{n-mr} y^r}{(n-mr)!r!}, \tag{22}$$

we can introduce the following generalization of telephone numbers

$$T_n^{(m)} = H_n^{(m)}\left(1, \frac{1}{m}\right), \tag{23}$$

with generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} T_n^{(m)} = e^{t+\frac{1}{m}t^m}, \tag{24}$$

which satisfy the recurrence

$$T_{n+1}^{(m)} = T_n^{(m)} + \frac{n!}{(n-m+1)!} T_{n-m+1}^{(m)}. \tag{25}$$

In the case of $m = 3$ the numbers $T_n^{(3)} = (1, 1, 1, 3, 9, 21, 81, 351, 1233, \ldots)$ are identified with OEIS sequence A001470, while for $m = 4$, the series $(1, 1, 1, 1, 7, 31, 91, 211, 1681, 12097, \ldots)$, corresponds to A118934. For $m = 5$ the associated series appears to be A052501, but should be more appropriately identified with the coefficients of the expansion (23). Finally the sequence $m = 6$ is not reported in OEIS.

A more detailed analysis of this family of numbers and the relevant interplay with Motzkin will be discussed elsewhere.

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