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Further Results on Paths in an n-Dimensional Cubic Lattice

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Abstract

We study paths formed by integer n-tuples in an n-dimensional cubic lattice. We establish some connections between these paths, Riordan arrays, coefficients of Chebyshev polynomials of the second kind, and k-colored Motzkin paths.

1 Introduction

A three-dimensional cubic lattice is a lattice in $\mathbb{R}^3$ formed by integer triplets. We study properties of three-dimensional lattice walks in the upper half of the three-dimensional cubic lattice. In particular, we count the number of paths of length k that are in three-dimensional cubic lattice beginning at the origin and ending on the xy-plane. This gives another answer to a question raised by Deutsch [5] in 2000. The problem asks: *A 3-dimensional lattice walk of length n takes n successive unit steps, each in one of the six coordinate directions. How many 3-dimensional lattice walks of length n are there that begin at the origin and never go below the horizontal plane?* The answer to Deutsch's question is [1, 5]

$$\sum_{k=0}^{n} \binom{n}{k} \binom{k}{\lceil k/2 \rceil} 4^{n-k}.$$

This problem generalizes, naturally, to an n-dimensional cubic lattice.

Deutsch's problem is a generalization, to three dimensions, of the following problem proposed by Sands [24] in 1990: *What is the number of different walks, in the plane, with n steps such that each step moves one unit either north, south, east, or west, starting at the origin and remaining in the upper half-plane?* Hirschhorn [12] proved in 1991 that the number of such paths is $\binom{2n+1}{n}$. Later, Guy [9] gave a short proof of the same result. In the same paper, Guy studied this problem from the point of view of one-dimensional and two-dimensional arrays using the Pascal semi-triangle and Pascal semi-pyramid, respectively. He found some interesting relations with Catalan numbers as well as several combinatorial identities that were used in the entries of the mentioned arrays. Guy, Krattenthaler, and Sagan [10] gave combinatorial proofs for several two-dimensional results. The three-dimensional case was also studied by Guy [9].

In this paper we answer an open question that Guy proposed in [10]. *Is it possible to find a closed formula for the number of paths, in the three-dimensional cube, that can not go below the plane $z = 0$?* We give such a formula as well as other closed formulas. In addition, Nkwanta [20] generalizes several problems proposed by Sands and Deutsch to n-dimensional case using Riordan arrays. He also shows a bijection between the lattice paths of length k and k-colored Motzkin paths. In 2017, Dershowitz [6] obtained a bijection between Dyck paths and 2-dimensional lattice paths that end in the x-axis. He also considered the n-dimensional case with the same condition. We provide several relations and counting results involving walks in the n-dimensional cubic lattice and Riordan arrays.

We also study Riordan arrays from the perspective of the *height* of a path. The *height* at which a path P ends, in the n-dimensional cubic lattice, is the last value of the last n-tuple (vertex or point) of the path. For simplicity, we call this value the *height* of P. Those heights give rise to a Riordan array $\mathcal{A}_3$. We have found that the entries of $\mathcal{A}_3^{-1}$ are the coefficients of Chebyshev polynomials of the second kind (shifted four units up). Using the generating function obtained from $\mathcal{A}_3$, we find a closed formula that answers the open question given by Guy in [9]. Since $\mathcal{A}_3$ and $\mathcal{A}_3^{-1}$ are infinite matrices, we found a fractal associated to those matrices. Visually, these fractals look like half of the Sierpiński fractal.

Finally, we provide several sequences and conjectures about paths in the 3-dimensional cubic lattice. These sequences/conjectures are based on numerical experimentation.

Most proofs in this paper use generating functions that are constructed using the symbolic method introduced by Flajolet and Sedgewick [7].

2 Background

An n-dimensional cubic lattice is a lattice L in $\mathbb{R}^n$ formed by points in $\mathbb{Z}^n$. If p_i is a point in $\mathbb{Z}^n$ with $p_0 = (0, 0, \ldots, 0)$, then a path $P = (p_0, p_1, \ldots, p_m)$ of length m is a concatenation of $p_0, p_1, \ldots, p_m$ where the distance between p_i and p_{i+1} is one for $i \in \{0, 1, \ldots, m-1\}$. A *step* in P is a pair of two consecutive points (p_i, p_{i+1}) for $i \in \{0, \ldots, m-1\}$. We identify the path $P = (p_0, p_1, \ldots, p_m)$ with its broken-line graph obtained by joining p_i to p_{i+1} with a line segment for $i \in \{0, \ldots, m-1\}$. We denote by e_i the n-tuple $(0, \ldots, 1, \ldots, 0)$ where 1 is in the i-th position and zeros elsewhere for $1 \leq i \leq n$. Since $p_0 = (0, 0, \ldots, 0)$, we have that $p_1 = e_i$ for some $1 \leq i \leq n$. It is easy to see that p_j can be written in either of the forms $p_j = p_{j-1} + e_r$ or $p_j = p_{j-1} - e_r$, for some $1 \leq r \leq n$ and $1 \leq j \leq m$. Note that e_r gives the orientation of the step $p_{j-1}p_j$. For example, if a path P is in $\mathbb{Z}^3$ with $p_5 = p_4 + (0, 1, 0)$, then the step p_4p_5 is parallel to the positive direction of the y-axis. Now it is easy to see that a path $P = (p_0, p_1, p_3, \ldots, p_{m-1}, p_m)$ can be written in the form $P = (p_0, p_0 \pm e_{j_1}, p_1 \pm e_{j_2}, \ldots, p_{m-1} \pm e_{j_m})$ where $\pm$ indicates the orientation of each step. For simplicity, we represent $P = (p_0, p_0 \pm e_{j_1}, p_1 \pm e_{j_2}, \ldots, p_{m-1} \pm e_{j_m})$ as $P = (\pm e_{j_1})(\pm e_{j_2}) \cdots (\pm e_{j_m})$ and we say that $(\pm e_{j_1}), (\pm e_{j_2}), \ldots, (\pm e_{j_m})$ are the components of P, (see Figures 1 and 2).

We use $\mathcal{C}_n^{\pm}(k)$ to mean the set of all paths of length k in the n-dimensional cubic lattice. We divide $\mathcal{C}_n^{\pm}(k)$ into subfamilies depending on the behavior of the path. We now give definitions and notation for those families. If $P = (\pm e_{j_1})(\pm e_{j_2}) \cdots (\pm e_{j_k})$, then we define $V_r := (\pm e_{j_1}) + (\pm e_{j_2}) + \cdots + (\pm e_{j_r})$, the algebraic combination of the first r components of P for $0 < r \leq k$, i.e., V_r is the sum of the components of any initial subpath of P with r steps. We denote $\mathcal{C}_n(k)$ the subset of $\mathcal{C}_n^{\pm}(k)$ formed by all paths $P = (\pm e_{j_1})(\pm e_{j_2}) \cdots (\pm e_{j_k})$ that satisfy that the nth coordinate of V_k is zero. We use $\mathcal{C}_n^{\geq}(k)$ to denote all paths in $\mathcal{C}_n^{\pm}(k)$ with $P = (\pm e_{j_1})(\pm e_{j_2}) \cdots (\pm e_{j_k})$ and that nth coordinate of V_r is non-negative for all $0 < r \leq k$. We now let $\mathcal{C}_n^{+}(k)$ be $\mathcal{C}_n^{\geq}(k) \cap \mathcal{C}_n(k)$. Finally, we let $\mathcal{C}_n = \bigcup_{i=0}^{\infty} \mathcal{C}_n(i)$; $\mathcal{C}_n^{\geq} = \bigcup_{i=0}^{\infty} \mathcal{C}_n^{\geq}(i)$ and $\mathcal{C}_n^{+} = \bigcup_{i=0}^{\infty} \mathcal{C}_n^{+}(i)$.

For example, Figure 1 depicts the 14 paths in $\mathcal{C}_2^{+}(3)$. Figure 2 depicts the 17 paths in $\mathcal{C}_3^{+}(2)$.

One of the goals of this paper is to study the enumerate problem of the families of lattice paths $\mathcal{C}_n^{+}(k)$, $\mathcal{C}_n(k)$, $\mathcal{C}_n^{\geq}(k)$ and $\mathcal{C}_n^{\pm}(k)$. Most of the proofs in this paper are done using generating functions according to paths length.

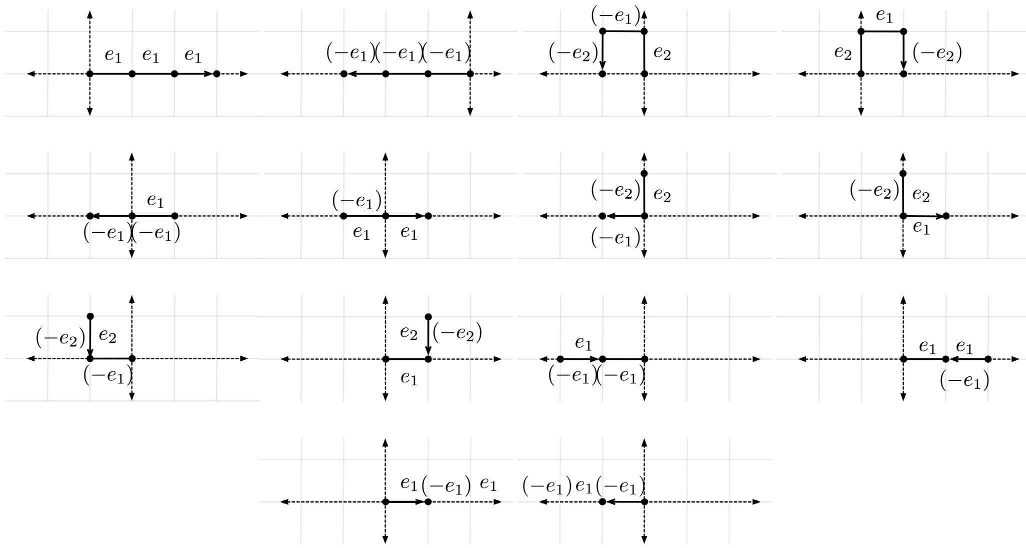

Figure 1: All paths of $\mathcal{C}_2^+(3)$.

3 Special case of Generating function for paths in a cube

In this section we count paths in the three-dimensional cube. We classify those paths in families. For example, we have the family of paths where each path never goes below of the plane $z = 0$ but ends on it. There is a family of paths where each path goes below $z = 0$ and ends on it. There is another family of paths where each path ends above the plane $z = 0$ and so on. Therefore, we divide the section in several subsections depending on the nature of the family of paths.

3.1 Counting paths that never go below the horizontal plane

The main theorem in this section counts the total number of lattice paths of length m, in the three-dimensional cube that end at the xy-plane and never go below it. The proof uses generating functions (the technique used to obtain the generating function is based on the symbolic method introduced in [7]). We give some necessary notation. We denote by $a_3(m)$ the total number of paths of length m in $\mathcal{C}_3^+(m)$. That is, $a_3(m) = |\mathcal{C}_3^+(m)|$.

Theorem 1. *If C_n is the nth Catalan number, then*

(1) the generating function for the total number of paths of length i in $\mathcal{C}_3^+$ is given by

$$T_3(z) := \sum_{i=0}^{\infty} a_3(i) z^i = \frac{1 - 4z - \sqrt{1 - 8z + 12z^2}}{2z^2}, \tag{1}$$

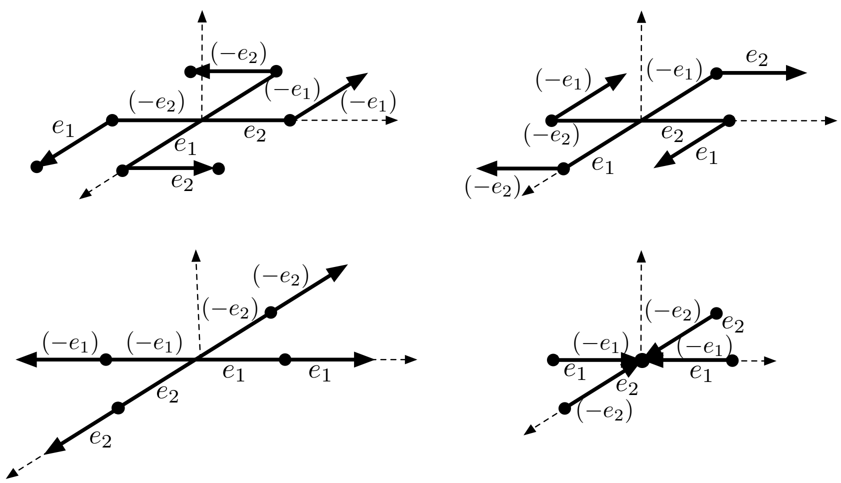

Figure 2: All paths of $\mathcal{C}_3^+(2)$.

(2) the number of paths, in the three-dimensional cube, of length m that end in the horizontal plane and never go below it is given by

$$a_3(m) = \sum_{i=0}^{\lfloor m/2 \rfloor} C_i \binom{m}{2i} 4^{m-2i}.$$

Proof. From the first return decomposition a nonempty three-dimensional lattice path P in $\mathcal{C}_3^+$ may be decomposed using one of the following forms

$$e_3P'(-e_3)P''; \quad e_1P'; \quad -e_1P'; \quad e_2P'; \quad -e_2P',$$

where P' and P'' are paths (possibly empty) in $\mathcal{C}_3^+$ (see Figure 3).

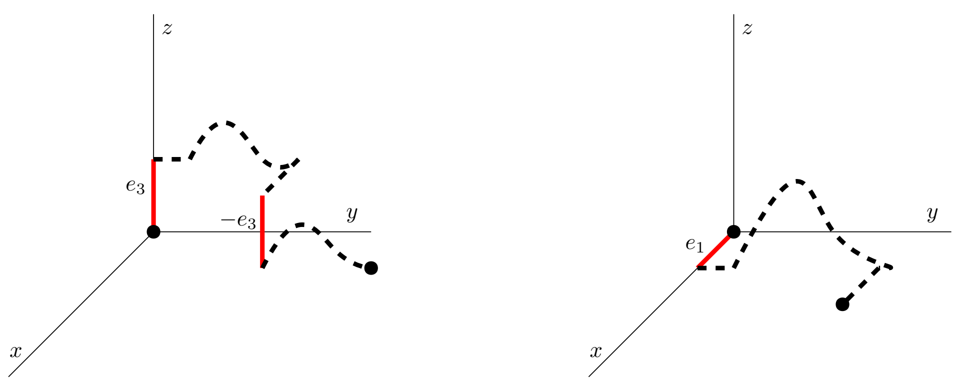

Figure 3: Factoring a path P in $\mathcal{C}_3^+$.

Using the symbolic method we obtain

$$T_3(z) = z^2T_3^2(z) + 4zT_3(z) + 1. \tag{2}$$

This proves part (1).

Proof of part (2): It is easy to see that

$$T_3(z) = \frac{1}{1-4z} \cdot \frac{1-\sqrt{1-4z^2/(1-4z)^2}}{2z^2/(1-4z)^2} = \frac{1}{1-4z} \cdot \frac{1-\sqrt{1-4u}}{2u} = \frac{1}{1-4z} \cdot C(u), \quad (3)$$

where $u = z^2/(1-4z)^2$ and $C(z)$ is the generating function of the Catalan numbers. Thus,

$$C(z) := \sum_{n=0}^{\infty} C_n z^n = \frac{1-\sqrt{1-4z}}{2z}.$$

Hence,

$$T_3(z) = \sum_{i=0}^{\infty} C_i \left(\frac{z^{2i}}{(1-4z)^{2i+1}} \right) = \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} C_i \binom{m+2i}{m} 4^m z^{2i+m}.$$

If we take $s = 2i + m$, then

$$T_3(z) = \sum_{i=0}^{\infty} \sum_{s=2i}^{\infty} C_i \binom{s}{s-2i} 4^{s-2i} z^s.$$

This proves part (2), completing the proof of the theorem. □

The following sequence gives some values for the total number of paths of length $0-9$ in the three-dimensional cube, that end at the xy-plane and never go below the mentioned plane. From (2) it is easy to see that

$$a_3(1) = 1, \text{ and } a_3(n) = 4a_3(n-1) + \sum_{i=1}^{n-1} a_3(i-1)a_3(n-i-1).$$

This sequence appears in OEIS as A005572 [28]. So, $a_3(m)$ for $m = 0, 1, \ldots, 9$ is given by

$$1, 4, 17, 76, 354, 1704, 8421, 42508, 218318, 1137400.$$

3.2 Riordan arrays for paths in three-dimensional space

We give an informal definition of *height of a path* P in $\mathcal{C}_3^{\geq}(n)$. It is the third entry of the last vertex (point) of the path. That is, the value of the z coordinate of the ending point. In this section we construct an infinite matrix using the height of all paths in $\mathcal{C}_3^{\geq}(n)$. The resulting matrix becomes a Riordan array. Theorem 6 gives a relationship between the sum of the entries on the nth rising diagonal of the Riordan array (found in this section) and the number of paths in $\mathcal{C}_3^{+}$ with no level steps at height 0 (it means no steps on the xy-plane). We also provide a formula (given a length) that counts the total number of paths, in three-dimensional cube, that never go below to xy-plane. Several authors have used Riordan arrays as a technique to study lattice paths (see for example [3, 15, 20, 21, 22, 27]).

3.2.1 A short background about Riordan arrays

We recall that an infinite lower triangular matrix is called a (proper) Riordan array [26] if its kth column satisfies the generating function $g(z)\left(f(z)\right)^k$ for $k \geq 0$, where $g(z)$ and $f(z)$ are formal power series with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. The matrix corresponding to the pair $f(z), g(z)$ is denoted by $(g(z), f(z))$. If we multiply (g, f) by a column vector $(c_0, c_1, \dots)^T$ with the generating function $h(z)$, then the resulting column vector has generating function $gh(f)$. This property is known as the Fundamental Theorem of Riordan arrays or summation property.

The product of two Riordan arrays $(g(z), f(z))$ and $(h(z), l(z))$ is defined by

$$(g(z), f(z)) * (h(z), l(z)) = (g(z)h\left(f(z)\right), l\left(f(z)\right)).$$

We recall that the set of all Riordan matrices is a group under the operator " $*$ " [26]. The identity element is $I = (1, z)$, and the inverse of $(g(z), f(z))$ is

$$(g(z), f(z))^{-1} = \left(1/\left(g \circ \overline{f}\right)(z), \overline{f}(z)\right), \tag{4}$$

where $\overline{f}(z)$ is the compositional inverse of $f(z)$.

Rogers [23] introduced the concept of an A-sequence. Specifically, Rogers observed that every element $d_{n+1,k+1}$ of a Riordan matrix (not belonging to 0 row or 0 column) could be expressed as a linear combination of the elements in the preceding row. Merlini et al. [14] introduced the Z-sequence, which characterizes 0 column, except for the element $d_{0,0}$. Therefore, the A-sequence, Z-sequence and the element $d_{0,0}$ completely characterize a proper Riordan array. Summarizing, we have the following theorem.

Theorem 2 ([14])**.** *An infinite lower triangular array $\mathcal{D} = [d_{n,k}]_{n,k\in\mathbb{N}}$ is a Riordan array if and only if $d_{0,0} \neq 0$ and there are sequences $A = (a_0 \neq 0, a_1, a_2, \dots)$ and $Z = (z_0, z_1, z_2, \dots)$ such that if $n, k \geq 0$, then*

$$\begin{aligned} d_{n+1,k+1} &= a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots, \\ d_{n+1,0} &= z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots, \end{aligned}$$

or equivalently

$$g(z) = \frac{g(0)}{1 - zZ(g(z))} \quad \text{and} \quad f(z) = z(A(f(z))),$$

where A and Z are the generating functions of the A-sequence and Z-sequence, respectively.

3.2.2 A Riordan array from heights of paths

If $P = (\pm e_{j_1})(\pm e_{j_2}) \cdots (\pm e_{j_k})$ is a path in the n-dimensional cube, then the *height* of P is the nth coordinate of V_k where $V_k = (\pm e_{j_1}) + (\pm e_{j_2}) + \cdots + (\pm e_{j_k})$ is the algebraic combination of all components of P. We denote by $A_3(n, k)$ the subset of $\mathcal{C}_3^{\pm}(n)$ of all paths having height k and not passing below the plane $z = 0$. If $a_3(n, k)$ denotes $|A_3(n, k)|$ and $b_3(n)$ denotes

$|\mathcal{C}_3^{\geq}(n)|$, then $a_3(n) = a_3(n,0)$ and $b_3(n) = \sum_{k=0}^{n} a_3(n,k)$. Note that the last component of the last step of any path in $A_3(n,k)$ is one element of $\{\pm e_1, \pm e_2, \pm e_3\}$. Therefore, $a_3(n,k)$ satisfies the following third order recurrence relation.

$$a_3(n,k) = a_3(n-1,k-1) + 4a_3(n-1,k) + a_3(n-1,k+1) \tag{5}$$

with $n, k \geq 1$, and the initial values $a_3(0,0) = 1$ and $a_3(n,k) = 0$ if $k > n$ (see Guy [9]). This sequence gives rise to an infinite lower triangular matrix. It is denoted by $\mathcal{A}_3 = [a_3(n,k)]_{n,k\geq 0}$. So,

$$\mathcal{A}_3 = [a_3(n,k)]_{n,k\geq 0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 17 & 8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 76 & 50 & 12 & 1 & 0 & 0 & 0 & 0 \\ 354 & 288 & 99 & 16 & 1 & 0 & 0 & 0 \\ 1704 & 1605 & 700 & 164 & 20 & 1 & 0 & 0 \\ 8421 & 8824 & 4569 & 1376 & 245 & 24 & 1 & 0 \\ 42508 & 48286 & 28476 & 10318 & 2380 & 342 & 28 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

From Theorem 2 we can prove that the matrix $\mathcal{A}_3$ is a Riordan matrix. Guy [9] also found a relation between this matrix and the sequences A005572, A005573, A052177, A052178 and A052179.

Theorem 3. *The infinite triangular matrix $\mathcal{A}_3 = [a_3(n,k)]_{n,k\geq 0}$ has a Riordan array expression given by*

$$\mathcal{A}_3 = \left(T_3(z), zT_3(z)\right),$$

where

$$T_3(z) = \frac{1 - 4z - \sqrt{1 - 8z + 12z^2}}{2z^2}.$$

Proof. From the recurrence relation (5), we know that the A-sequence is $(1, 4, 1, 0, 0, \ldots)$. Therefore, $A(z) = 1 + 4z + z^2$. This implies that $f(z) = z(1 + 4f(z) + f(z)^2)$. Therefore,

$$f(z) = \frac{1 - 4z - \sqrt{1 - 8z + 12z^2}}{2z} = zT_3(z).$$

Now, it is easy to see that the generating function of the 0th column is $T_3(x)$. ☐

The proof of Theorem 3 shows that the Riordan array $\mathcal{A}_3$ has A-sequence $(1, 4, 1)$. Merlini et al. [18] studied a lattice path model in the plane that has an associated Riordan matrix with A-sequence (a, b, c). Merlini and Sprugnoli [16] study again this type model. In Section 5, we show a relationship between the coloured Motzkin path and the paths in the n-dimensional cube.

Corollary 4. *In the three-dimensional cube the following hold:*

(1) the generating function for the total number of paths of length i in $\mathcal{C}_3^{\geq}$ is given by

$$H_3(z) := \sum_{i=0}^{\infty} b_3(i) z^i = \frac{T_3(z)}{1 - zT_3(z)} = \frac{1 - 6z - \sqrt{1 - 8z + 12z^2}}{2z(6z-1)},$$

(2) the number of paths of length m such that the paths never go below the horizontal plane is given by

$$b_3(m) = \sum_{n=0}^{m} \sum_{k=0}^{\lfloor \frac{m-n}{2} \rfloor} \binom{n+2k+1}{k} \binom{m}{n+2k} \left(\frac{n+1}{n+2k+1} \right) 4^{m-n-2k}.$$

Proof. From the Fundamental Theorem of Riordan arrays we have

$$(T_3(z), zT_3(z)) \frac{1}{1-z} = \frac{T_3(z)}{1 - zT_3(z)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_3(n,k) z^n = \sum_{n=0}^{\infty} b_3(n) z^n = H_3(z).$$

After simplification we obtain the result that proves part (1).

Proof of part (2): It is easy to see that

$$H_3(z) = \frac{T_3(z)}{1 - zT_3(z)} = \frac{(1/(1-4z))\, C(u)}{1 - (z/(1-4z))\, C(u)},$$

where $u = z^2/(1-4z)^2$ and $C(z)$ is the generating function of the Catalan numbers. Therefore,

$$H_3(z) = \frac{1}{1-4z} \sum_{n=0}^{\infty} \left(\frac{z}{1-4z} \right)^n C^{n+1}(u).$$

From [8, equation 5.70] we know that

$$C^n(z) = \sum_{k=0}^{\infty} \binom{n+2k}{k} \frac{n}{n+2k} z^k. \tag{6}$$

This implies that

$$\begin{aligned} H_3(z) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n+2k+1}{k} \left(\frac{n+1}{n+2k+1} \right) \left(\frac{z^n}{(1-4z)^{n+1}} \right) u^k \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n+2k+1}{k} \left(\frac{n+1}{n+2k+1} \right) \left(\frac{z^{n+2k}}{(1-4z)^{n+2k+1}} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{n+2k+1}{k} \binom{n+2k+l}{l} \left(\frac{n+1}{n+2k+1} \right) 4^l z^{n+2k+l}. \end{aligned}$$

This with $t = n + 2k + l$ implies

$$H_3(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=n+2k}^{\infty} \binom{n+2k+1}{k}\binom{t}{n+2k}\left(\frac{n+1}{n+2k+1}\right)4^{t-n-2k}z^t.$$

Therefore,

$$b_3(m) = [z^m]H_3(z) = \sum_{n=0}^{m}\sum_{k=0}^{\lfloor\frac{m-n}{2}\rfloor} \binom{n+2k+1}{k}\binom{m}{n+2k}\left(\frac{n+1}{n+2k+1}\right)4^{m-n-2k}.$$

This proves part (2). $\square$

The following sequence gives some values for the total number of paths of length $0-9$ in three-dimensional cube that never go below the xy-plane. This sequence appears in OEIS as A005573. So, $b_3(m)$ for $m = 0, 1, \ldots, 9$ is given by

$$1, 5, 26, 139, 758, 4194, 23460, 132339, 751526, 4290838.$$

The previous theorem gives a closed formula that answers Guy's question. Theorem 5 gives a closed formula for all entries of $\mathcal{A}_3$.

Theorem 5. *If m and n are non-negative integers, then*

$$a_3(n,k) = \frac{k+1}{n+1}\sum_{l=0}^{\lfloor\frac{n-k}{2}\rfloor}\binom{n+1}{n-k-l}\binom{n-k-l}{l}4^{n-k-2l}.$$

Proof. Let $M(z) = zT_3(z)$ then from (2) we obtain

$$M(z) = z(M(z)^2 + 4M(z) + 1) = zP(z),$$

where $P(z) = z^2 + 4z + 1$. From the Lagrange Inversion Formula (cf. [17])

$$\begin{aligned}
a_3(n,k) &= [z^n]\left(z^kT_3(z)^{k+1}\right) = [z^{n+1}]M(z)^{k+1}\\
&= \frac{k+1}{n+1}[z^{n-k}]\left(z^2+4z+1\right)^{n+1}\\
&= \frac{k+1}{n+1}[z^{n-k}]\left(\sum_{i=0}^{n+1}\binom{n+1}{i}z^i(z+4)^i\right)\\
&= \frac{k+1}{n+1}[z^{n+1}]\left(\sum_{i=0}^{n+1}\sum_{l=0}^{i}\binom{n+1}{i}\binom{i}{l}4^{i-l}z^{i+l+k+1}\right)\\
&= \frac{k+1}{n+1}[z^{n+1}]\left(\sum_{m=0}^{2n+2}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}\binom{n+1}{m-l}\binom{m-l}{l}4^{m-2l}z^{m+k+1}\right).
\end{aligned}$$

If we take $m = n-k$, we obtain the desired result. $\square$

Theorem 6. *In the three-dimensional cube the following hold:*

(1) the generating function $U_3(z)$ for the total number of paths of length i in $\mathcal{C}_3^+$ with no level steps at height 0 is given by

$$U_3(z) := \sum_{n=0}^{\infty} u_3(n) z^n = \frac{1}{1 - z^2 T_3(z)},$$

(2) the sum of entries of the rising diagonal of the Riordan array $\mathcal{A}_3 = [a_3(n,k)]_{n,k\geq 0}$ is

$$u_3(n+2) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a(n-i, i).$$

Proof. From the first return decomposition a nonempty three-dimensional lattice path P in $\mathcal{C}_3^+$ with no level steps at height 0 may be decomposed as $e_3 P'(-e_3)P''$ where P' and P'' are three dimensional lattice paths (possibly empty) in $\mathcal{C}_3^+$ such that P'' does not have level steps at height 0. Then

$$U_3(z) = 1 + z^2 T_3(z) U_3(z).$$

This proves part (1).

Proof of part (2): From the definition of the Riordan array we have

$$\begin{aligned}
L(z) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a(n-k, k) z^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} [z^{n-k}] g f^k z^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g f^k z^k \\
&= \frac{g}{1 - zf} = \frac{T_3(z)}{1 - z^2 T_3(z)}.
\end{aligned}$$

Therefore, comparing coefficients we get that

$$U_3(z) = z^2 L(z) + 1.$$

This proves part (2). □

The following sequence gives some values for the total number of paths of length $0-10$ in paths $\mathcal{C}_3^+$ with no level steps at height 0. This sequence appears in OEIS as A185132. So, $u_3(n)$ for $n = 0, 1, \ldots, 10$ is given by

$$1, 4, 18, 84, 405, 2004, 10126, 52048, 271338, 1431400, 7627348.$$

3.2.3 The inverse matrix of the Riordan array

Since $\mathcal{A}_3 = [a_3(n,k)]_{n,k\geq 0}$ is a Riordan matrix and the set of all Riordan matrices is a group, the inverse matrix of $\mathcal{A}_3$ exists. So, it is natural to ask: what properties does the inverse matrix of $\mathcal{A}_3$ satisfy? In this section we analyze the inverse matrix $\mathcal{A}_3^{-1}$ and in particular, we study the combinatorial interpretation of the unsigned entries of the matrix $\mathcal{A}_3^{-1}$. We found that there is a combinatorial relation between the entries of $\mathcal{A}_3^{-1}$ and the words over the alphabet $\{\texttt{0, 1, 2, 3}\}$. There is also another relation between the matrix $\mathcal{A}_3^{-1}$ and the matrix of coefficients of Chebyshev's polynomials of the second kind. Finally we present two fractals resulting from both matrices $\mathcal{A}_3$ and $\mathcal{A}_3^{-1}$.

From (4) we obtain that the inverse matrix $\mathcal{A}_3^{-1}$ is given by the Riordan matrix

$$\widetilde{\mathcal{F}} := \left[\widetilde{f}(n,k)\right]_{n,k\geq 0} = \mathcal{A}_3^{-1} = \left(\frac{1}{1+4z+z^2}, \frac{z}{1+4z+z^2}\right). \tag{7}$$

In general if a Riordan matrix $[a_{n,k}]_{n,k\geq 0} = (g(z), f(z))$ is given, then the alternating Riordan matrix defined by $[(-1)^{n+k}a_{n,k}]_{n,k\geq 0}$ can be found by the product of Riordan matrices as follows

$$(1,-z)(g(z), f(z))(1,-z) = (g(-z), -f(-z)).$$

Now from (7) it is easy to see that $g(z) = 1/(1+4z+z^2)$ and $f(z) = z/(1+4z+z^2)$. Therefore,

$$\mathcal{F} := [f(n,k)]_{n,k\geq 0} := \left[(-1)^{n+k}\widetilde{f}(n,k)\right]_{n,k\geq 0} = \left(\frac{1}{1-4z+z^2}, \frac{z}{1-4z+z^2}\right).$$

We now express this matrix explicitly as follows (see also A207823).

$$\mathcal{F} = [f(n,k)]_{n,k\geq 0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 15 & 8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 56 & 46 & 12 & 1 & 0 & 0 & 0 & 0 \\ 209 & 232 & 93 & 16 & 1 & 0 & 0 & 0 \\ 780 & 1091 & 592 & 156 & 20 & 1 & 0 & 0 \\ 2911 & 4912 & 3366 & 1200 & 235 & 24 & 1 & 0 \\ 10864 & 21468 & 17784 & 8010 & 2120 & 330 & 28 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

From the Fundamental Theorem of Riordan arrays we have

$$\begin{aligned} F(x,y) := \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m)x^m z^n &= \left(\frac{1}{1-4z+z^2}, \frac{z}{1-4z+z^2}\right)\frac{1}{1-xz} \\ &= \frac{1}{1-(4+x)z+z^2}. \end{aligned}$$

Therefore, the element $f(n,m)$ satisfies the following recurrence relation.

$$f(n,m) = 4f(n-1,m) + f(n-1,m-1) - f(n-2,m), \tag{8}$$

with $n, m \geq 1$, and the initial values $f(0,0) = 1$ and $f(n,m) = 0$ if $m > n$.

Note that from [11, Theorems 4.1 and 4.2] we obtain generating functions for the A-sequence and the Z-sequence of the Riordan array $\widetilde{\mathcal{F}}$ (and therefore for $\mathcal{F}$). So, we have

$$\begin{aligned} A_{\widetilde{\mathcal{F}}}(z) &= \frac{z}{zT_3(z)} = \frac{1-4z+\sqrt{1-8z+12z^2}}{2} \\ Z_{\widetilde{\mathcal{F}}}(z) &= \frac{1}{zT_3(z)}(1-T_3(z)) = \frac{-1-4z+\sqrt{1-8z+12z^2}}{2z} \end{aligned}$$

and

$$\begin{aligned} A_{\mathcal{F}}(z) &= \frac{z}{zT_3(-z)} = \frac{1+4z+\sqrt{1+8z+12z^2}}{2} \\ &= 1+4z-z^2+4z^3-17z^4+76z^5-354z^6+1704z^7-8421z^8+\cdots \end{aligned}$$

$$\begin{aligned} Z_{\mathcal{F}}(z) &= \frac{1}{zT_3(-z)}(1-T_3(-z)) = \frac{-1+4z+\sqrt{1+8z+12z^2}}{2z} \\ &= 4-z+4z^2-17z^3+76z^4-354z^5+1704z^6-8421z^7+\cdots \end{aligned}$$

Therefore, the element $f(n,m)$ can be calculated by a complex linear combination of elements in the previous row:

$$f(n+1,k+1) = f(n,k) + 4f(n,k+1) - f(n,k+2) + 4f(n,k+3) - 17f(n,k+4) + \cdots$$

This observation was made by the anonymous referee. The generating function $A_{\mathcal{F}}(z)$ can be also obtain from Theorem 3.2 of [14].

We observe that the matrix $\mathcal{F}$ is related to the coefficients of Chebyshev polynomials of the second kind $U_n(x)$. Recall that those polynomials are defined recursively as follows: $U_0(x) = 1$, $U_1(x) = 2x$ and $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$ for $n \geq 2$. Equivalently,

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \tag{9}$$

$$= \sum_{k=0}^{n} (-1)^{(n-k)/2} \binom{(n+k)/2}{k} \left(\frac{1+(-1)^{n-k}}{2}\right) (2x)^k. \tag{10}$$

The ordinary generating function of the polynomials $U_n(x)$ is

$$U(x, z) := \sum_{n=0}^{\infty} U_n(x) z^n = \frac{1}{1 - 2xz + z^2}.$$

Thus,

$$U\left(\frac{x+4}{2}, z\right) = \frac{1}{1 - (x+4)z + z^2} = F(x, y).$$

Therefore, this proves the following theorem.

Theorem 7. *If m and n are non-negative integers, then*

$$f(n, m) = u(n, m), \text{ where } u(n, m) = [z^n x^m] U\left((x+4)/2, z\right).$$

The following theorem gives an explicit expression for the entries $f(n, m)$.

Theorem 8. *If m and n are non-negative integers, then*

$$f(n, m) = \sum_{k=m}^{n} (-1)^{(n-k)/2} \binom{(n+k)/2}{k} \binom{k}{m} \left(\frac{1 + (-1)^{n-k}}{2}\right) 4^{k-m}.$$

Proof. Theorem 7 and Equation (10) imply that

$$\begin{aligned}
f(n, m) &= [x^m] U_n\left(\frac{x+4}{2}\right) \\
&= [x^m] \sum_{k=0}^{n} (-1)^{(n-k)/2} \binom{(n+k)/2}{k} \left(\frac{1 + (-1)^{n-k}}{2}\right) (x+4)^k \\
&= [x^m] \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{(n-k)/2} \binom{(n+k)/2}{k} \binom{k}{j} \left(\frac{1 + (-1)^{n-k}}{2}\right) 4^{k-j} x^j \\
&= [x^m] \sum_{j=0}^{n} \left[\sum_{k=j}^{n} (-1)^{(n-k)/2} \binom{(n+k)/2}{k} \binom{k}{j} \left(\frac{1 + (-1)^{n-k}}{2}\right) 4^{k-j}\right] x^j \\
&= \sum_{k=m}^{n} (-1)^{(n-k)/2} \binom{(n+k)/2}{k} \binom{k}{m} \left(\frac{1 + (-1)^{n-k}}{2}\right) 4^{k-m}.
\end{aligned}$$

This completes the proof. □

Using the combinatorial interpretation given in the sequence A261711 we obtain the following theorem.

Theorem 9. *The entry $f(n, m)$ of the matrix $\mathcal{F}$ counts the number of words over the alphabet $\Sigma := \{$`0, 1, 2, 3`$\}$ of length $n + m$ having exactly m occurrences of the word 01.*

Proof. Let $g(n,m)$ be the number of words over the alphabet $\Sigma := \{\texttt{0, 1, 2, 3}\}$ of length $n+m$ with exactly m occurrences of the word `01`. For example, $g(3,2) = 12$ with the words being

01010	01011	01012	01013	01001	01101
01201	01301	00101	10101	20101	30101.

We show that $g(n,m)$ satisfies the recurrence relation given in (8) with the same initial values. Let w be a word over the alphabet Σ where w has length $n+m$ and with exactly m occurrences of the word `01`. Thus, $|w| = n+m$ and $|w|_{\texttt{01}} = m$ (the number of `01`'s in w). The word w can be written in the form $w = w_1 a$ where a and w are words over Σ with $|a| = 1$, $|w_1| = n+m-1$ and $|w_1|_{\texttt{01}} = m$, then there are $4g(n-1,m)$ ways. However, we have to subtract the cases where the last symbol of w_1 is `0` and $a = \texttt{1}$. In this case we have $g(n-2,m)$ ways.

On the other hand, the word w can also be written as $w = w_1\texttt{01}$ with $|w_1| = n+m-2$ and $|w_1|_{\texttt{01}} = m-1$. So, there are $g(n-1,m-1)$ ways to do it. Hence,

$$g(n,m) = 4g(n-1,m) - g(n-2,m) + g(n-1,m-1) \text{ for } n,m \geq 1.$$

It is easy to see that that $g(0,0) = 1$ and $g(n,m) = 0$ for $m > n$. Therefore, this holds $g(n,m) = f(n,m)$. □

3.2.4 Fractals from the Riordan arrays

A natural question for infinite numerical arrays is: what is the parity behavior between the entries of the numerical array? (See, for example, the Sierpiński fractal.) Trying to answer this question for our matrices $\mathcal{A}_3$ and $\mathcal{A}_3^{-1}$ we evaluated (using *Mathematica*®) their entries mod 2 and we found two interesting fractals (see Figure 4). An easy way to construct both fractals –without using *Mathematica*®– is as follows. The entries of the fractal in Figure 4 part (a) are obtained by evaluating the equation (5) mod 2. The entries of the fractal in Figure 4 part (b) are obtained by evaluating the equation (8) mod 2. Merlini and Nocentini [13] have studied some relations between Riordan arrays and fractal patterns.

3.3 Counting paths in three-dimensional cube

The main theorem of this section counts the total number of paths, in three-dimensional cube, of length m that end in the horizontal plane. Again the proof uses generating functions.

Theorem 10. *In the three-dimensional cube the following hold:*

(1) the generating function for the total number of paths of length i in $\mathcal{C}_3$ is given by

$$G_3(z) := \sum_{i=0}^{\infty} g_3(i) z^i = \frac{1}{\sqrt{1-8z+12z^2}},$$

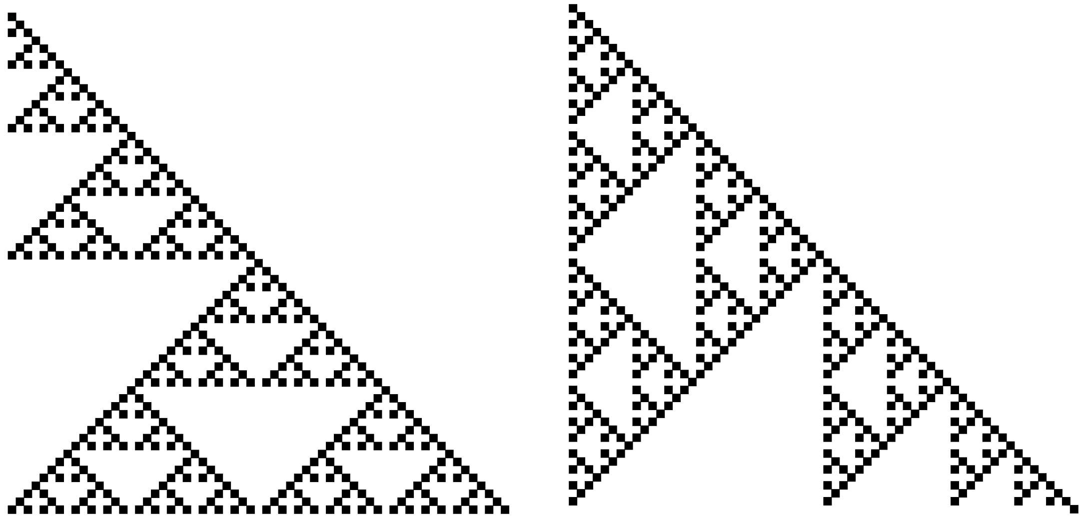

Figure 4: (a) Matrix $\mathcal{A}_3$ mod 2 (b) Matrix $\mathcal{A}_3^{-1}$ mod 2.

(2) the number of paths, in three-dimensional cube, of length m that end in the horizontal plane is given by

$$g_3(m) = \frac{1}{2^m} \sum_{k=0}^{m} \binom{2m-2k}{m-k} \binom{2k}{k} 3^k.$$

Proof. From the first return decomposition a nonempty three-dimensional lattice path T in $\mathcal{C}_3$ may be decomposed as

$$e_3 P(-e_3)T'; \quad (-e_3)P(e_3)T'; \quad e_1 T'; \quad -e_1 T'; \quad e_2 T'; \quad -e_2 T',$$

where P is a path (possibly empty) in $\mathcal{C}_3^+$ and T' is a path (possibly empty) in $\mathcal{C}_3$. Therefore,

$$G_3(z) = 2z^2 T_3(z) G_3(z) + 4z G_3(z) + 1.$$

So, from equation (1) on page 4 we obtain

$$G_3(z) = \frac{1}{1 - 4z - 2z^2 T_3(z)} = \frac{1}{\sqrt{1 - 8z + 12z^2}}. \tag{11}$$

This proves part (1).

Proof of part (2): Noe [19] showed that

$$\sum_{i=0}^{\infty} T_i x^i = \frac{1}{\sqrt{1 - 2bx - (b^2 - 4c)x^2}},$$

where

$$T_n = \frac{1}{4^n} \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2k}{k} (b + 2\sqrt{c})^k (b - 2\sqrt{c})^{n-k}.$$

Therefore, setting $b = 4$ and $c = 1$ we obtain the equation in part (2), completing the proof of the theorem. $\square$

Note that the number $g_3(m)$ is equal to the generalized central trinomial coefficient of $(1 + 4x + x^2)^n$. Then

$$g_3(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k} \binom{n}{2k} 4^{n-2k}.$$

This sequence appears in OEIS as A081671. So, $g_3(n)$ for $n = 0, 1, \ldots, 10$ is given by

$$1, 4, 18, 88, 454, 2424, 13236, 73392, 411462, 2325976, 13233628.$$

In the following theorem we obtain another formula to the sequence $g_3(m)$.

Theorem 11. *The number of paths, in three-dimensional cube, of length m that end in the horizontal plane is given by*

$$g_3(m) = 4^m + \sum_{n=1}^{m} \sum_{k=0}^{\left\lfloor \frac{m-2n}{2} \right\rfloor} \binom{n+2k}{k} \binom{s}{2n+2k} \left(\frac{n}{n+2k}\right) 2^{2m-4k-3n}.$$

Proof. The Equations (3) and (11) imply that

$$\begin{aligned} G_3(z) &= \frac{1}{1 - 4z - 2z^2 T_3(z)} = \frac{1}{1 - 4z - 2z^2\left(1/(1-4z)\right) C(u)} \\ &= \frac{1}{1-4z} \left(\frac{1}{1 - 2uC(u)}\right) = \frac{1}{1-4z} \sum_{n=0}^{\infty} (2uC(u))^n, \end{aligned}$$

where $u = z^2/(1-4z)^2$ and $C(z)$ is the generating function of the Catalan numbers. From identity (6) we have

$$\begin{aligned} G_3(z) &= \frac{1}{1-4z} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{n+2k}{k} \left(\frac{n}{n+2k}\right) \left(\frac{z^{2n+2k}}{(1-4z)^{2n+2k+1}}\right) 2^n \\ &= \frac{1}{1-4z} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{n+2k}{k} \binom{2n+2k+l}{l} \left(\frac{n}{n+2k}\right) 2^{n+2l} z^{2n+2k+l}. \end{aligned}$$

This with $t = 2n + 2k + l$ implies that

$$G_3(z) = \frac{1}{1-4z} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=t-2n-2k}^{\infty} \binom{n+2k}{k} \binom{t}{2n+2k} \left(\frac{n}{n+2k}\right) 2^{2t-3n-4k} z^t.$$

Therefore,

$$[z^m]G_3(z) = 4^m + \sum_{n=1}^{m} \sum_{k=0}^{\left\lfloor \frac{m-2n}{2} \right\rfloor} \binom{n+2k}{k} \binom{m}{2n+2k} \left(\frac{n}{n+2k}\right) 2^{2m-3n-4k}.$$

This proves the theorem. $\square$

Theorem 12 studies the case in which a path does not need to end in the horizontal plane. That is, we study the family of paths in $\mathcal{C}_3^{\pm}$. Theorem 12 was originally proved by Guy.

Theorem 12. *In three-dimensional cube it holds that*

(1) The generating function for the total number of paths of length i in $\mathcal{C}_3^{\pm}$ is given by

$$W_3(z) := \sum_{i=0}^{\infty} w_3(i) z^i = \frac{1}{1-6z},$$

(2) the number of paths, in three-dimensional cube, of length m is given by

$$w_3(m) = 6^m.$$

Proof. First of all we note that from the first return decomposition a nonempty three-dimensional lattice path J in $\mathcal{C}_3^{\pm}$ may be decomposed as

$$e_3J'; \quad (-e_3)J'; \quad e_1J'; \quad (-e_1)J'; \quad e_2J'; \quad (-e_2)J',$$

where J' is a path (possibly empty) in $\mathcal{C}_3^{\pm}$. Therefore, we have that

$$W_3(z) = 6zW_3(z) + 1.$$

Therefore

$$W_3(z) = \frac{1}{1-6z}.$$

This proves part (1) and (2). $\square$

4 Generating functions for paths in the n-space

In this section we generalize the results given in Section 3 for paths in three-dimensional cube to paths in the n-dimensional cube. In Section 3 we classify the families of paths depending on the plane $z = 0$. The generalization is focused depending on the hyperplane $x_n = 0$. Since the results here are a natural generalization of Section 3, we omit some details. Whoever is interested in these problems from the point of view of Riordan arrays can see Nkwanta [20].

Theorem 13. *If C_n is the nth Catalan number, then*

(1) the generating function for the total number of paths of length i in $\mathcal{C}_n^+$ is given by

$$\begin{aligned} T_n(z) &:= \sum_{i=0}^{\infty} a_n(i)z^i = \frac{1-2(n-1)z-\sqrt{1-4(n-1)z+4(n-2)nz^2}}{2z^2} \\ &= \cfrac{1}{1-2(n-1)z-\cfrac{z^2}{1-2(n-1)z-\cfrac{z^2}{1-2(n-1)z-\cfrac{z^2}{\ddots}}}} \end{aligned}$$

(2) the number of paths in $\mathcal{C}_n^+(m)$ is given by

$$a_n(m) = \sum_{i=0}^{\lfloor m/2 \rfloor} C_i \binom{m}{2i} (2(n-1))^{m-2i}. \tag{12}$$

Proof. We prove part (1), the proof of part (2) is similar to Theorem 1 and we omit it. From the first return decomposition a nonempty n-dimensional lattice path P in $\mathcal{C}_n^+$ may be decomposed as

$$e_nP'(-e_n)P'', \quad \pm e_1P', \quad \pm e_2P', \quad \dots \quad , \pm e_{n-1}P',$$

where P' is a path (possibly empty) in $\mathcal{C}_n^+$.

Therefore

$$T_n(z) = z^2T_n^2(z) + 2(n-1)zT_n(z) + 1.$$

This proves part (1). ◻

For example, if $n = 2$, we obtain

$$T_2(z) = \frac{1-2z-\sqrt{1-4z}}{2z^2} = \sum_{n=0}^{\infty} C_{n+1}z^n.$$

Therefore, $a_2(m) = C_{m+1}$, where C_m is the mth Catalan number. Moreover, from Equation (12) we get

$$C_{m+1} = \sum_{i=0}^{\lfloor m/2 \rfloor} C_i \binom{m}{2i} 2^{m-2i}.$$

This identity is known as Touchard's formula. In 2017 Dershowitz [6] found a bijection between the paths of $\mathcal{C}_2^+$ and Dyck paths.

If $a_n(m,k)$ is the total number of lattice paths with height k in $\mathcal{C}_n^+(m)$, then $a_n(m,k)$ satisfies the third order recurrence relation

$$a_n(m,k) = a_n(m-1,k-1) + 2(n-1)a_3(m-1,k) + a_3(m-1,k+1).$$

Therefore, the infinite triangular matrix $\mathcal{A}_n = [a_n(m,k)]_{m,k\geq 0}$ has the Riordan array expression

$$\mathcal{A}_n = (T_n(z), zT_n(z))\,,$$

with A-sequence equal to $(1, 2(n-1), 1)$.

Theorem 14. *In the n-dimensional cube the following hold:*

(1) the generating function for the total number of paths of length i in $\mathcal{C}_n^{\geq}$ is given by

$$\begin{aligned} H_n(z) &:= \sum_{i=0}^{\infty} b_n(i)z^i = \frac{T_n(z)}{1-zT_n(z)} \\ &= \frac{1-2nz-\sqrt{1-4(n-1)z+4(n-2)nz^2}}{2z(2nz-1)} \\ &= \cfrac{1}{1-2(n-1)z-\cfrac{z^2}{1-2(n-1)z-\cfrac{z^2}{1-2(n-1)z-\cfrac{z^2}{\ddots}}}} \end{aligned}$$

(2) the number of paths that belong to $\mathcal{C}_n^{\geq}(m)$ is given by

$$b_n(m) = \sum_{i=0}^{m} \sum_{k=0}^{\lfloor \frac{m-i}{2} \rfloor} \binom{i+2k+1}{k} \binom{m}{i+2k} \left(\frac{i+1}{i+2k+1}\right) (2(n-1))^{m-2k-i}. \qquad (13)$$

Proof. We prove part (1), the proof of part (2) is analogous to the proof of Corollary 4 and we omit it.

From the first return decomposition a nonempty n-dimensional lattice path P in $\mathcal{C}_n^{\geq}$ may be decomposed as P', $\quad P'e_nH'$ where P' is a path (possibly empty) in $\mathcal{C}_n^+$ and H' is a path (possibly empty) in $\mathcal{C}_n^{\geq}$.

Therefore

$$H_n(z) = T_n(z) + zT_n(z)H_n(z).$$

This proves part (1). $\square$

It is easy to see that using Guy [9, Equation (1)], the Equation (13) for $n = 2$ becomes

$$b_2(m) = \binom{2m+1}{m} = \sum_{i=0}^{m} \sum_{k=0}^{\lfloor \frac{m-i}{2} \rfloor} \binom{i+2k+1}{k} \binom{m}{i+2k} \left(\frac{i+1}{i+2k+1} \right) 2^{m-2k-i}.$$

Theorem 14 part (2) with $n = 3$ proves the the problem **10795** [5]. This problem was originally proposed by Deutsch and solved by Brawner [1] (without using generating functions).

Theorem 15. *In the n-dimensional cube the following hold:*

(1) the generating function for the total number of paths of length i in $\mathcal{C}_n$ is given by

$$G_n(z) := \sum_{i=0}^{\infty} g_n(i) z^i = \frac{1}{\sqrt{1 - 4(n-1)z + 4(n-2)nz^2}},$$

(2) the number of paths, in three-dimensional cube, of length m that end in the horizontal plane is given by

$$g_n(m) = \frac{1}{2^m} \sum_{k=0}^{m} \binom{2m-2k}{m-k} \binom{2k}{k} n^k (n-2)^{n-k}.$$

Theorem 16. *In the n-dimensional cube the following hold:*

(1) the generating function for the total number of paths of length i in $\mathcal{C}_n^{\pm}$ is given by

$$W_n(z) := \sum_{i=0}^{\infty} w_n(i) z^i = \frac{1}{1 - 2nz},$$

(2) the number of paths, in the n-dimensional cube, of length m is given by

$$w_n(m) = (2n)^m.$$

5 A relation with the k-colored Motzkin paths

A *Motzkin path* of length n is a lattice path of $\mathbb{Z} \times \mathbb{Z}$ running from $(0, 0)$ to $(n, 0)$ that never passes below the x-axis and whose permitted steps are the up diagonal step $U = (1, 1)$, the down diagonal step $D = (1, -1)$ and the horizontal step $H = (1, 0)$, called rise, fall, and level step, respectively. The number of Motzkin paths of length n is the nth *Motzkin number* m_n, (see A001006). A *grand Motzkin path* of length n is a Motzkin path without the condition that it never passes below the x-axis. The number of grand Motzkin paths of

n	Sequence	A-Sequence
2	1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796	A000108
3	1, 4, 17, 76, 354, 1704, 8421, 42508, 218318	A005572
4	1, 6, 37, 234, 1514, 9996, 67181, 458562, 3172478	A025230
5	1, 8, 65, 536, 4482, 37968, 325509, 2821400	-

Table 1: Sequence $a_n(m)$ for $n = 2, 3, 4, 5$.

length n is the nth *grand Motzkin number* gm_n (see A002426). A *k-colored Motzkin path* is a Motzkin path such that each horizontal step is colored with one of k specific colors. The number of k-colored Motzkin paths of length n is the nth *k-colored Motzkin number* $m_{k,n}$. The generating functions of these type of lattices where also studied by Callan [2]. Analogously, we have *k-colored grand Motzkin paths*, the number of k-colored grand Motzkin paths of length n is denoted by $gm_{k,n}$.

It is easy to obtain a bijection between the 4-colored Motzkin paths of length m and three-dimensional lattice path of $\mathcal{C}_3^+(m)$. That is, we identify the north-east step (U) with $e_3 = (1, 0, 0)$, the south-east step (D) with $-e_3$ and the 4-colored horizontal steps with $\pm e_1$ and $\pm e_2$. Therefore, we obtain Theorem 17 (it was proved originally by Nkwanta using Riordan arrays). By similar reasons as we obtained Theorem 17, we obtain also Theorem 18. From Theorem 13 part (2), the bijection given in Theorem 17 and the recurrence relation given in [29] (see also [25]) we obtain the Corollary 19.

Theorem 17. *The number of lattice paths of length k in $\mathcal{C}_n^+$ is equal to the number of $2(n-1)$-colored Motzkin paths of length k. Thus, $a_n(k) = m_{2(n-1),k}$.*

Theorem 18. *The number of lattice paths of length k in $\mathcal{C}_n$ is equal to the number of $2(n-1)$-colored grand Motzkin paths of length k. Thus, $g_n(k) = gm_{2(n-1),k}$.*

Corollary 19. *The numbers $a_n(m)$ given in Theorem 13 part (2), satisfy the recurrence relation*

$$(m + 2)a_n(m) = 2(n - 1)(2m + 1)a_n(m - 1) + (6 - 2n)(m - 1)a_n(m - 2).$$

6 Tables and sequences from experimentation

From formula (12) we obtain the Table 1 (see Theorem 1 part (2), Theorem 13). That is, we show the first few terms of the sequence $a_n(m)$ for $n = 2, 3, 4, 5$. Note that $a_2(3) = 14$ and $a_3(3) = 17$, (see Figures 1 and 2). From Theorem 14 part (2) we obtain the Table 2. That is, we show the first few terms of the sequence $b_n(m)$ for $n = 2, 3, 4, 5$. From Theorem 15 part (2) we obtain the Table 3. That is, we show the first few terms of the sequence $g_n(m)$ for $n = 2, 3, 4, 5$. Note that the sequence A098410 is equal to the number of paths from $(0, 0)$ to $(n, 0)$ using steps $U = (1, 1), H = (1, 0)$ and $D = (1, -1)$, the H steps can have 6 colors.

n	Sequence	A-Sequence
2	1, 3, 10, 35, 126, 462, 1716, 6435, 24310, 92378, 352716	A001700
3	1, 5, 26, 139, 758, 4194, 23460, 132339, 751526, 4290838	A005573
4	1, 7, 50, 363, 2670, 19846, 148772, 1122995, 8525398, 65030706	-
5	1, 9, 82, 755, 7014, 65658, 618612, 5860611, 55784710, 533147438	-

Table 2: Sequence $b_n(m)$ for $n = 2, 3, 4, 5$.

n	Sequence	A-Sequence
2	1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620	A000984
3	1, 4,18, 88, 454, 2424, 13236, 73392, 411462	A081671
4	1, 6, 38, 252, 1734, 12276, 88796, 652728	A098410
5	1, 16, 258, 4192, 68614, 1130976, 18766356	-

Table 3: Sequence $g_n(m)$ for $n = 2, 3, 4, 5$.

Proposition 20. *For $k \geq 1$, the number of paths in $\mathcal{C}_3^+(k)$ that are completely contained in the xy-plane is 4^k.*

Proposition 20 is easy to prove, we omit it. Motivated by Proposition 20, we have the following conjectures.

Conjecture 1: For $k \geq 1$, the number of paths in $\mathcal{C}_3^+(k)$ that are completely contained in the $xz-$plane is (see Table 4 first line)

$$\sum_{i=1}^{k+1} \frac{\binom{2\,i}{i}\binom{k}{i-1}}{i+1}.$$

Conjecture 2: For $k \geq 1$, the number of paths in $\mathcal{C}_3^+(k)$ that are completely contained in the $yz-$plane is

$$\sum_{i=1}^{k+1} \frac{\binom{2\,i}{i}\binom{k}{i-1}}{i+1}.$$

For the Conjecture 1 see Table 4 first line and for the Conjecture 2 see Table 4 second line. Notice that first and second lines in Tables 4 are exactly the same.

The following sequences/conjectures are based on our experimentation. We do not prove and/or provide any closed formulas for any of them here. We leave them as conjectures for future work.

If we define the *altitude* of a path P in $\mathcal{C}_3^+(k)$ as the largest z-value of all the points in P, then we have the following conjecture.

Conjecture 3: The number of paths in $\mathcal{C}_3^+(k)$ that have altitude 1 is

$$\frac{3^k}{2} - 4^k + \frac{5^k}{2}.$$

plane	Sequence	A-Sequence
xz-plane	3, 10, 36, 137, 543, 2219, 9285, 39587, 171369	A002212$(k+1)$
yz-plane	3, 10, 36, 137, 543, 2219, 9285, 39587, 171369	A002212$(k+1)$

Table 4: Paths in $\mathcal{C}_3^+(k)$ completely contained in the xz-plane or the yz-plane.

h	Sequence	A-Sequence
0	4, 16, 64, 256, 1024, 4096, 16384, 65536	A002212
1	0, 1, 12, 97, 660, 4081, 23772, 133057, 724260	A016753
2	0, 0, 0, 1, 20, 243, 2324, 19271, 145404	-
3	0, 0, 0, 0, 0, 1, 28, 453, 5556	-

Table 5: All paths of altitude h in $\mathcal{C}_3^+(k)$.

We say that a path $P = (p_0, p_1, \dots, p_k)$ in $\mathcal{C}_3^+(k)$ has a *right corner*, if there are three points $p_i, p_{i+1}, p_{i+2} \in \{p_0, p_1, \dots, p_k\}$ such that the vectors $\overrightarrow{p_i p_{i+1}}$ and $\overrightarrow{p_{i+1} p_{i+2}}$ are perpendicular. Using this definition, we have Table 6.

We say that a path $P = (p_0, p_1, \dots, p_k)$ in $\mathcal{C}_3^+(k)$ has an *overlap*, if there are four points $p_i, p_{i+1}, p_{i+2}, p_{i+3} \in \{p_0, p_1, \dots, p_k\}$ such that $p_i = p_{i+3}$ and $p_{i+1} = p_{i+2}$. Using this definition, we have Table 7.

Remark 21. Some of the results of this paper were discovered by using the counting automata methodology (see De Castro et al. [4]).

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r	Sequence	A-Sequence
0	4, 9, 16, 34, 64, 133, 256, 526, 1024	-
1	0, 8, 40, 112, 304, 736, 1768, 4048, 9232	-
2	0, 0, 20, 136, 552, 1808, 5380, 14760, 38936	-
3	0, 0, 0, 72, 512, 2576, 9856, 33832, 104832	-

Table 6: All paths in $\mathcal{C}_3^+(k)$ with exactly "r" right corners.

t	Sequence	A-Sequence
0	4, 12, 40, 152, 608, 2476, 10240, 42972, 182904	-
1	0, 5, 32, 132, 580, 2764, 13420, 64260, 306388	-
2	0, 0, 4, 65, 416, 2052, 10448, 55688, 297516	-
3	0, 0, 0, 5, 96, 953, 6212, 34904, 197824	-

Table 7: All paths in $\mathcal{C}_3^+(k)$ with exactly "t" overlaps.

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