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The Star of David and Other Patterns in Hosoya Polynomial Triangles

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Abstract

We define two types of second-order polynomial sequences. A sequence is of *Fibonacci-type* (*Lucas-type*) if its Binet formula is similar in structure to the Binet formula for the Fibonacci (Lucas) numbers. Familiar examples are Fibonacci polynomials, Chebyshev polynomials, Morgan-Voyce polynomials, Lucas polynomials, Pell polynomials, Fermat polynomials, Jacobsthal polynomials, Vieta polynomials and other known sequences of polynomials.

We generalize the numerical recurrence relation given by Hosoya to polynomials by constructing a Hosoya triangle for polynomials where each entry is either a product of two polynomials of Fibonacci-type or a product of two polynomials of Lucas-type. For every such choice of polynomial sequence we obtain a triangular array of polynomials. In this paper we extend the star of David property, also called the Hoggatt-Hansell identity, to these types of triangles. In addition, we study other geometric patterns in these triangles and as a consequence we obtain geometric interpretations for the Cassini identity, the Catalan identity, and other identities for Fibonacci polynomials.

1 Introduction

A second-order polynomial sequence is of *Fibonacci-type* (*Lucas-type*) if its Binet formula has a structure similar to that for Fibonacci (Lucas) numbers. Familiar examples of such polynomials are Fibonacci polynomials, Chebyshev polynomials, Morgan-Voyce polynomials, Lucas polynomials, Pell polynomials, Fermat polynomials, Jacobsthal polynomials, Vieta polynomials, and other sequences of polynomials. Most of the polynomials mentioned here are discussed by Koshy [14, 15].

The *Hosoya triangle*, formerly called the Fibonacci triangle, [3, 6, 11, 14], consists of a triangular array of numbers where each entry is a product of two Fibonacci numbers (see A058071). If in this triangle we replace the Fibonacci numbers with the corresponding polynomials from the sequences mentioned above, we obtain a Hosoya-like polynomial triangle (see Tables 2 and 3). Therefore, for every choice of a polynomial sequence we obtain a distinct Hosoya polynomial triangle. So, every polynomial evaluation gives rise to a numerical triangle (see Table 8). In particular the classic Hosoya triangle can be obtained by evaluating the entries of Hosoya polynomial triangle arising from Fibonacci polynomials evaluated at $x = 1$. For brevity we call these triangles Hosoya polynomial triangles and if there is no ambiguity we call them Hosoya triangles.

The Hosoya polynomial triangle provides a good geometric way to study algebraic and combinatorial properties of products of recursive sequences of polynomials. In this paper we study some of its geometric properties. Note that any geometric property in this triangle is automatically true for the classic (numerical) Hosoya triangle.

A hexagon gives rise to the star of David — connecting its alternating vertices with a continuous line as in Figure 1. Given a hexagon in a Hosoya triangle can one determine whether the vertices of the two triangles of the star of David have the same greatest common divisor (gcd)? If both greatest common divisors are equal, then we say that the star of David has the *gcd property*. Several authors have been interested in this property, see for example [4, 6, 8, 13, 17, 18, 21]. For instance, in 2014 Flórez et al. [5] proved the star of David property in the generalized Hosoya triangle. Koshy [13, 15] defined the gibonomial triangle and proved one of the fundamental properties of the star of David in this triangle. In a short

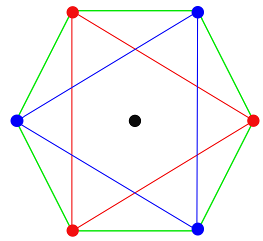

Figure 1: Star of David from a hexagon.

comment we establish the gcd property of the star of David for the gibonomial triangle.

Since every polynomial sequence of *Fibonacci-type* or of *Lucas-type* gives rise to a Hosoya triangle, the above question seems complicated to answer. We prove that the star of David property holds for most of the cases (depending on the locations of its points in the Hosoya triangle). We also prove that if the star of David property does not hold, then the two gcds are proportional. We give a characterization of the members of the family of Hosoya triangles that satisfy the star of David property. From Table 1, we obtain a sub-family of fourteen distinct Hosoya triangles. We provide a complete classification of the members that satisfy the star of David property.

We also study other geometric properties that hold in a Hosoya triangle, called the *rectangle property* and the *zigzag property*. A rectangle in the Hosoya polynomial triangle is a set of four points in the triangle that are arranged as the vertices of a rectangle. Using the rectangle property we give geometric interpretations and proofs of the Cassini, Catalan, and Johnson identities for *Fibonacci-type* or for *Lucas-type* sequences.

2 Preliminaries and the main theorem

In this section we summarize some concepts given by the authors in earlier articles. For example, the authors [2] have studied the polynomial sequences given here. The authors [3] have also studied polynomial triangular array. Throughout the paper we consider polynomials in $\mathbb{Q}[x]$. The polynomials in the Subsection 2.1 are presented in a formal way. However, for brevity and if there is no ambiguity after Subsection 2.1 and throughout the paper we avoid these formalities. Thus, we present the polynomials without explicit use of "x". We return to this formality if we need to evaluate a polynomial at a particular value. Another exception of this mentioned informality are the familiar examples of Fibonacci-type and Lucas-type polynomials. We adhere to the conventional formality as they appear in the literature (see, for example, Table 1).

2.1 Second-order polynomial sequences

We now define two types of second-order polynomial recurrence relations:

$$\mathcal{F}_0(x) = 0,\ \mathcal{F}_1(x) = 1,\ \text{and}\ \mathcal{F}_n(x) = d(x)\mathcal{F}_{n-1}(x) + g(x)\mathcal{F}_{n-2}(x)\ \text{for}\ n \geq 2, \tag{1}$$

where $d(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$.

We say a polynomial recurrence relation is of *Fibonacci-type* if it satisfies the relation given in (1), and of *Lucas-type* if

$$\mathcal{L}_0(x) = p_0,\ \mathcal{L}_1(x) = p_1(x),\ \text{and}\ \mathcal{L}_n(x) = d(x)\mathcal{L}_{n-1}(x) + g(x)\mathcal{L}_{n-2}(x)\ \text{for}\ n \geq 2, \tag{2}$$

where $|p_0| = 1$ or 2 and $p_1(x)$, $d(x) = \alpha p_1(x)$, and $g(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$ with α an integer of the form $2/p_0$.

To use similar notation for (1) and (2) on certain occasions we write $p_0 = 0$, $p_1(x) = 1$ to indicate the initial conditions of Fibonacci-type polynomials. Some familiar examples of Fibonacci-type polynomials and of Lucas-type polynomials are in Table 1 (see also [2, 3, 9, 10, 14]).

If G_n is either $\mathcal{F}_n$ or $\mathcal{L}_n$ for all $n \geq 0$ and $d^2(x) + 4g(x) > 0$ then the explicit formula for the recurrence relations in (1) and (2) is given by

$$G_n(x) = t_1 a^n(x) + t_2 b^n(x),$$

where $a(x)$ and $b(x)$ are the solutions of the quadratic equation associated to the second-order recurrence relation $G_n(x)$. That is, $a(x)$ and $b(x)$ are the solutions of $z^2 - d(x)z - g(x) = 0$. If $\alpha = 2/p_0$, then the Binet formula for Fibonacci-type polynomials is stated in (3) and the Binet formula for Lucas-type polynomials is stated in (4) (for details on the construction of the two Binet formulas see [2]).

$$\mathcal{F}_n(x) = \frac{a^n(x) - b^n(x)}{a(x) - b(x)} \tag{3}$$

and

$$\mathcal{L}_n(x) = \frac{a^n(x) + b^n(x)}{\alpha}. \tag{4}$$

Note that for both types of sequences the identities

$$a(x) + b(x) = d(x), \quad a(x)b(x) = -g(x), \quad \text{and} \quad a(x) - b(x) = \sqrt{d^2(x) + 4g(x)}$$

hold, where $d(x)$ and $g(x)$ are the polynomials defined in (1) and (2).

A sequence of Lucas-type (Fibonacci-type) is *equivalent* or *conjugate* to a sequence of Fibonacci-type (Lucas-type), if their recursive sequences are determined by the same polynomials $d(x)$ and $g(x)$. Notice that two equivalent polynomials have the same $a(x)$ and $b(x)$ in their Binet representations. Examples of equivalent polynomials are given in Table

1. Note that the leftmost polynomials in Table 1 are of Lucas-type and their equivalent Fibonacci-type polynomials are in the fifth column on the same line.

For most of the proofs involving these sequences it is required that

$$\gcd(p_0, p_1(x)) = 1, \quad \gcd(p_0, d(x)) = 1, \quad \gcd(p_0, g(x)) = 1, \text{ and } \gcd(d(x), g(x)) = 1. \quad (5)$$

Therefore, for the rest the paper we suppose that these four conditions hold for both types of sequence studied here. We use ρ to denote $\gcd(d(x), G_1(x))$. Notice that in the definition of Pell-Lucas we have $Q_0(x) = 2$ and $Q_1(x) = 2x$. Thus, the $\gcd(2, 2x) = 2 \neq 1$. Therefore, Pell-Lucas does not satisfy the extra conditions that we imposed in (5). So, to resolve this inconsistency we use $Q'_n(x) = Q_n(x)/2$ instead of $Q_n(x)$. Flórez et al. [4] have worked on similar problems for numerical sequences.

Polynomial of Lucas-type	$\mathcal{L}_n(x)$	$d(x)$	$g(x)$	Polynomial of Fibonacci-type	$\mathcal{F}_n(x)$	$d(x)$	$g(x)$
Lucas	$D_n(x)$	x	1	Fibonacci	$F_n(x)$	x	1
Pell-Lucas	$Q_n(x)$	$2x$	1	Pell	$P_n(x)$	$2x$	1
Fermat-Lucas	$\vartheta_n(x)$	x	-2	Fermat	$\Phi_n(x)$	x	-2
Chebyshev first kind	$T_n(x)$	$2x$	-1	Chebyshev second kind	$U_n(x)$	$2x$	-1
Jacobsthal-Lucas	$j_n(x)$	1	$2x$	Jacobsthal	$J_n(x)$	1	$2x$
Morgan-Voyce	$C_n(x)$	$x+2$	-1	Morgan-Voyce	$B_n(x)$	$x+2$	-1
Vieta-Lucas	$v_n(x)$	x	-1	Vieta	$V_n(x)$	x	-1

Table 1: $\mathcal{F}_n$ equivalent to $\mathcal{L}_n$.

2.2 Hosoya polynomial triangle

We now give a precise definition of both the Hosoya polynomial sequence and the Hosoya polynomial triangle. We recall that for brevity throughout the paper we present the polynomials without specifying the variable "x". For example, instead of $\mathcal{F}_n(x)$ we use $\mathcal{F}_n$.

Let p_0, p_1, d, and g be fixed polynomials as defined in (1) and (2). Then the *Hosoya polynomial* sequence $\{H(r,k)\}_{r,k\geq 0}$ is defined using the double recursion

$$H(r,k) = dH(r-1,k) + gH(r-2,k)$$

and

$$H(r,k) = dH(r-1,k-1) + gH(r-2,k-2),$$

where $r > 1$ and $0 \leq k \leq r-1$, with initial conditions

$$H(0,0) = p_0^2; \quad H(1,0) = p_0p_1; \quad H(1,1) = p_0p_1; \quad H(2,1) = p_1^2.$$

This sequence gives rise to the *Hosoya polynomial triangle*, where the entry in position k (taken from left to right), of the rth row is equal to $H(r,k)$ (see Table 2).

$$
\begin{array}{ccccccccccc}
 & & & & & H(0,0) & & & & & \\
 & & & & H(1,0) & & H(1,1) & & & & \\
 & & & H(2,0) & & H(2,1) & & H(2,2) & & & \\
 & & H(3,0) & & H(3,1) & & H(3,2) & & H(3,3) & & \\
 & H(4,0) & & H(4,1) & & H(4,2) & & H(4,3) & & H(4,4) & \\
H(5,0) & & H(5,1) & & H(5,2) & & H(5,3) & & H(5,4) & & H(5,5)
\end{array}
$$

Table 2: Hosoya polynomial triangle.

We say that the Hosoya triangle is of Fibonacci-type, denoted $H_{\mathcal{F}}$, if $p_0 = 0$ and $p_1 = 1$, and d and g are as in (1). Similarly, the Hosoya triangle of Lucas-type (denoted $H_{\mathcal{L}}$) can be defined.

In the definition of the Hosoya polynomial sequence the polynomials d, g, p_0, and p_1 can be any four polynomials in $\mathbb{Q}[x]$. Thus, these four polynomials need not be as defined in (1) and (2). However, in this paper we impose the restrictions above since we want a relationship between the sequences of Fibonacci-type and of Lucas-type for the Hosoya polynomial triangles. This relation is given by Proposition 1 (see Flórez et al. [3, 5]).

Proposition 1 ([3])**.** *If G_n is either $\mathcal{F}_n$ or $\mathcal{L}_n$ for all $n \geq 0$, then $H(r,k) = G_k G_{r-k}$.*

Proposition 1 implies that the entries of a Hosoya polynomial triangle are the product of two polynomials that are of the form as described in (1) or in (2). We observe that Table 2 together with this proposition give rise to Table 3.

$$
\begin{array}{ccccccccccc}
 & & & & & G_0\,G_0 & & & & & \\
 & & & & G_0\,G_1 & & G_1\,G_0 & & & & \\
 & & & G_0\,G_2 & & G_1\,G_1 & & G_2\,G_0 & & & \\
 & & G_0\,G_3 & & G_1\,G_2 & & G_2\,G_1 & & G_3\,G_0 & & \\
 & G_0\,G_4 & & G_1\,G_3 & & G_2\,G_2 & & G_3\,G_1 & & G_4\,G_0 & \\
G_0\,G_5 & & G_1\,G_4 & & G_2\,G_3 & & G_3\,G_2 & & G_4\,G_1 & & G_5\,G_0
\end{array}
$$

Table 3: $H(r,k) = G_k G_{r-k}$.

Some examples of $H(r,k)$ are in Table 4, obtained from Table 1 using Proposition 1. Therefore, some examples of Hosoya polynomial triangles can be constructed using Tables 3 and 4. It is enough to substitute each entry in Table 2 or Table 3 by the corresponding entry in Table 4. Thus, we obtain a Hosoya polynomial triangle for each of the specific polynomials mentioned in Table 1. So, Table 4 gives rise to 14 examples of Hosoya polynomial triangles.

For example, using the first polynomial in Table 4 and Proposition 1 in Table 3 we obtain the Hosoya polynomial triangle H_F where the entry $H(r,k)$ is equal to $F_k(x)F_{r-k}(x)$. This is represented in Table 5 without the points that contain the factor $F_0(x) = 0$.

Observe that $H(r,k)$ in the first column of Table 4 is a product of polynomials of Fibonacci-type. Therefore, $G_0 = 0$. So, the edges containing G_0 as a factor in Table 3,

$H(r,k)$	p_0	p_1	d	g	$H(r,k)$	p_0	p_1	d	g
$F_k(x)F_{r-k}(x)$	0	1	x	1	$D_k(x)D_{r-k}(x)$	2	$2x$	x	1
$P_k(x)P_{r-k}(x)$	0	1	$2x$	1	$Q_k(x)Q_{r-k}(x)$	2	$2x$	$2x$	1
$\Phi_k(x)\Phi_{r-k}(x)$	0	1	x	-2	$\vartheta_k(x)\vartheta_{r-k}(x)$	2	$3x$	x	-2
$U_k(x)U_{r-k}(x)$	0	1	$2x$	-1	$T_k(x)T_{r-k}(x)$	1	x	$2x$	-1
$J_k(x)J_{r-k}(x)$	0	1	1	$2x$	$j_k(x)j_{r-k}(x)$	2	1	1	$2x$
$B_k(x)B_{r-k}(x)$	0	1	$x+2$	-1	$C_k(x)C_{r-k}(x)$	2	$x+2$	$x+2$	-1
$V_k(x)V_{r-k}(x)$	0	1	x	-1	$v_k(x)v_{r-k}(x)$	2	x	x	-1

Table 4: Terms $H(r,k)$ of the Hosoya polynomial triangle.

will have entries equal to zero. From the sixth column of Table 4 we see that $H(r,k)$ is a product of polynomials of Lucas-type. So, the edges containing G_0 as a factor in Table 3 will not have entries equal to zero.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & x & & x & & & \\
 & & x^2+1 & & x^2 & & x^2+1 & & \\
 & x^3+2x & & x(x^2+1) & & x(x^2+1) & & x^3+2x & \\
x^4+3x^2+1 & & x(x^3+2x) & & (x^2+1)^2 & & x(x^3+2x) & & x^4+3x^2+1
\end{array}
$$

Table 5: The Hosoya triangle H_F where $H(r,k) = F_k(x)F_{r-k}(x)$.

2.3 Star of David property in the Hosoya polynomial triangle

In this subsection we state the main results, namely the star of David properties for both type Hosoya polynomial triangle, Lucas-type and Fibonacci-type. These properties hold in the Pascal triangle, the Fibonomial triangle, the gibonomial triangle, and in both the Hosoya and the generalized Hosoya triangle.

Koshy [16, Chapters 6 and 26] discussed that some properties of star of David are present in several triangular arrays. These properties — called the *Hoggatt-Hansell* identity, the *Gould* property, or *gcd* property — were also proved in [5, 6] for Hosoya and generalized Hosoya triangles. The results in this paper generalize several results in the articles [5, 6, 11, 16] that were proved for numerical sequences.

Those familiar with the gibonomial triangle (see Koshy [13] or Sagan [19]), may find it interesting that the gcd property also holds in this triangle. The proof of this fact follows by adapting the numerical proof in Hillman and Hoggatt [7], to polynomials.

From Figure 2 we can see that the star of David is formed by two triangles. For the rest of paper when we refer to the star of David we assume that it is embedded in a Hosoya polynomial triangle. We show that the product of points in one triangle equals the product of points in the second triangle. We also find conditions that ensure that the gcd of the points in the leftmost triangle are equal to the gcd of the points in the rightmost triangle (this is true if $\gcd(\rho, G_n/\rho) = 1$, where $\rho = \gcd(d, G_1)$ and G_n is either $\mathcal{F}_n$ or $\mathcal{L}_n$). For example, the polynomials in Table 1 that satisfy this condition are: Fibonacci, Lucas, Pell-Lucas, Chebyshev first kind, Jacobsthal, Jacobsthal-Lucas, and both Morgan-Voyce polynomials. The polynomials in Table 1 that satisfy $\gcd(\rho^2, G_n) \neq 1$ are: Pell, Fermat, Fermat-Lucas, and Chebyshev second kind.

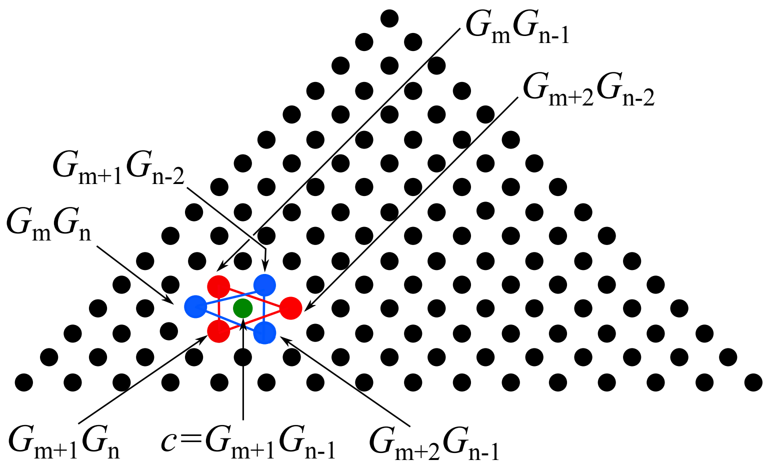

Figure 2: Star of David in a Hosoya triangle where G_k is either $\mathcal{F}_k$ or $\mathcal{L}_k$ for all $k \geq 0$.

In the following three theorems we generalize the *Hoggatt-Hansell* identity and *Gould* property to polynomials. We also analyze the relationship between the point that is within the two triangles of the star of David (see the point c in Figure 2) and the two diagonals of the star of David. We now state the main results — for their proofs see Section 3 page 12. We recall that for brevity we always suppose that the star of David is embedded in a Hosoya polynomial triangle.

Theorem 2. *Suppose that $\mathcal{F}_{m+1}\mathcal{F}_{n-2}$, $\mathcal{F}_m\mathcal{F}_n$, and $\mathcal{F}_{m+2}\mathcal{F}_{n-1}$ are the points in a triangle of the star of David and $\mathcal{F}_m\mathcal{F}_{n-1}$, $\mathcal{F}_{m+2}\mathcal{F}_{n-2}$, and $\mathcal{F}_{m+1}\mathcal{F}_n$ are the points in the second triangle of the star of David. If $m \geq 1$ and $n > 1$, then*

(1) $\gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_m\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-1})$ *is equal to*

$$\begin{cases} \beta \gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+2}\mathcal{F}_{n-2}, \mathcal{F}_{m+1}\mathcal{F}_n), & \text{if } m \text{ and } n \text{ are both even;} \\ \gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+2}\mathcal{F}_{n-2}, \mathcal{F}_{m+1}\mathcal{F}_n), & \text{otherwise,} \end{cases}$$

where β is a constant that depends on d, m, and n.

(2) Let $c = \mathcal{F}_{m+1}\mathcal{F}_{n-1}$ be the point within the two triangles of the star of David. Then $\gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_{m+1}\mathcal{F}_n) \cdot \gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+2}\mathcal{F}_{n-1})$ *is equal to either c, $c\mathcal{F}_2$, or $c\mathcal{F}_2^2$.*

Theorem 3. *Suppose that $\mathcal{L}_{m+1}\mathcal{L}_{n-2}$, $\mathcal{L}_m\mathcal{L}_n$, and $\mathcal{L}_{m+2}\mathcal{L}_{n-1}$ are the points in a triangle of the star of David and $\mathcal{L}_m\mathcal{L}_{n-1}$, $\mathcal{L}_{m+2}\mathcal{L}_{n-2}$, and $\mathcal{L}_{m+1}\mathcal{L}_n$ are the points in the second triangle of the star of David. If $m \geq 0$ and $n \geq 0$ and $\mathcal{L}_m\mathcal{L}_n \neq \mathcal{L}_0\mathcal{L}_0$, then*

(1) $\gcd(\mathcal{L}_{m+1}\mathcal{L}_{n-2}, \mathcal{L}_m\mathcal{L}_n, \mathcal{L}_{m+2}\mathcal{L}_{n-1})$ *is equal to*

$$\begin{cases} \beta' \gcd(\mathcal{L}_m\mathcal{L}_{n-1}, \mathcal{L}_{m+2}\mathcal{L}_{n-2}, \mathcal{L}_{m+1}\mathcal{L}_n), & \text{if } m \text{ and } n \text{ are both even;} \\ \gcd(\mathcal{L}_m\mathcal{L}_{n-1}, \mathcal{L}_{m+2}\mathcal{L}_{n-2}, \mathcal{L}_{m+1}\mathcal{L}_n), & \text{otherwise,} \end{cases}$$

where β' is a constant that depends on $\mathcal{L}_1, m$, and n.

(2) Let $c = \mathcal{L}_{m+1}\mathcal{L}_{n-1}$ be the point within the two triangles of the star of David. Then $\gcd(\mathcal{L}_{m+1}\mathcal{L}_{n-2}, \mathcal{L}_{m+1}\mathcal{L}_n) \cdot \gcd(\mathcal{L}_m\mathcal{L}_{n-1}, \mathcal{L}_{m+2}\mathcal{L}_{n-1})$ *is equal to either c, $c\mathcal{L}_1$, or $c\mathcal{L}_1^2$.*

Theorem 4. *Suppose that G_k is either $\mathcal{F}_k$ or $\mathcal{L}_k$ for all $k \geq 0$. If $G_{m+1}G_{n-2}$, G_mG_n, and $G_{m+2}G_{n-1}$ are the points in a triangle of the star of David and G_mG_{n-1}, $G_{m+2}G_{n-2}$, and $G_{m+1}G_n$ are the points in the second triangle of the star of David, where $G_m\, G_n \neq G_0\, G_0$, then*

$$(G_{m+1}G_{n-2}) \cdot (G_mG_n) \cdot (G_{m+2}G_{n-1}) = (G_mG_{n-1}) \cdot (G_{m+2}G_{n-2}) \cdot (G_{m+1}G_n).$$

3 Proof of the main theorems

In this section we prove Theorems 2 and 3. The proof of Theorem 4 is straightforward. In addition, we present some corollaries of the main theorems, a few divisibility properties, and gcd properties that are true for both types of polynomial sequences. Proposition 5 is a generalization of [6, Proposition 2.2], both proofs are similar.

Proposition 5. *Let a, b, c and d be polynomials in $\mathbb{Q}[x]$.*

(1) If $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$, then

$$\gcd(ab, cd) = \gcd(a, c) \cdot \gcd(a, d) \cdot \gcd(b, c) \cdot \gcd(b, d).$$

(2) If $\gcd(a, c) = \gcd(b, d) = 1$, then $\gcd(ab, cd) = \gcd(a, d) \cdot \gcd(b, c)$.

Proof. The proof of Part (1) follows from the multiplication property of the gcd. The proof of Part (2) follows from [6, Proposition 2.2] by replacing a, b, c and d integers by a, b, c and d polynomials in $\mathbb{Q}[x]$. □

Proposition 6. *If G_i is either $\mathcal{F}_i$ or $\mathcal{L}_i$ for all $i \geq 0$, then*

$$G_m \bmod d^2 = \begin{cases} g^{k-1}\left(kdG_1 + gG_0\right), & \text{if } m = 2k; \\ g^k\left(kdG_0 + G_1\right), & \text{if } m = 2k+1. \end{cases}$$

Proof. We use mathematical induction. Let $S(m)$ be the statement

$$G_m \bmod d^2 = \begin{cases} g^{t-1}\left(tdG_1 + gG_0\right), & \text{if } m = 2t; \\ g^{t}\left(tdG_0 + G_1\right), & \text{if } m = 2t+1. \end{cases}$$

The basis step, $S(1)$ and $S(2)$, follows from the following two facts;

$$G_1 \equiv G_1 = g^0\left(0dG_0 + G_1\right) \bmod d^2$$

and

$$G_2 \equiv G_2 = g^0\left(dG_1 + gG_0\right) \bmod d^2.$$

We suppose that $S(m)$ is true for $m = 2k$ and $m = 2k+1$. The proof of $S(m+1)$ requires two cases, we prove the case for $m+1 = 2k+2$, the case $m+1 = 2k+3$ is similar and is omitted. We know that $G_{m+1} = dG_m + gG_{m-1}$. Thus, $G_{2k+2} = dG_{2k+1} + gG_{2k}$. This and the inductive hypothesis imply that $G_{2k+2} \bmod d^2$ is

$$d\left(g^k\left(kdG_0 + G_1\right)\right) + g\left(g^{k-1}\left(kdG_1 + gG_0\right)\right).$$

Simplifying, we obtain

$$G_{2(k+1)} \equiv g^k\left((k+1)dG_1 + gG_0\right) \pmod{d^2}.$$

This completes the proof. $\square$

Lemma 7 ([2]). *If m and n are positive integers, $\mathcal{F}_t$ is a Fibonacci-type polynomial, and $\mathcal{L}_t$ is a Lucas-type polynomial, then these hold*

(1) $\gcd(d, \mathcal{F}_{2n+1}) = \mathcal{F}_1$ *and* $\gcd(d, \mathcal{L}_{2n+1}) = \mathcal{L}_1$.

(2) $\gcd(d, \mathcal{F}_{2n}) = d$ *and* $\gcd(d, \mathcal{L}_{2n}) = 1$.

(3) $\gcd(g, \mathcal{F}_n) = \gcd(g, \mathcal{F}_1) = 1$ *and* $\gcd(g, \mathcal{L}_n) = \gcd(g, \mathcal{L}_1) = 1$.

(4) If $0 < |m-n| \le 2$, *then*

$$\gcd(\mathcal{L}_m, \mathcal{L}_n) = \begin{cases} \alpha^{-1}d, & \textit{if } m \textit{ and } n \textit{ are both odd;} \\ 1, & \textit{otherwise.} \end{cases}$$

(5) If $0 < |m-n| \le 2$, *then*

$$\gcd(\mathcal{F}_m, \mathcal{F}_n) = \begin{cases} d, & \textit{if } m \textit{ and } n \textit{ are both even;} \\ 1, & \textit{otherwise.} \end{cases}$$

Lemma 8. *Suppose that G_k is either $\mathcal{F}_k$ or $\mathcal{L}_k$ for all $k \geq 0$. Let $G_{m+1}G_{n-2}$, G_mG_n, and $G_{m+2}G_{n-1}$ be the points in a triangle of the star of David and G_mG_{n-1}, $G_{m+2}G_{n-2}$, and $G_{m+1}G_n$ be the points in the second triangle of the star of David, with m and n positive integers where $G_m\, G_n \neq G_0\, G_0$. If $\Delta_t = \gcd(G_t, G_{t-2})$, then*

$$\gcd(G_mG_{n-1}, G_{m+1}G_n, G_{m+2}G_{n-2}) = \gcd(G_n, G_m, \Delta_m\Delta_n)$$

and

$$\gcd(G_{m+1}G_{n-2}, G_mG_n, G_{m+2}G_{n-1}) = \gcd(G_{n-2}, G_{m+2}, \Delta_n\Delta_m).$$

Proof. We prove that

$$\gcd(G_mG_{n-1}, G_{m+1}G_n, G_{m+2}G_{n-2}) = \gcd(G_n, G_m, \Delta_m\Delta_n).$$

From Lemma 5 Part (2) we have

$$\gcd(G_mG_{n-1}, G_{m+1}G_n) = \gcd(G_m, G_n) \cdot \gcd(G_{n-1}G_{m+1}).$$

Therefore,

$$\begin{aligned}
\gcd\left(G_mG_{n-1}, G_{m+1}G_n, G_{m+2}G_{n-2}\right) &= \gcd\left(\gcd\left(G_mG_{n-1}, G_{m+1}G_n\right), G_{m+2}G_{n-2}\right)\\
&= \gcd\left(\left(\gcd(G_m, G_n) \cdot \gcd\left(G_{n-1}G_{m+1}\right)\right), G_{m+2}G_{n-2}\right).
\end{aligned}$$

From Lemma 7 Parts (4) and (5) we know that

$$\gcd(G_{m+2}G_{n-2}, \gcd(G_{n-1}, G_{m+1})) = 1.$$

So,

$$\begin{aligned}
\gcd\left(G_mG_{n-1}, G_{m+1}G_n, G_{m+2}G_{n-2}\right) &= \gcd\left(\gcd\left(G_m, G_n\right), G_{m+2}, G_{n-2}\right)\\
&= \gcd\left(G_m, G_n, G_{m+2}G_{n-2}\right)\\
&= \gcd(G_m, \gcd(G_n, G_{m+2}G_{n-2})).
\end{aligned}$$

This and Lemma 5 imply that

$$\begin{aligned}
\gcd(G_mG_{n-1}, G_{m+1}G_n, G_{m+2}G_{n-2}) &= \gcd(G_m, \gcd(G_n, G_{m+2}\Delta_n))\\
&= \gcd(G_n, \gcd(G_m, G_{m+2}\Delta_n))\\
&= \gcd(G_n, \gcd(G_m, \Delta_m\Delta_n))\\
&= \gcd(G_n, G_m, \Delta_m\Delta_n).
\end{aligned}$$

Similarly, we have $\gcd(G_{m+1}G_{n-2}, G_mG_n, G_{m+2}G_{n-1}) = \gcd(G_{n-2}, G_{m+2}, \Delta_n\Delta_m)$. $\square$

3.1 Proof of the main theorems

Proof of Theorem 2. If in Lemma 8 we consider $G_n = \mathcal{F}_n$, we have

$$\gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+1}\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-2}) = \gcd(\mathcal{F}_n, \mathcal{F}_m, \Delta_m\Delta_n) \tag{6}$$

and

$$\gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_m\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-1}) = \gcd(\mathcal{F}_{n-2}, \mathcal{F}_{m+2}, \Delta_m\Delta_n), \tag{7}$$

where $\Delta_t = \gcd(\mathcal{F}_t, \mathcal{F}_{t-2})$.

For this proof we consider three cases depending on the parity of m and n.

Case m and n are odd. From Lemma 7 Part (5) we have $\Delta_m = \Delta_n = 1$. This, (6), and (7) imply that

$$\gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+1}\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-2}) = 1$$

and

$$\gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_m\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-1}) = 1.$$

Case m and n have different parity. From Lemma 7 Part (5) we have $\Delta_m\Delta_n = d$. This, (6), and (7) imply that

$$\gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+1}\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-2}) = \gcd(\mathcal{F}_n, \mathcal{F}_m, d)$$

and

$$\gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_m\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-1}) = \gcd(\mathcal{F}_{n-2}, \mathcal{F}_{m+2}, d).$$

From Lemma 7 Part (1) we have $\gcd(\mathcal{F}_n, \mathcal{F}_m, d) = 1 = \gcd(\mathcal{F}_{n-2}, \mathcal{F}_{m+2}, d)$. Therefore, $\gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+1}\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-2}) = \gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_m\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-1}) = 1$.

Case both m and n are even. Suppose that $n = 2k_1$ and $m = 2k_2$ for some $k_1, k_2 \in \mathbb{N}$. So, from Lemma 7 Part (5) we have that $\Delta_m = \Delta_n = d$. Since $\mathcal{F}_0 = 0$ and $\mathcal{F}_1 = 1$, by Proposition 6 we have

$$\mathcal{F}_{2k_1} \equiv k_1 g^{k_1-1} d \pmod{d^2},$$
$$\mathcal{F}_{2k_2} \equiv k_2 g^{k_2-1} d \pmod{d^2}.$$

This and $\gcd(d, g) = 1$ imply that

$$\gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+1}\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-2}) = \gcd(k_1 g^{k_1-1} d, k_2 g^{k_2-1} d, d^2) = d\gcd(d, k_1, k_2).$$

Similarly we have that $\gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_m\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-1}) = d\gcd(d, k_1 - 1, k_2 + 1)$.

Let $\beta = (\gcd(d, k_1 - 1, k_2 + 1)) \,/\, (\gcd(d, k_1, k_2))$. Therefore,

$$\gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_m\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-1}) = \beta\gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+1}\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-2}).$$

We now prove Part (2). Factoring, we have that

$$\gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_{m+1}\mathcal{F}_n) \cdot \gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+2}\mathcal{F}_{n-1})$$

is equal to

$$\mathcal{F}_{m+1}\mathcal{F}_{n-1} \cdot \gcd(\mathcal{F}_{n-2}, \mathcal{F}_n) \cdot \gcd(\mathcal{F}_m, \mathcal{F}_{m+2}).$$

The conclusion follows using Lemma 7 Part (5). $\square$

Proof of Theorem 3. In Lemma 8 if we take $G_n = \mathcal{L}_n$, we have

$$\gcd(\mathcal{L}_m\mathcal{L}_{n-1}, \mathcal{L}_{m+1}\mathcal{L}_n, \mathcal{L}_{m+2}\mathcal{L}_{n-2}) = \gcd(\mathcal{L}_n, \mathcal{L}_m, \Delta_m\Delta_n) \tag{8}$$

and

$$\gcd(\mathcal{L}_{m+1}\mathcal{L}_{n-2}, \mathcal{L}_m\mathcal{L}_n, \mathcal{L}_{m+2}\mathcal{L}_{n-1}) = \gcd(\mathcal{L}_{n-2}, \mathcal{L}_{m+2}, \Delta_n\Delta_m),$$

where $\Delta_t = \gcd(\mathcal{L}_t, \mathcal{L}_{t-2})$. If m and n are not both odd, then the proof follows in a similar way as in the proof of Theorem 2.

Suppose that both m and n are odd, that is $n = 2k_1 + 1$ and $m = 2k_2 + 1$ where k_1, k_2 are non-negative integers. Therefore, by Lemma 7 Part (4) we know that $\Delta_m = \Delta_n = \mathcal{L}_1$. Since $\mathcal{L}_1 | d$, by Proposition 6 we have

$$\mathcal{L}_n \equiv n g^{k_1} \mathcal{L}_1 \pmod{\mathcal{L}_1^2}$$

and

$$\mathcal{L}_m \equiv m g^{k_2} \mathcal{L}_1 \pmod{\mathcal{L}_1^2}.$$

This and (8) imply that

$$\gcd(\mathcal{L}_m\mathcal{L}_{n-1}, \mathcal{L}_{m+1}\mathcal{L}_n, \mathcal{L}_{m+2}\mathcal{L}_{n-2}) = \gcd(n g^{k_1}\mathcal{L}_1, m g^{k_2}\mathcal{L}_1, (\mathcal{L}_1)^2).$$

This and $\gcd(d, g) = 1$ imply that $\gcd(\mathcal{L}_m\mathcal{L}_{n-1}, \mathcal{L}_{m+1}\mathcal{L}_n, \mathcal{L}_{m+2}\mathcal{L}_{n-2}) = \mathcal{L}_1 \gcd(n, m, \mathcal{L}_1)$.

Similarly we can prove that

$$\gcd(\mathcal{L}_{m+1}\mathcal{L}_{n-2}, \mathcal{L}_m\mathcal{L}_n, \mathcal{L}_{m+2}\mathcal{L}_{n-1}) = \mathcal{L}_1 \gcd(\mathcal{L}_1, n-2, m+2).$$

Let $\beta' = (\gcd(\mathcal{L}_1, n-2, m+2)) \,/\, (\gcd(\mathcal{L}_1, n, m))$. Then,

$$\gcd(\mathcal{L}_{m+1}\mathcal{L}_{n-2}, \mathcal{L}_m\mathcal{L}_n, \mathcal{L}_{m+2}\mathcal{L}_{n-1}) = \beta' \gcd(\mathcal{L}_m\mathcal{L}_{n-1}, \mathcal{L}_{m+1}\mathcal{L}_n, \mathcal{L}_{m+2}\mathcal{L}_{n-2}).$$

We now prove Part (2). Factoring, we have that

$$\gcd(\mathcal{L}_{m+1}\mathcal{L}_{n-2}, \mathcal{L}_{m+1}\mathcal{L}_n) \gcd(\mathcal{L}_m\mathcal{L}_{n-1}, \mathcal{L}_{m+2}\mathcal{L}_{n-1})$$

is equal to

$$\mathcal{L}_{m+1}\mathcal{L}_{n-1} \gcd(\mathcal{L}_{n-2}, \mathcal{L}_n) \gcd(\mathcal{L}_m, \mathcal{L}_{m+2}).$$

The conclusion follows using Lemma 7 Part (4). $\square$

3.2 Corollaries of the main theorem

Theorems 2, 3, and 4 are also true for the star of David with a vertical configuration as depicted in Figure 3 (with similar proofs). The following corollaries are a formalization of some results that are in the proofs of Theorems 2 and 3. For the following three corollaries we suppose that the points are as given in Theorems 2 and 3 and Figure 2.

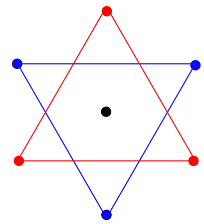

Figure 3: Vertical star of David.

Corollary 9. *Let G_t be one of the following polynomials: Fibonacci, Lucas, Jacobsthal, Jacobsthal-Lucas, Chebyshev first kind polynomials, Pell-Lucas, and both Morgan-Voyce polynomials, for every $t \in \mathbb{N}$. If $G_{m+1}G_{n-2}$, G_mG_n, and $G_{m+2}G_{n-1}$ are the points in a triangle of the star of David and G_mG_{n-1}, $G_{m+2}G_{n-2}$, and $G_{m+1}G_n$ are the points in the second triangle of the star of David, then*

$$\gcd(G_{m+1}G_{n-2}, G_mG_n, G_{m+2}G_{n-1}) = \gcd(G_mG_{n-1}, G_{m+2}G_{n-2}, G_{m+1}G_n).$$

Corollary 10. *Suppose that $\mathcal{F}_{m+1}\mathcal{F}_{n-2}$, $\mathcal{F}_m\mathcal{F}_n$, and $\mathcal{F}_{m+2}\mathcal{F}_{n-1}$ are the points in a triangle of the star of David and $\mathcal{F}_m\mathcal{F}_{n-1}$, $\mathcal{F}_{m+2}\mathcal{F}_{n-2}$, and $\mathcal{F}_{m+1}\mathcal{F}_n$ are the points in the second triangle of the star of David. If $n = 2k_1$ and $m = 2k_2$ where $k_1, k_2 \in \mathbb{N}$, then the these hold*

(1) if $n \geq 0$ and $\mathcal{F}_n$ is a Pell polynomial or a Chebyshev polynomial of the second kind with $k_1k_2 \not\equiv 0 \pmod 4$ and $k_1 \not\equiv k_2 \pmod 2$, then

$$\gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_m\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-1}) = \gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+2}\mathcal{F}_{n-2}, \mathcal{F}_{m+1}\mathcal{F}_n).$$

(2) If $n \geq 0$ and $\mathcal{F}_n$ is a Fermat polynomial with $k_1k_2 \not\equiv 0 \pmod 9$ and $k_1 \not\equiv 2k_2 \pmod 3$, then

$$\gcd(\mathcal{F}_{m+1}\mathcal{F}_{n-2}, \mathcal{F}_m\mathcal{F}_n, \mathcal{F}_{m+2}\mathcal{F}_{n-1}) = \gcd(\mathcal{F}_m\mathcal{F}_{n-1}, \mathcal{F}_{m+2}\mathcal{F}_{n-2}, \mathcal{F}_{m+1}\mathcal{F}_n).$$

Corollary 11. *Suppose that $\mathcal{L}_{m+1}\mathcal{L}_{n-2}$, $\mathcal{L}_m\mathcal{L}_n$, and $\mathcal{L}_{m+2}\mathcal{L}_{n-1}$ are the points in a triangle of the star of David and $\mathcal{L}_m\mathcal{L}_{n-1}$, $\mathcal{L}_{m+2}\mathcal{L}_{n-2}$, and $\mathcal{L}_{m+1}\mathcal{L}_n$ are the points in the second triangle of the star of David. If $m, n \geq 0$, $\mathcal{L}_t$ is a Fermat-Lucas polynomial for $t \geq 0$, and $\mathcal{L}_m\mathcal{L}_n \neq \mathcal{L}_0\mathcal{L}_0$, then*

$$\gcd(\mathcal{L}_{m+1}\mathcal{L}_{n-2}, \mathcal{L}_m\mathcal{L}_n, \mathcal{L}_{m+2}\mathcal{L}_{n-1}) = \gcd(\mathcal{L}_m\mathcal{L}_{n-1}, \mathcal{L}_{m+2}\mathcal{L}_{n-2}, \mathcal{L}_{m+1}\mathcal{L}_n).$$

4 The geometry of some identities

The aim of this section is to give geometric interpretations of some polynomial identities that are known for the Fibonacci numbers. The novelty of this section is that we extend some well-known numerical identities to $\{\mathcal{F}_k\}$ and to $\{\mathcal{L}_k\}$ sequences and provide geometric proofs for these identities instead of the classical mathematical induction proofs.

Hosoya-type triangles (polynomial and numeric) are good tools to discover, prove, or represent theorems geometrically. Some properties that have been found and proved algebraically are easy to understand when interpreted geometrically using these triangles.

4.1 Identities in the Hosoya polynomial triangle

Lemma 12. *If i, j, k, and r are nonnegative integers with $k + j \leq r$, then in the Hosoya polynomial triangle this holds*

$$H(r + 2i, k + j + i) - H(r + 2i, k + i) = (-1)^i g(H(r, k + j) - H(r, k)).$$

The proof of the Lemma 12 follows using induction and the rectangle property which states that $H(n, m) = dH(n - 1, m) + gH(n - 2, m)$ (see Figure 4).

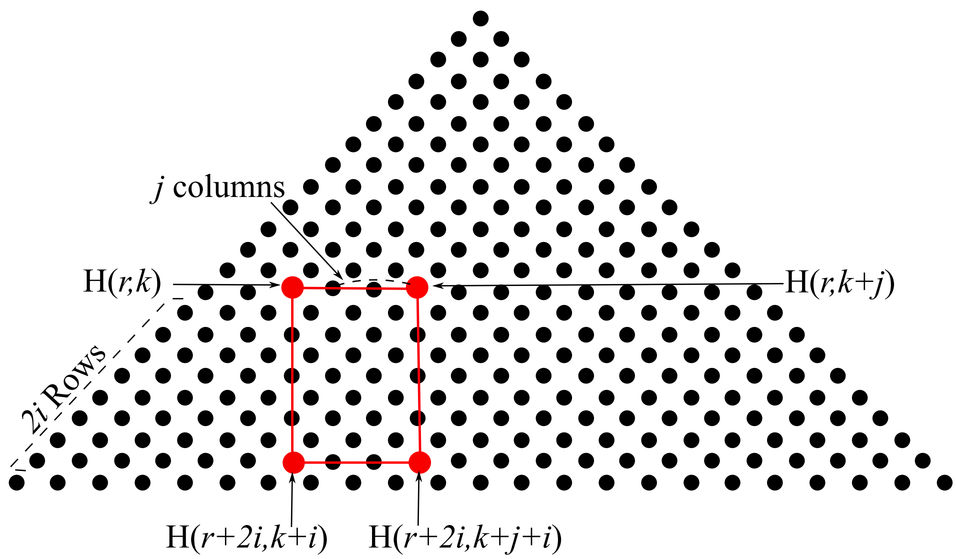

Figure 4: Property of Rectangle.

It is well known that the Catalan identity is a generalization of the Cassini identity. Johnson [12], gives another numerical generalization of the Cassini and Catalan identities, called the Johnson identity. It states that for the Fibonacci number sequence $\{F_n\}$,

$$F_a F_b - F_c F_d = (-1)^r \left(F_{a-r} F_{b-r} - F_{c-r} F_{d-r}\right)$$

where a, b, c, d, and r are arbitrary integers with $a + b = c + d$.

The example in Figure 5 gives a geometric representation of the numeric identities (the same representation holds for polynomials). To represent the Cassini identity we take two consecutive points in the Hosoya triangle along a horizontal line such that one point is located in the central column of the triangle, see Figure 5. We then pick two other arbitrary consecutive points P_1 and P_2 such that they form a vertical rectangle along with the first pair of points. The subtraction of the horizontal points P_1 and P_2 gives ± 1. Since the entries of the triangle are products of Fibonacci numbers, we obtain the Cassini identity.

The second example in Figure 5 represents the Catalan identity. In this case we take any two horizontal points Q_1 and Q_2 where Q_1 is located (arbitrarily) in the central column of the triangle. We then pick other two arbitrary points P_1 and P_2 which form a rectangle with Q_1 and Q_2. The subtraction of the horizontal points P_1 and P_2 gives $\pm(Q_1 - Q_2)$. Since the entries of the triangle are products of Fibonacci numbers, we obtain the Catalan identity. Note that if we eliminate the condition that Q_1 must be in the central column, we obtain the Johnson identity.

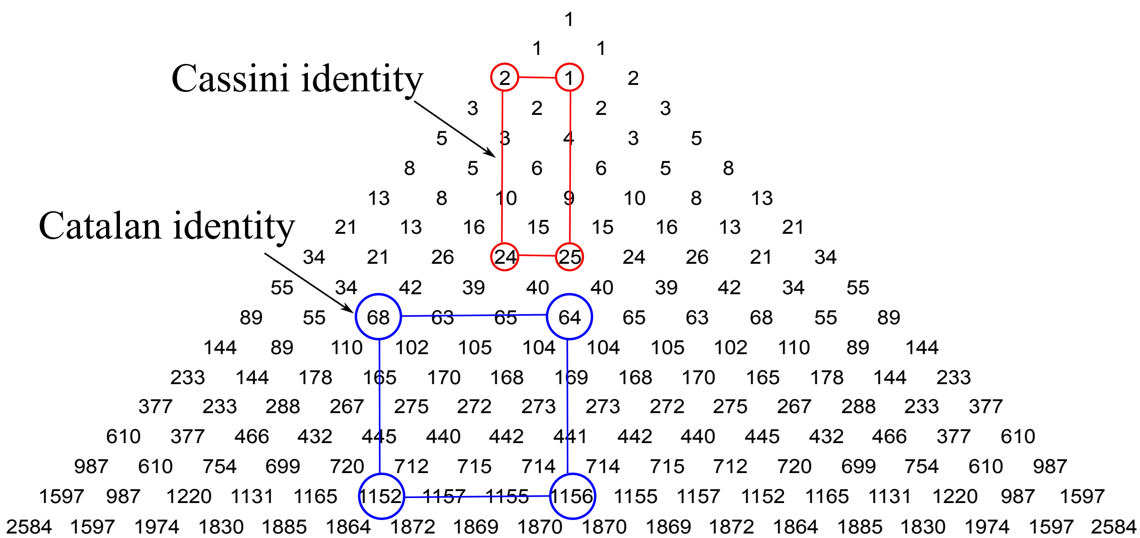

Figure 5: Cassini and Catalan identities in the Hosoya Triangle.

Theorem 13. *Let a, b, c, d and t be nonnegative integers with $\min\{a, b, c, d\} - t$ non-negative. Suppose that G_k is either $\mathcal{F}_k$ or $\mathcal{L}_k$ for all $k \geq 0$. If $a + b = c + d$, then*

$$\begin{vmatrix} G_a & G_c \\ G_d & G_b \end{vmatrix} = (-1)^t g^t \begin{vmatrix} G_{a-t} & G_{c-t} \\ G_{d-t} & G_{b-t} \end{vmatrix}.$$

Proof. Let i, j, k, and r be nonnegative integers such that $a = k + j + i$, $b = r + i - k - j$, $c = k + i$, $d = r + i - k$, and $t = i$. Therefore, by Lemma 12 and Proposition 1 the equality holds. $\square$

Theorem 13 is a generalization of Johnson identity [12] and Falcón and Plaza identity [1]. As a consequence of Theorem 13 we state Corollary 14 — this generalizes the Catalan identity to $\mathcal{F}_k$ and $\mathcal{L}_k$. If in Corollary 14 we take $r = 1$, then we obtain a generalization of Cassini identity.

Corollary 14 (Catalan identity)**.** *Suppose that m, r are non-negative integers. If G_k is either $\mathcal{F}_k$ or $\mathcal{L}_k$ for all $k \geq 0$, then*

$$\begin{vmatrix} G_m & G_{m+r} \\ G_{m-r} & G_m \end{vmatrix} = (-1)^{m-r} g^{m-r} \begin{vmatrix} G_r & G_{2r} \\ G_0 & G_r \end{vmatrix}.$$

Proof. The proof is straightforward when the appropriate values of m and r are substituted in Theorem 13 (see Figure 5). If we evaluate both determinants in Theorem 13 we obtain four summands that are four points in the Hosoya polynomial triangle. Note that these four points are the vertices of a rectangle in the Hosoya triangle. □

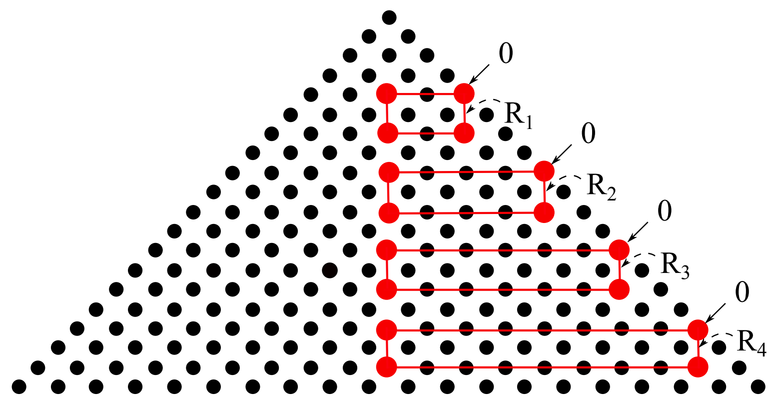

Figure 6: Geometric interpretation of Theorem 15.

We observe that if we have a Hosoya triangle where the entries are products of two polynomial of $\{\mathcal{F}_k\}$, then we can draw rectangles with two vertices in the central line (the perpendicular bisector) of the triangle and a third vertex on the edge of the triangle (see Figure 6). For a fixed $i \in \mathbb{N}$ let R_i be a rectangle with the extra condition that the upper vertex points are multiplied by g, then Lemma 12 guarantees that the sum of the two top vertices of R_i is equal to the sum of the remaining vertices of R_i. Since the points in the edge of this triangle are equal to zero, one of the vertices of R_i is equal to zero. The other vertex in the same vertical line is a polynomial $\mathcal{F}_i$ multiplied by one. This geometry gives rise to Theorem 15.

For the next result we introduce the following function. We recall that g is as defined in (1).

$$I(n) = \begin{cases} g, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 15. *If n and k are positive integers, then*

$$\sum_{j=2}^{2n+1} I(j)\mathcal{F}_j^2 = \sum_{j=1}^{n} \mathcal{F}_{4j+1}$$

and

$$\sum_{j=2}^{2n+1} (-1)^{j+1} I^2(j)\mathcal{F}_{2j}^2 = d\sum_{j=1}^{n} \mathcal{F}_{8j+2}.$$

Proof. First of all we recall that $\mathcal{F}_1 = 1$. We prove the first identity.

$$\begin{aligned}\sum_{j=2}^{2n+1} I(j)\mathcal{F}_j^2 &= \sum_{j=1}^{n}(\mathcal{F}_{2j+1}^2 + g\mathcal{F}_{2j}^2) = \sum_{j=1}^{n}(\mathcal{F}_{4j+1}\mathcal{F}_1 + \mathcal{F}_0\mathcal{F}_{4j}) \\ &= \sum_{j=1}^{n} \mathcal{F}_{4j+1}.\end{aligned}$$

We now prove the second identity. Let $S := \sum_{j=2}^{2n+1}(-1)^{j+1}I^2(j)\mathcal{F}_{2j}^2$. Lemma 12 implies that

$$\begin{aligned} S &= \sum_{j=1}^{n}(\mathcal{F}_{4j+2}^2 - g^2\mathcal{F}_{4j}^2) \\ &= \sum_{j=1}^{n}\left(\left(\mathcal{F}_{4j+2}^2 + g\mathcal{F}_{4j+1}^2\right) - g\left(\mathcal{F}_{4j+1}^2 + g\mathcal{F}_{4j}^2\right)\right).\end{aligned}$$

Since $\mathcal{F}_1 = 1$, we have

$$\begin{aligned} S &= \sum_{j=1}^{n}\left(\left(\mathcal{F}_{8j+3} + g\mathcal{F}_{8j+1}\mathcal{F}_0\right) - g\left(\mathcal{F}_{8j+1} + g\mathcal{F}_{8j-1}\mathcal{F}_0\right)\right) \\ &= \mathcal{F}_1\sum_{j=1}^{n}\left(\mathcal{F}_{8j+3} - g\mathcal{F}_{8j+1}\right) = d\sum_{j=1}^{n}\mathcal{F}_{8j+2}.\end{aligned}$$

This completes the proof. $\square$

Corollary 16 provides a closed formula for special cases of Theorem 15. We use Figure 7 to give a geometric interpretation of Corollary 16. For brevity we only give an algebraic proof of Part (1), the algebraic proof of Part (2) is similar, therefore it is omitted, and instead we provide a geometric proof of Part (2). This gives us the geometric behavior of a zigzag pattern of points. Thus, Corollary 16 Part (2) states that the sum of all points that are in the intersection of a finite zigzag configuration and the central line of the triangle is the last point of the zigzag configuration (see Figure 7).

Corollary 16. *Suppose that g is as defined in (1). Then these hold*

(1)

$$\sum_{j=1}^{n} g^{2n-j}\mathcal{F}_{4j-3} = \frac{\mathcal{F}_{2n-1}\mathcal{F}_{2n}}{d}.$$

(2) If (in particular) the sequence $\{\mathcal{F}_k\}$ satisfies that $g = 1$, then

$$\sum_{j=1}^{2n-1} \mathcal{F}_j^2 = \frac{\mathcal{F}_{2n-1}\mathcal{F}_{2n}}{d}.$$

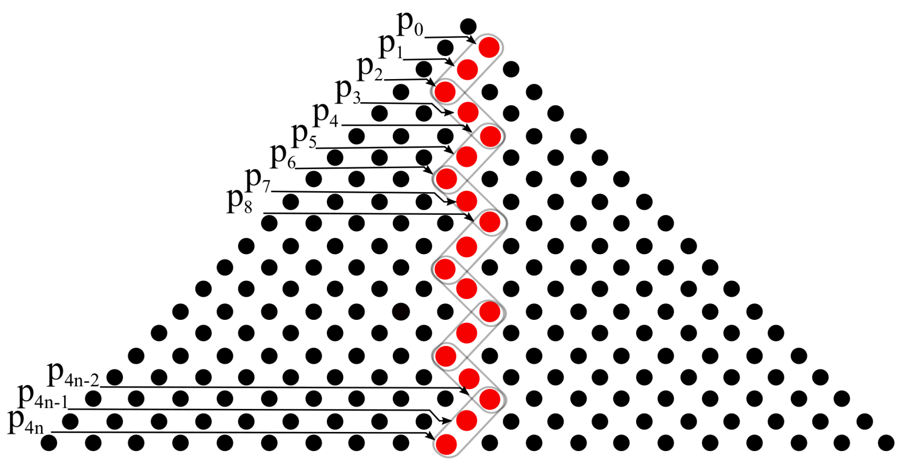

Figure 7: Geometric interpretation of Corollary 16.

Proof. Since $H(2n, n) = \mathcal{F}_n^2$, we have that $g^n H(1,1) + \sum_{j=1}^{n} dg^{n-j}\mathcal{F}_j^2$ is equal to

$$\begin{aligned}
\sum_{j=1}^{n} dg^{n-j}H(2j, j) &= g^{n-1}(gH(1,1) + dH(2,1)) + \sum_{j=2}^{n} dg^{n-j}\mathcal{F}_j^2 \\
&= g^{n-1}H(3,1) + dg^{n-2}H(4,2) + \sum_{j=3}^{n} dg^{n-j}\mathcal{F}_j^2 \\
&= g^{n-2}H(5,3) + \sum_{j=3}^{n} dg^{n-j}\mathcal{F}_j^2 \\
&= g^{n-2}\mathcal{F}_3\mathcal{F}_2 + \sum_{j=3}^{n} dg^{n-j}\mathcal{F}_j^2.
\end{aligned}$$

Similarly, we find that

$$\sum_{j=1}^{n} dg^{n-j}\mathcal{F}_j^2 = H(2n+1, n+1) = \mathcal{F}_{n+1}\mathcal{F}_n - g^n\mathcal{F}_1\mathcal{F}_0. \tag{9}$$

Note that

$$\begin{aligned}\sum_{j=2}^{2n+1} g^{2n+1-j}\mathcal{F}_j^2 &= \sum_{j=1}^{n} g^{2n-j}(\mathcal{F}_{2j+1}^2 + g\mathcal{F}_{2j}^2)\\ &= \sum_{j=1}^{n} g^{2n-j}(\mathcal{F}_{4j+1}\mathcal{F}_1 + \mathcal{F}_0\mathcal{F}_{4j})\\ &= \sum_{j=1}^{n} g^{2n-j}\mathcal{F}_{4j+1}.\end{aligned}$$

This, Equation (9), and $\mathcal{F}_0 = 0$ complete the proof of Part (1).

Proof of Part (2). From the hypothesis of Part (2), $g = 1$, we see that the sequence $\{\mathcal{F}_n\}$ defined in (1) satisfies that $g = 1$ and that $H(0, k) = H(k, 0) = 0$ for every k. This and the definition of the Hosoya polynomial sequence (page 5), imply that

$$H(r, k) = dH(r-1, k) + H(r-2, k) \text{ and } H(r, k) = dH(r-1, k-1) + H(r-2, k-2).$$

Therefore the points depicted in Figure 7 have the properties described in Table 6.

$p_0 = 0,$	$p_2 = dp_1 + p_0,$	$p_4 = dp_3 + p_2$	$p_6 = dp_5 + p_4$
$p_8 = dp_7 + p_6,$	$p_{10} = dp_9 + p_8,$	$\dots$	$p_{4n} = dp_{4n-1} + p_{4n-2}.$

Table 6: Properties of points in the Zigzag Figure 7.

Since $g = 1$, we have that $I(j) = 1$ for all j. Therefore, $\sum_{j=1}^{2n+1} I(j)\mathcal{F}_j^2 = \sum_{j=1}^{2n+1} \mathcal{F}_j^2$ is actually the sum of all points that are in the intersection of the zigzag diagram with central line of the triangle (see Figure 7). Thus,

$$d\sum_{j=1}^{2n+1} \mathcal{F}_j^2 = p_0 + dp_1 + dp_3 + dp_5 + dp_7 + \cdots + dp_{4n-1}.$$

The sum of the first two terms in the right side is equal to the third point of the zigzag diagram (see Table 6 and Figure 7). Therefore, substituting them with p_2 we have

$$d\sum_{j=1}^{2n+1} \mathcal{F}_j^2 = p_2 + dp_3 + dp_5 + dp_7 + \cdots + dp_{4n-1}.$$

Now the sum of the first two terms of the right side of the previous equation is equal to the fifth point, p_4, of the zigzag diagram (see Table 6 and Figure 7). Therefore, substituting them with p_4 we have

$$d\sum_{j=1}^{2n+1}\mathcal{F}_j^2 = p_4 + dp_5 + dp_7 + \cdots + dp_{4n-1}.$$

Similarly, we substitute $p_4 + dp_5$ with the seventh point of the zigzag diagram. Thus,

$$d\sum_{j=1}^{2n+1}\mathcal{F}_j^2 = p_6 + dp_7 + \cdots + dp_{4n-1}.$$

Continuing this process, systematically substituting the terms, we obtain

$$d\sum_{j=1}^{2n+1}\mathcal{F}_j^2 = p_{4n} = \mathcal{F}_{2n-1}\mathcal{F}_{2n}.$$

This completes the geometric proof of Part (2). □

4.2 Integration in the Hosoya triangle

We now discuss some examples on how the geometry of the triangle can be used to represent identities. The examples given in the following discussion are only for the case in which the Hosoya triangle (denoted by H_F) has products of Fibonacci polynomials as entries. With this triangle in mind we introduce a notation that will be used in following examples. We define an n-initial triangle as the finite triangular arrangement formed by the first n-rows of H_F with non-zero entries. Note that the initial triangle is the equilateral sub-triangle of the Hosoya triangle as in Table 3 on page 6 without the entries containing the factor G_0. For instance, Table 5 on page 7 represents the 5-initial triangle of H_F.

If $F_n'(x)$ represents the derivative of the Fibonacci polynomial $F_n(x)$, then $F_n'(x) = \sum_{k=1}^{n-1} F_k(x)F_{n-k}(x)$ (see [1, 14]). The geometric representation of this property in an n-initial triangle is as follows: the derivative of the first entry of the last row of a given n-initial triangle is equal to the sum of all points of the penultimate row of this triangle (see Table 5 on page 7). We have observed that this property implies that the integral of all points of the first $n-1$ rows of a given n-initial triangle is equal to the sum of all points of one edge of this triangle, where the constant of integration is $\lceil n/2 \rceil$. This result is stated formally in Proposition 17. Similar results can be obtained using Table 7.

Proposition 17. *Let n be a positive number, then*

(1)

$$H(n,1) \;=\; \sum_{k=1}^{n-1}\int H(n-1,k).$$

Equivalently,

$$F_n(x) = \sum_{k=1}^{n-1} \int F_k(x) F_{n-k}(x),$$

where the constant of integration is $C = 1$ if n is odd and zero otherwise.

(2)

$$H(n+1,1) + H(n,1) - 1 = x \sum_{r=1}^{n} \sum_{k=1}^{r-1} \int H(r-1,k).$$

Equivalently,

$$F_{n+1}(x) + F_n(x) - 1 = x \sum_{r=1}^{n} \sum_{k=1}^{r-1} \int F_k(x) F_{r-k}(x),$$

where the constant of integration is $C = \lceil n/2 \rceil$.

Proof. The proof of Part (1) is straightforward using the geometric interpretation of $F_n'(x)$.

We prove Part (2). From Part (1) and from the geometry of the $(n-1)$-initial triangle we have $\sum_{r=1}^{n} H(k,1) = \sum_{r=1}^{n-1} \sum_{k=1}^{r-1} \int H(r-1,k)$. From Koshy [14, Theorem 37.1] we know that $F_{n+1}(x) + F_n(x) - 1 = x \sum_{i=1}^{n} F_i(x)$. This and the fact that $H(t,1) = F_t(x)$ for all $t \geq 1$ completes the proof. ☐

Derivative
$F_n'(x) = \sum_{k=1}^{n-1} F_k(x) F_{n-k}(x)$
$P_n'(x) = 2 \sum_{k=1}^{n-1} P_k(x) P_{n-k}(x)$
$\Phi_n'(x) = 3 \sum_{k=1}^{n-1} \Phi_k(x) \Phi_{n-k}(x)$
$U_n'(x) = 2 \sum_{k=1}^{n-1} P_k(x) P_{n-k}(x)$
$B_n'(x) = \sum_{k=1}^{n-1} B_k(x) B_{n-k}(x)$

Table 7: Derivatives of Fibonacci-type polynomials.

5 Appendix. Numerical types of Hosoya triangle

In this section we study some connections of the Hosoya polynomial triangles with some numeric sequences that may be found in [20]. We show that when we evaluate the entries of a Hosoya polynomial triangle at $x = 1$ they give a triangle that is in http://oeis.org/. The first Hosoya triangle is the classic Hosoya triangle formerly called the Fibonacci triangle.

We now introduce some notation that is used in Table 8. Recall that H_F is the Hosoya triangle with products of Fibonacci polynomials as entries. Similarly we define the notation for the Hosoya polynomial triangle of the other types — Chebyshev polynomials, Morgan-Voyce polynomials, Lucas polynomials, Pell polynomials, Fermat polynomials, Jacobsthal polynomials. The star of David property holds obviously for all these numeric triangles.

Triangle type	Notation	Entries	Sloane
Fibonacci	$H_F(1)$	$F_k(1)F_{r-k}(1)$	A058071
Lucas	$H_D(1)$	$D_k(1)D_{r-k}(1)$	A284115
Pell	$H_P(1)$	$P_k(1)P_{r-k}(1)$	A284127
Pell-Lucas	$H_Q(1)$	$Q_k(1)Q_{r-k}(1)$	A284126
Fermat	$H_\Phi(1)$	$\Phi_k(1)\Phi_{r-k}(1)$	A143088
Fermat-Lucas	$H_\vartheta(1)$	$\vartheta_k(1)\vartheta_{r-k}(1)$	A284128
Jacobsthal	$H_J(1)$	$J_k(1)J_{r-k}(1)$	A284130
Jacobsthal-Lucas	$H_j(1)$	$j_k(1)j_{r-k}(1)$	A284129
Morgan-Voyce	$H_B(1)$	$B_k(1)B_{r-k}(1)$	A284131
Morgan-Voyce	$H_C(1)$	$C_k(1)C_{r-k}(1)$	A141678

Table 8: Numerical Hosoya triangles present in Sloane [20].

We also observe some curious numerical patterns when we compute the gcd of the coefficients of polynomials discussed in this paper. In particular, the gcd of the coefficients of $\Phi_n(x)$, the nth Fermat polynomial, is 3^{a_n} where a_n is the nth element of A168570. The gcd of the coefficients of $\vartheta_n(x)$, the nth Fermat-Lucas polynomial, is 3^{a_n} where a_n is the nth element of A284413. We also found that the gcd of the coefficients of the $P_{2n}(x)$, the $2n$th Pell polynomial, is 2^{a_n} where a_n is the nth element of A001511. Finally, the gcd of the coefficients of the $U_n(x)$, the nth Chebyshev polynomial of second kind, is 2^{a_n} where a_n is the nth element of A007814.

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