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On Some Sequences Related to Sums of Powers

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Abstract

Automorphic numbers (in a specified base) have the property that the expansion of n^2 ends in that of n; Fairbairn characterized these numbers for all bases in 1969. Here we consider some related sequences: those n for which the sum of the first n natural numbers, squares, or cubes ends in n. For sums of natural numbers, these are Trigg's "trimorphic" numbers; for sums of squares, Pickover's "square pyramorphic" numbers. We characterize the trimorphic numbers for all bases, and the other two for base 10 and prime powers. We also solve a related problem due to Pickover.

1 Introduction

In decimal notation, the number 76 appears as the final digit string of its square, 5776. Such numbers are called *automorphic*, *circular*, or *cyclic* (the inconsistent nomenclature was already noted by Kraitchik [6, pp. 77–78], in 1942.) The sequence of decimal automorphic numbers appears in the *On-line Encyclopedia of Integer Sequences* as sequence A003226. Such numbers were characterized, in all bases, by Fairbairn [2] in 1969. He showed that if B has k distinct prime factors, then there are $2^k - 2$ d-digit automorphic numbers (base B), possibly including leading zeros, as well as the trivial 0 and 1. In particular, if B is a prime power, there are no nontrivial automorphic numbers.

When we wish a string of digits to be interpreted as a number in base B, we express the base (as a decimal number) as a subscript. Thus, 10010_2, 200_3, and 16_{12} all represent 18. On one occasion we use A for the digit after 9.

It is clear that a number that ends its own square must end its own cube and all higher powers, too. On the other hand, there are numbers (such as 49) which end their own cubes and not their own squares. This justifies the 1974 introduction by Hunter [3] of the separate term *trimorphic* for such numbers (A033819.) However, these numbers have also been called "perissomorphic" [9], while "trimorphic" has also been used [1, 9, 10] for a number n that ends the decimal representation of the nth triangular number.

To avoid ambiguity, given any function $F : \mathbb{N} \to \mathbb{N}$, we say that the number n is F-*morphic* in base B if the digit string for $F(n)$ terminates in that for n. Thus, the decimal automorphic numbers are n^2-morphic (base 10). We state the following obvious lemmas for completeness.

Lemma 1. *A d-digit number is F-morphic (base B) if and only if*

$$B^d \mid (F(n) - n).$$

Lemma 2. *If $d > 1$, then $B^{d-1} < n < B^d$.*

In this paper we consider the sequences of such numbers generated in this way by the following four functions:

- the *triangular numbers* $T(n) := \sum_{k=0}^{n} k = n(n+1)/2$ (A000217);
- the *pyramidal numbers* $P(n) := \sum_{k=0}^{n} k^2 = n(n+1)(2n+1)/6$ (A000330);
- the *hyperpyramidal numbers* $H(n) := \sum_{i=0}^{n} i^3 = (T(n))^2$ (A000537); and
- Pickover's *cake numbers* $K(n) := (n^2 + n + 2)/2 = T(n) + 1$ (A000124).

In Section 2 we characterize the T-morphic numbers to any base B. If B is a prime power, only 0 and 1 are T-morphic; otherwise infinitely many numbers have the property. In Section 3, we characterize P-morphic numbers for prime power bases and for base 10. In the latter case, we will see that the sequence of P-morphic numbers is the union of twelve simpler sequences. In Section 4 we characterize H-morphic numbers, again for prime power bases and base 10, and show that the sequence of decimal H-morphic numbers is the union of ten simpler sequences. Finally, we settle a conjecture of Pickover by showing that there are no decimal K-morphic numbers.

2 T-morphic numbers

If the digit string (base B) representing $T(n)$ ends in the digit string representing n, we call n T-*morphic*. For instance, $T(25) = 3255$, so 25 is T-morphic in base 10. The sequence of decimal T-morphic numbers is A067270; a comment in this OEIS entry credits David Wilson with proving that every T-morphic number is automorphic. We show below that, conversely, for odd bases all automorphic numbers are T-morphic.

Trigg [10] listed all the decimal T-morphic numbers with five or fewer digits: 1, 5, 25, 625, 9376, and 90625. He also listed the first five base-6 T-morphic numbers, and noted (without proof, but correctly) that there are no T-morphic numbers in other bases less than 10. (This follows from Wilson's observation, combined with Fairbairn's proof that there are no automorphic numbers to a prime-power base.)

Let $T(n) = n(n+1)/2$ be the nth triangular number. To characterize the T-morphic numbers, we follow the approach used by Fairbairn [2] for his study of automorphic numbers.

Lemma 3. *Suppose that n is a d-digit T-morphic number (base B). Then:*

(i) $2B^d \mid n(n-1)$;

(ii) *if B is odd,* $B^d \mid n(n-1)$;

(iii) *if $p \mid B$, it follows that p divides exactly one of n and $n-1$.*

Proposition 4. *For a prime or prime power base, the only T-morphic numbers are* 0 *and* 1*.*

Proof. For a prime power base $B = p^r$, Lemma 3(i) implies that $2p^{dr} \mid n(n-1)$. As n and $n-1$ have no common factors, $B^d = p^{dr}$ divides either n or $n-1$. But, for $0 \le n < B^d$, $B^d \mid n$ implies $n = 0$, and $B^d \mid n-1$ implies $n = 1$. $\square$

Proposition 5. *If B is odd and has k distinct prime factors, there are 2^k d-digit T-morphic numbers (base B) if we permit leading zeros. If we do not permit leading zeros, there are at most 2^k one-digit T-morphic numbers and, for $d > 1$, at most $2^k - 2$ d-digit T-morphic numbers.*

Proof. If B is odd, then as shown above, a necessary and sufficient condition for a d-digit number n (possibly with one or more leading zeros, which must also appear in $T(n)$) to be T-morphic (base B) is that $B^d \mid n(n-1)$. If this is so, then by Lemma 3(iii), n determines a unique factorization $B^d = CD$, where $C \mid n$ and $D \mid (n-1)$ are coprime.

There are 2^k such factorizations. The improper factorization $C = 1$, $D = B^d$ gives $B^d \mid n-1$; this has no solutions with $B^{d-1} \le n < B^d$, except when $d = 1$ and $n = 1$. For $d > 1$ this is still a solution if we permit leading zeros. The other improper factorization, with $C = B^d$ and $D = 1$, gives $B^d \mid n$, which has the one-digit solution $n = 0$. There are $2^k - 2$ proper ordered factorizations; given any such factorization, the Chinese remainder theorem says that the system

$$\begin{aligned} C &\mid n \\ D &\mid n-1 \end{aligned} \tag{1}$$

has a unique solution $n \in [0, B^d)$. If this solution is less than B^{d-1}, then n (without leading zeros) is not a d-digit number, although n is still T-morphic. $\square$

Example 6. Let $B = 15$. For $d = 1$, we have the special solutions 0 and 1 from improper factorizations. The proper factorization $C = 3$, $D = 5$ gives the system of congruences $3 \mid n$, $5 \mid (n-1)$, with solution $n = 6$.

$$6 = 6_{15}; \quad T(6) = 21 = 16_{15}.$$

The proper factorization $C = 5$, $D = 3$ gives the system of congruences $5 \mid n$, $3 \mid n-1$, with solution $n = 10$ (which we represent in base 15 with the digit 'A'.)

$$10 = A_{15}; \quad T(10) = 55 = 3A_{15}.$$

For $d = 2$, we consider proper factorizations of 15^2 into coprime factors. The proper factorization $C = 9$, $D = 25$ gives the system $9 \mid n$, $25 \mid n-1$ with solution $n = 126$.

$$126 = 86_{15}; \quad T(126) = 16002 = 2586_{15}.$$

The proper factorization $C = 25$, $D = 9$ gives the system $25 \mid n$, $9 \mid n-1$ with solution $n = 100$.

$$100 = 6A_{15}; \quad T(100) = 5050 = 176A_{15}.$$

($T(100)$ is, of course, the value apocryphally computed by the youthful Gauß.)

In base 10, Lemma 3(i) yields $2^{d+1} \mid n(n-1)$ and $5^d \mid n(n-1)$. By Lemma 3(iii) we have either

$$\begin{aligned} 2^{d+1} &\mid n \\ 5^d &\mid n-1 \end{aligned} \tag{2}$$

or

$$\begin{aligned} 2^{d+1} &\mid n-1 \\ 5^d &\mid n. \end{aligned} \tag{3}$$

For any d, the systems (2) and (3) each have a unique solution in the interval $[0, 2 \cdot 10^d)$. We call these $g(d)$ and $g'(d)$ respectively;

they are T-morphic if and only if they are less than 10^d. The first few solutions are

$$(g(d)) = (16, 176, 1376, \mathbf{9376}, 109376, 1109376, \mathbf{7109376}, 187109376, \ldots);$$

$$(g'(d)) = (\mathbf{5}, \mathbf{25}, \mathbf{625}, 10625, \mathbf{90625}, \mathbf{890625}, 12890625, \mathbf{12890625}, \ldots).$$

The T-morphic numbers are shown in boldface; we note that, for any given g, exactly one of $\{g(d), g'(d)\}$ appears to be T-morphic. In fact this is the case: adding the congruences (2,3), we get

$$\begin{aligned} 2^{d+1} &\mid g(d) + g'(d) - 1, \\ 5^d &\mid g(d) + g'(d) - 1, \end{aligned} \tag{4}$$

whence $g(d) + g'(d) \equiv 1 \pmod{2 \cdot 10^d}$. As $0 < g(d), g'(d) < 2 \cdot 10^d$, we must have

$$g(d) + g'(d) = 2 \cdot 10^d + 1.$$

By examining the last digits, we rule out the solution $\{10^d, 10^d + 1\}$.

Proposition 7. *The decimal numbers* 0*,* 1*, and* 5 *are* T*-morphic; and for every* $d > 1$ *there is at most one decimal* T*-morphic number with* d *digits.* □

Remark 8. If we do not allow leading zeros, there may be no d-digit decimal T-morphic number. This occurs first when $d = 12$: $g(12) = 81787109376 < 10^{11}$ while $g'(12) = 1918212890625 > 10^{12}$. If, e.g., $g(d) < 10^{d-1}$, then $g(d)$ is equal to $g(d-1)$, and is T-morphic. In fact, more is true; $g(d)$ is T-morphic even when written with a leading zero:

$$T(g(12)) = 334456563008178710937 6.$$

In fact, as may be seen, $g(11) = g(12) = g(13)$. If $g(d) > 10^d$, then $g(d)$ is T-morphic if and only if $g(d) = g(d+1)$. We conclude that every T-morphic number with d digits appears as $g(d)$ or $g'(d)$.

Proposition 7 generalizes straightforwardly to bases with more prime factors.

Proposition 9. *If* B *is even with exactly* k *distinct prime factors, for every* $d > 1$ *there are at most* $2^{k-1} - 1$ T*-morphic numbers (base* B*) with* d *digits.* □

If we consider strings that differ only by leading zeros to be distinct, we have shown that there are infinitely many T-morphic numbers. If we do not, we must show that there does not exist some $g(d)$ such that, for all $d' > d$, $g(d')$ is obtained by prefixing $d' - d$ leading zeros to $g(d)$. But if this were so, we would have $g(d)^2 = g(d')^2 \equiv g(k) \equiv g(d) \pmod{B^k}$ for all k. Hence $g(d)^2 = g(d)$, so that $g(d)$ must be either 0 or 1. We conclude that even if we do not consider numbers that differ only by a string of leading zeros to be distinct, there must be infinitely many T-morphic numbers.

However, there appears to be no simple pattern determining which of $g(d)$ and $g'(d)$ is T-morphic. The first thousand terms of the sequence A067270 show no obvious sign of periodicity, asymptotic dominance by last digit 5 or 6, or other structure.

3 P-morphic numbers

The n^{th} *square pyramidal number* $P(n)$ (A000330) is defined to be $\sum_{i=0}^{n} i^2 = n(n+1)(2n+1)/6$. This sequence is so called because $P(n)$ counts the number of spheres in a pyramidal pile on an $n \times n$ square base. Pickover [8, p.160] and the notes on A093534 call the decimal P-morphic numbers *square pyramorphic.* We have

$$P(n) - n = \frac{n(n+1)(2n+1)}{6} - n = \frac{2n^3 + 3n^2 - 5n}{6} = \frac{n(n-1)(2n+5)}{6},$$

and so n is P-morphic if and only if

$$6B^d \mid n(n-1)(2n+5). \tag{5}$$

The following lemma gathers useful and easily-proved facts about the right-hand side of (5).

Lemma 10. *Of the three numbers n, $n-1$, and $2n+5$:*

(i) *$2n+5$ is always odd, and exactly one of the other two is even;*

(ii) *exactly one is divisible by 3;*

(iii) *only n and $2n+5$ can both be divisible by 5, and they cannot both be divisible by 25;*

(iv) *only $n-1$ and $2n+5$ can both be divisible by 7, and they cannot both be divisible by 49;*

(v) *no two have a common prime factor greater than 7.*

We first consider the numbers which are P-morphic to prime power bases.

Proposition 11. *The only P-morphic numbers (base 2^r) are the trivial cases 0 and 1.*

Proof. As observed above, only one of the first two factors of $n(n-1)(2n+5)$ can be even, and $2n+5$ must be odd. So if (5) is satisfied, $2 \cdot (2^r)^d$ must divide n or $n-1$, both of which are, by Lemma 2, less than $(2^r)^d$. This is only possible in the trivial cases where $n = 0$ or 1. □

Proposition 12. *The only P-morphic numbers (base 3^r) are the trivial cases 0 and 1, and the special case $a = 2$ when $r = 1$.*

Proof. For a number n to be P-morphic with d digits base 3^r, it is necessary that one of the following hold:

$$\begin{aligned} 3 \cdot 3^{rd} &\mid n \\ 3^{rd} &> n; \end{aligned} \tag{6}$$

$$3 \cdot 3^{rd} \mid n - 1 \qquad (7)$$
$$3^{rd} > n;$$

or

$$3 \cdot 3^{rd} \mid 2n + 5 \qquad (8)$$
$$3^{rd} > n.$$

Neither the system (6) nor the system (7) have any solution except for the trivial $n = 0$ and $n = 1$ respectively. The system (8) requires that $3^{rd} < 5$, whence $r = d = 1$. Thus no number is P-morphic base 3^r except for 0 and 1 (any r) and 2 (base 3 only.) $\square$

Proposition 13. *The only d-digit P-morphic numbers (base 5^r) are the trivial cases 0 and 1, and, for $rd > 1$, the numbers $c \cdot 5^{rd-1}$ for $c \in \{1, 2, 3, 4\}$, and the numbers $c \cdot 5^{rd-1} + (5^{rd-1} - 5)/2$ for $c \in \{0, 1, 2, 3, 4\}$.*

Proof. 5^{rd} cannot (nontrivially) divide any of n, $n - 1$, or $2n + 5$, for the same reasons as above; but if $rd > 1$, then, by Lemma 10(iii), one of n and $2n + 5$ can be divisible by 5^{rd-1}, the other by 5. (In the case $r = d = 1$, 5^1 cannot be factorized.) In general, we have, for $rd > 1$, the following two sets of solutions:

- $5^{rd-1} \mid n$, whence $5 \mid 2n + 5$; the constraint $n < 5^{rd}$ gives us solutions $p_5(c, rd) := c \cdot 5^{rd-1}, c \in \{1, 2, 3, 4\}$;

- $5^{rd-1} \mid 2n + 5$, whence $5 \mid n$; the constraint $n < 5^{rd}$ gives us solutions $p'_5(c, rd) := c \cdot 5^{rd-1} + (5^{rd-1} - 5)/2, c \in \{0, 1, 2, 3, 4\}$.

By Lemma 10(i,ii), 2 and 3 each divide one of n, $(n - 1)$, and $(2n + 5)$; so (5) follows. $\square$

Corollary 14.

- *There are, in base 5, two P-morphic numbers with one digit, four with two digits, and eight with d digits for any $d > 2$.*
- *There are, in base 25, six P-morphic numbers with one digit and nine with d digits for any $d > 1$.*
- *For $r > 2$ there are, in base 5^r, eleven P-morphic numbers with one digit and nine with d digits for any $d > 1$.*

Proof. There are no one-digit base-5 P-morphic numbers except for 0 and 1, because $5^{1\cdot 1}$ does not factor. There are only four two-digit base-5 P-morphic numbers, because $(5^{2-1} - 5) = 0$, so that $p_5(c, 2) = p'_5(c, 2)$ for $c \in \{1, 2, 3, 4\}$, while $p'_5(0, 2) = 0$. For $r > 2$ the numbers

$p_5(c,d)$ and $p'_5(c,d)$ are all distinct for $c \in \{1,2,3,4\}$, but $p'_5(0,d) = p_5(2,d-1)$ (and has $d-1$ digits.)

In base 25, one-digit numbers have $rd = 2$; again, $p_5(c,2) = p'_5(c,2)$ for $c \in \{1,2,3,4\}$, and there are also the trivial solutions 0 and 1. For $d > 2$, $p'_5(0,2d) = (5^{2d-1} - 5)/2 > 25^{d-1}$, so that $p'_5(0,2d)$ is a d-digit number. There are thus nine distinct d-digit solutions.

For $k > 3$, we always have $rd > 2$, so all five $p'_5(c,r)$ and four $p_5(c,r)$ are distinct from each other and from the trivial solutions 0 and 1. For $d > 1$ the solutions are exactly the five $p'_5(c,rd)$ and four $p_5(c,rd)$. ☐

Example 15. The sequence of base-5 P-morphic numbers begins

$$(0_5, 1_5, 10_5, 20_5, 30_5, 40_5, 100_5, 120_5, 200_5, 220_5, 300_5, 320_5, 400_5, 420_5, \ldots)$$

The situation for powers of 7 is very similar.

Proposition 16. *The only P-morphic numbers (base 7^r) are the trivial cases 0 and 1, and, for $rd > 1$, the numbers $c \cdot 7^{rd-1} + 1$ for $c \in \{1,2,3,4,5,6\}$, and $c \cdot 7^{rd-1} + (7^{rd-1} - 5)/2 + 1$ for $c \in \{0,1,2,3,4,5,6\}$.*

Proof. This proceeds like the proof above, noting that, of the three factors, only $n-1$ and $2n+5$ can be simultaneously divisible by 7. ☐

Corollary 17.

- *There are, in base 7, two P-morphic numbers with one digit, six with two digits, and twelve with d digits for any $d > 2$.*
- *There are, in base 49, eight P-morphic numbers with one digit and, for $d > 1$, thirteen with d digits.*
- *For $r > 2$ there are, in base 7^r, fifteen P-morphic numbers with one digit. For any $d > 1$ there are thirteen with d digits.* ☐

Proposition 18. *If p is a prime greater than 7, the only P-morphic numbers (base p^r) are the trivial cases 0 and 1, and the numbers $(p^{rd} - 5)/2$.*

Proof. No power of a prime $p > 7$ can divide two of the three factors of the right-hand side of (5) nontrivially; it follows that we must have one of the following:

$$\begin{aligned} &p^{rd} \mid n \\ &p^{rd} > n; \end{aligned} \tag{9}$$

$$\begin{aligned} &p^{rd} \mid n-1 \\ &p^{rd} > n; \end{aligned} \tag{10}$$

or

$$p^{rd} \mid 2n+5 \qquad (11)$$
$$p^{rd} > n.$$

The first two systems give 0 and 1 respectively. In the third system, for $m \geq 3$ we have $mp^{rd} > 3n > 2n+5$, while $2p^{rd}$ is even. We conclude that $p^{rd} = 2n+5$, whence the conclusion follows.

□

Corollary 19. *For any base which is a power of a prime greater than 7, there are three one-digit P-morphic numbers and one P-morphic number of every other length.* □

Example 20. The base-11 P-morphic numbers are

$$(0, 1, 3, 53_{11}, 553_{11}, 5553_{11}, 55553_{11}, \ldots, \frac{11^n - 5}{2}, \ldots).$$

The fact that there are four "special" prime numbers in the theory of P-morphic numbers adds significantly to the complication when the base has more than one prime factor. We shall consider the decimal case in detail; the same techniques can be applied to other composite bases. We define the following sequences:

- $a(d) := 4 \cdot 10^{d-1}$;
- $b(d)$ is the unique solution in $[0, 2 \cdot 10^d)$ of the system $2^{d+1} \mid b(d)$, $5^d \mid b(d) - 1$;
- $c(d)$ is the unique solution in $[0, 4 \cdot 10^{d-1})$ of the system $2^{d+1} \mid c(d)$, $5^{d-1} \mid 2c(d) + 5$;
- $c'(d)$ is the unique solution in $[0, 4 \cdot 10^{d-1})$ of the system $2^{d+1} \mid c'(d) - 1$, $5^{d-1} \mid 2c'(d) + 5$;
- $c''(d)$ is the unique solution in $[0, 4 \cdot 10^{d-1})$ of the system $2^{d+1} \mid c''(d) - 1$, $5^{d-1} \mid c''$.

Theorem 21. *Every decimal P-morphic number belongs to one of the following twelve sequences:*

1. $(a(d) : d \geq 2)$;
2. $(2a(d) : d \geq 2)$;
3. $(b(d) : d \geq 1)$;
4. $(c(d) : d \geq 2)$;
5. $(c(d) + a(d) : d \geq 2)$;
6. $(c(d) + 2a(d) : d \geq 2, c(d) + 2a(d) < 10^d)$;

7. $(c'(d) : d \geq 2)$*;*

8. $(c'(d) + a(d) : d \geq 2)$*;*

9. $(c'(d) + 2a(d) : d \geq 2, c(d) + 2a(d) < 10^d)$*;*

10. $(c''(d) : d \geq 2)$*;*

11. $(c''(d) + a(d) : d \geq 2)$*;*

12. $(c''(d) + 2a(d) : d \geq 2, c(d) + 2a(d) < 10^d)$*.*

Proof. For n to be P-morphic to the base 10, we must have

$$6 \cdot 10^d = 2^{d+1} \cdot 3 \cdot 5^d \mid (P(n) - n).$$

By Lemma 10(ii), the factor of 3 is always present. By Lemma 10(i), 2^{d+1} must divide n or $n-1$. We consider these two cases separately.

Case 1 ($2^{d+1} \mid n$)*.* When $d = 1$, 5 may divide either $n-1$ or both of n and $2n+5$; neither of these yields a solution in $[0, 10)$. For $d > 1$, either $5^d \mid n-1$, or 5^{d-1} divides one of $\{n, 2n+5\}$, and 5 divides the other.

Subcase 1.1 ($2^{d+1} \mid n$, $5^{d-1} \mid n$, and $d \geq 2$)*.* We cannot have $5^d \mid n$ (except in the trivial case $n = 0$), as this would require $2 \cdot 10^d \mid n < 10^d$. For $d > 1$, however, we have $5 \mid (2n+5)$, so that $5^d \mid P(n)$. The system

$$\begin{aligned} 2^{d+1} &\mid n \\ 5^{d-1} &\mid n \end{aligned} \tag{12}$$

has solutions

$$a(d) := 4 \cdot 10^{d-1} \text{ and } 2a(d) = 8 \cdot 10^{d-1},$$

both always less than 10^d. The first few values of the sequences are

$$(a(d) : d \geq 2) = (-, 40, 400, 4000, \ldots) \text{ and } (2a(d) : d \geq 2) = (-, 80, 800, 8000, \ldots)$$

respectively; note that $a(1) = 4$ and $2a(1) = 8$ are not P-morphic.

Subcase 1.2 ($2^{d+1} \mid n$, $5^{d-1} \mid n-1$, and $d \geq 2$)*.* In this case, no other factor of $P(n) - n$ can be divisible by 5, so we must have $5^d \mid n-1$. The system

$$\begin{aligned} 2^{d+1} &\mid n \\ 5^d &\mid n-1 \end{aligned} \tag{13}$$

has a unique solution $b(d) \in [0, 2 \cdot 10^d)$. The first few values in the sequence are

$$(b(d) : d \geq 1) \in (16, 176, 1376, \mathbf{9376}, 109376, 1109376, \mathbf{7109376}, 187109376, \ldots)\ ;$$

values less than 10^d, shown in bold, are P-morphic.

Subcase 1.3 ($2^{d+1} \mid n$, $5^{d-1} \mid 2n+5$, and $d \geq 2$). If $d \geq 2$ we have $5 \mid n$, whence $5^d \mid P(n)-n$; and it suffices to solve the system

$$\begin{aligned} 2^{d+1} &\mid n \\ 5^{d-1} &\mid 2n+5. \end{aligned} \tag{14}$$

This system has a unique solution $c(d) \in [0, 4\cdot 10^{d-1})$ and an arithmetic sequence of solutions $c(d)+ka(d)$. Of these, $c(d)$ always has d or fewer digits; $c(d)+a(d)$ always has exactly d digits; $c(d)+2a(d)$ has d or $d+1$ digits; and the others always have more than d digits. The first few terms in the sequence $(c(d))$ are

$$(c(d) : d \geq 1) = (0, 0, 160, 2560, 26560, 226560, \mathit{0226560}, 12226560, \ldots).$$

The value $c(6) = c(7) = 226560$ may be considered as P-morphic with or without a leading zero (see remark 8.)

Case 2 ($2^{d+1} \mid n-1$).

Subcase 2.1 ($2^{d+1} \mid n-1$ and $5^{d-1} \mid n-1$). In this case, 5 cannot divide n or $2n+5$, so we must have $5^d \mid n-1$. But then $2\cdot 10^d \mid n-1 < 10^d$, giving only the trivial solution $n=1$.

Subcase 2.2 ($2^{d+1} \mid n-1$ and $5^{d-1} \mid 2n+5$). The system

$$\begin{aligned} 2^{d+1} &\mid n-1 \\ 5^{d-1} &\mid 2n+5 \end{aligned} \tag{15}$$

yields solutions in $[0, 4\cdot 10^{d+1})$ of the form

$$(c'(d) : d \geq 1) = (1, 25, 385, 1185, 37185, 317185, 1117185, 25117185, \ldots)$$

along with $c'(d)+a(d)$ and sometimes $c'(d)+2a(d)$.

Subcase 2.3 ($2^{d+1} \mid n-1$ and $5^{d-1} \mid n$). The system

$$\begin{aligned} 2^{d+1} &\mid n-1 \\ 5^{d-1} &\mid n \end{aligned} \tag{16}$$

has solutions in $[0, 4\cdot 10^{d-1})$ of the form

$$(c''(d) : d \geq 1) \in (5, 25, 225, 2625, 10625, \mathit{090625}, \mathit{0890625}, 12890625, \ldots),$$

along with $c''(d)+a(d)$ and sometimes $c''(d)+2a(d)$.

□

Remark 22. For $d > 3$, if a d-digit P-morphic number n ends in

- **00**: then $n = a(d)$ or $2a(d)$;

- **76**: then $n = b(k)$ for $k \geq d-1$;
- **60**: then $n = c(k)$ for $k \geq d$, or $n = c(d) + a(d)$, or $n = c(k) + 2a(k)$ for $k \in \{d-1, d\}$;
- **85**: then $n = c'(k)$ for $k \geq d$, or $n = c'(d) + a(d)$, or $n = c'(k) + 2a(k)$ for $k \in \{d-1, d\}$;
- **25**: then $n = c''(k)$ for $k \geq d$, or $n = c''(d) + a(d)$, or $n = c''(k) + 2a(k)$ for $k \in \{d-1, d\}$;

and these are the only two-digit strings that a P-morphic number larger than 100 can end in.

Remark 23. It is possible for $b(d)$, $c(d)$, $c'(d)$, or $c''(d)$ to be less than 10^{d-1}. In such a case, the number is always P-morphic; and if, e.g.,

$$10^{n-1} < b(d) < 10^n \text{ for } n < d,$$

then $b(n) = b(n+1) = \cdots = b(d)$, and $b(n)$ appears in $P(b(n))$ prefixed by $d - n$ zeros. To take the example above,

$$P(c(6)) = P(226560) = 387642449\mathbf{0}226560.$$

It is also possible for $b(d)$, $c(d) + 2a(d)$, $c'(d) + 2a(d)$, or $c''(d) + 2a(d)$ to be greater than 10^d. In such a case the number is usually not P-morphic, but may be. For instance,

$$b(9) = 1787109376 > 10^9 \text{ but } P(b(9)) = 1902532768569804241787019376.$$

If $b(d)$ is P-morphic and greater than 10^d, then $b(d)$ must equal its successor in the same sequence. If $c(d) + 2a(d)$ is P-morphic and greater than 10^d, then $c(d) + 2a(d) = c(d+1)$; analogous statements hold for $c'(d) + 2a(d)$ and for $c''(d) + 2a(d)$. We conclude that every P-morphic number with d digits appears *in its proper place* in one of these sequences.

The only other identities between elements of these sequences are

$$\begin{aligned}
&a(1) = 2a(1) = c(1) = c(1) + a(1) = c(1) + 2a(1) = c(2) = 0;\\
&c'(1) = c'(1) + a(1) = c'(1) + 2a(1) = 1;\\
&c''(1) = c''(1) + a(1) = c''(1) + 2a(1) = 5;\\
&c'(2) = c''(2) = 25.
\end{aligned}$$

Remark 24. The sequence of decimal P-morphic numbers appears in the OEIS as A093534. The initial terms are (0, 1, 5, 25, 40, 65, 80, 160, 225, 385, 400, 560, 625, 785, 800, 960, 1185, 2560, 2625, 4000, 5185, 6285,6625, 8000, 9185, 9376, ...).

Corollary 25. *There are three single-digit P-morphic decimal numbers and four two-digit P-morphic decimal numbers.*

For any $d > 2$, there are at least eight and at most eleven distinct P-morphic decimal numbers with d digits.

Proof. For $d = 1$ and $d = 2$ this is true by inspection. For $d > 2$ the sequences $a(d)$, $2a(d)$, $c(d) + a(d)$, $c'(d) + a(d)$, and $c''(d) + a(d)$ always yield distinct P-morphic numbers of d digits; and, additionally, at least one of $\{c(d), c(d) + 2a(d)\}$, of $\{c'(d), c'(d) + 2a(d)\}$, and of $\{c''(d), c''(d) + 2a(d)\}$ must have exactly d digits. This minimum is attained (see below): however, at most five of

$$\{c(d), c(d) + 2a(d), c'(d), c'(d) + 2a(d), c''(d), c''(d) + 2a(d)\}$$

can be d-digit numbers. We note that

$$10^{d-1} < c(d), c(d) + 2a(d) < 10^d \Leftrightarrow 10^{d-1} < c(d) < 2 \cdot 10^{d-1} \,;$$

and similarly for $c'(d)$, $c''(d)$. It thus suffices to show that not all of $c(d), c'(d), c''(d)$ can be in this interval. Suppose, for a contradiction, that they are; then

$$0 < c(d) - c'(d) + c''(d) < 4 \cdot 10^{d-1}.$$

But if we combine the three congruences (14,15,16) we get

$$\begin{aligned} 2^{d+1} &\mid c(d) - c'(d) + c''(d) \\ 5^{d-1} &\mid c(d) - c'(d) + c''(d), \end{aligned} \tag{17}$$

whence $4 \cdot 10^{d-1} \mid c(d) - c'(d) + c''(d)$, a contradiction. □

Remark 26. The minimum of eight d-digit P-morphic decimal numbers is attained for $d = 6$, when we have only

$$226560, 317185, 400000, 490625, 626560, 717185, 800000, \text{ and } 890625.$$

The maximum of eleven is first attained for $d = 49$; only $c''(49) + 2a(49) \approx 1.0982 \times 10^{49}$ is out of range.

4 H-morphic numbers

Let $H(n) := \sum_{k=0}^{n} k^3$ (the "hyperpyramidal numbers", A000537). If the base-B expansion of $H(n)$ ends in the base-B expansion of n, we call n *H-morphic*. By what may be viewed as coincidence (or "the strong law of small polynomials"), $H(n) = T(n)^2$; so as every T-morphic number is automorphic, any such number must also be H-morphic.

Are there other H-morphic numbers? As

$$H(n) - n = \frac{n(n-1)(n^2 + 3n + 4)}{4}$$

has an irreducible quadratic factor $h(n) := n^2 + 3n + 4$, our methods to date will not work. The following result from p-adic analysis will be useful.

Theorem 27. [4, §1.7] or [5, p.10]*: If F is a polynomial, $F(x) = 0$ in $\mathbb{Z}/p^d\mathbb{Z}$, and $F'(x) \neq 0$ in $\mathbb{Z}/p^d\mathbb{Z}$, then x lifts to a zero $\hat{x}$ of F in the field $\mathbb{Q}_p$ of p-adic numbers.*

Let π_d be the canonical projection $\mathbb{Q}_p \to \mathbb{Z}/p^d(\mathbb{Z})$.

Proposition 28. *For every d, the interval $[0, 2^d - 1]$ contains one even number $n_0(d)$, and one odd number $n_1(d)$, satisfying $2^d \mid h(n)$.*

Proof. In $\mathbb{Z}/2\mathbb{Z}$, $h(0) = h(1) = 0$ and $h'(0) = h'(1) = 1$; thus, by Hensel's lemma, there exist $\hat{0} := \cdots 11110100100$ and $\hat{1} := \cdots 00001011001$ in $\mathbb{Q}_2$ with $h(\hat{0}) = h(\hat{1}) = 0$; and these are the only dyadic zeros of h. Then $n_0(d) = \pi_d(\hat{0})$ and $n_1(d) = \pi_d(\hat{1})$.

□

The first few values are

$$\begin{aligned}(n_0(d) : d \geq 1) &= (0, 0, 4, 4, 4, 36, 36, 164, 420, 932, \ldots);\\ (n_1(d) : d \geq 1) &= (1, 1, 1, 9, 25, 25, 89, 89, 89, 89, \ldots).\end{aligned} \tag{18}$$

Remark 29. Note that $\hat{0}+\hat{1} = -3$; thus their digits beyond the "2's place" are complementary. Similarly, $n_0(d) + n_1(d) = 2^d - 3$ for $d > 1$.

Proposition 30.

(i) *The only base-7 H-morphic numbers are 0, 1, and 2. For $r > 1$ the only H-morphic numbers in base 7^r are 0 and 1.*

(ii) *If an odd prime p is congruent to 3, 5, or 6 (mod 7), the only H-morphic numbers in base p^r are 0 and 1.*

(iii) *If an odd prime p is congruent to 1, 2, or 4 (mod 7) there is at least one and at most two d-digit H-morphic numbers in base p^r.*

Proof. For $p > 2$, at most one of n, $n - 1$, or $h(n)$ can be divisible by p^r. But the system $p^{rd} \mid n$, $0 \leq n < p^{rd}$ has no solution except for $n = 0$; and $p^{rd} \mid n - 1$, $0 \leq n < p^{rd}$ has no solution except for $n = 1$. For any other H-morphic numbers to exist, we must have $p^{kd} \mid h(n)$. We can use the quadratic formula to solve $h(n) \equiv 0 \bmod p^d$ if and only if -7 is a quadratic residue mod p; that is, by the quadratic reciprocity theorem, when $p \equiv 1$, 2, or 4 (mod 7). We also have the special case $p = 7$ where $h(2) \equiv h'(2) \equiv 0 \pmod 7$ but $h(n)$ has no zeros mod 49. Thus, 0, 1, and 2 are the only H-morphic numbers in base 7, and 2 is not H-morphic base 7^r for $r > 1$.

Suppose that $p \equiv 1$, 2, or 4 (mod 7). Then $h(n)$ has two zeros, n and n', in $\mathbb{Z}/p\mathbb{Z}$; and for each d, Hensel's lemma gives liftings to $\hat{n}, \hat{n'} \in \mathbb{Q}_p$. These project to $n(d) := \pi_d(\hat{n})$ and $n'(d) := \pi_d(\hat{n'})$ in $[0, p^{rd})$, such that p^{rd} divides both $h(n(d))$ and $h(n'(d))$. It follows that $n(d)$ and $n'(d)$ are H-morphic, although they do not necessarily have d digits; and that no other d-digit numbers are H-morphic.

By Vieta's formula, $n + n' = -3$ in $\mathbb{Z}/p\mathbb{Z}$. Thus $\hat{n} + \hat{n'} = -3$ in $\mathbb{Q}_p$, and $n(d) + n'(d) = p^{rd} - 3$; it follows that at least one of $n(d)$ and $n'(d)$ is greater than $p^{r(d-1)}$, and has d digits.

□

Example 31. The first odd prime base for which infinitely many H-morphic numbers exist is 11. The zeros of $h(n)$, mod 11, are 3 and 5. Hensel's lemma lifts these to $\hat{3}, \hat{5} \in \mathbb{Q}_{11}$, giving base-11 H-morphic numbers

$$(\pi_d(\hat{3})) = (3, 13_{11}, 113_{11}, 7113_{11}, 57113_{11} \ldots)$$

and

$$(\pi_d(\hat{5}) = (5, 95_{11}, 995_{11}, 3995_{11}, 53995_{11} \ldots).$$

As for T-morphic and P-morphic numbers, the situation is more complicated for bases with more than one prime factor. We again restrict our attention to the decimal case. The following lemma summarizes useful facts about $H(n) - n$, all easily proved.

Lemma 32.

(i) *No two of n, $n-1$ and $h(n)$ can be divisible by 5.*

(ii) *Only one of n and $n-1$ is even; $h(n)$ is always even.*

(iii) *We have $4 \mid h(n)$ if and only if $4 \mid n$ or $4 \mid (n-1)$.*

(iv) *We have $8 \mid h(n)$ if and only if $8 \mid (n-4)$ or $8 \mid (n-1)$.*

(v) *It is not possible for 16 to divide both $n-1$ and $h(n)$.*

We define the following sequences, where $n_0(d)$, $n_1(d)$ are as defined in (18):

- $p(d)$ is the unique solution in $[0, 10^d)$ of the system $2^d \mid p(d) - n_0(d)$, $5^d \mid p(d)$;
- For $d \geq 4$, $q(d)$ is the unique solution in $[0, 5 \cdot 10^{d-1})$ of the system $2^{d-1} \mid q(d) - 1$, $5^d \mid q(d)$;
- $q'(d)$ is the unique solution in $[0, 5 \cdot 10^{d-1})$ of the system $2^{d-1} \mid q'(d) - n_1(d)$, $5^d \mid q'(d)$;
- $r(d)$ is the unique solution in $[0, 10^d)$ of the system $2^d \mid r(d)$, $5^d \mid r(d) - 1$;
- $r'(d)$ is the unique solution in $[0, 10^d)$ of the system $2^d \mid r'(d) - n_0(d)$, $5^d \mid r'(d) - 1$;
- $s(d) := 5 \cdot 10^{d-1} + 1$;
- $t(d)$ is the unique solution in $[0, 5 \cdot 10^{d-1})$ of the system $2^{d-1} \mid t(d) - n_0(d)$, $5^d \mid t(d) - 1$.

Theorem 33. *The numbers 1, 5, and 25 are H-morphic base 10. The other base-10 H-morphic numbers are those in the union of the following ten sequences:*

1. *$(p(d) : d \geq 1)$;*
2. *$(q(d) : d \geq 4)$;*

3. $(q(d) + 5 \cdot 10^{d-1} : d \geq 4)$;

4. $(q'(d) : d \geq 4)$;

5. $(q'(d) + 5 \cdot 10^{d-1} : d \geq 4)$;

6. $(r(d) : d \geq 2)$;

7. $(r'(d) : d \geq 2)$;

8. $(s(d) : d \geq 4)$;

9. $(t(d) : d \geq 5)$;

10. $(t(d) + 5 \cdot 10^{d-1} : d \geq 5)$.

Proof.
Case 1 ($5^d \mid n$).

Subcase 1.1 ($5^d \mid n$ and n is even).

By Lemma 32(ii) we have $2^{d+2} \mid nh(n)$. But by Lemma 2, $2^d \nmid n$, and by Lemma 32(iv), 8 cannot divide both of $\{n, h(n)\}$. It follows that if there is such a solution, $2^d \mid h(n)$ (and, if $d \geq 2$, $4 \mid n$.) By Proposition 28, $n \equiv n_0(d) \pmod{2^d}$. The system

$$\begin{aligned} 2^d &\mid n - n_0(d) \\ 5^d &\mid n \end{aligned} \tag{19}$$

has a unique solution $p(d) \in [0, 10^d)$; $p(d)$ is always H-morphic, but may have fewer than d digits. The first few terms of this sequence are

$$(p(d) : d \geq 1) = (0, \mathit{00}, 500, 2500, 62500, \mathit{062500}, 4062500, 14062500, 414062500, \ldots).$$

Italics indicate terms (other than $p(1) = 0$) that are less than 10^{d-1}. These are nonetheless H-morphic, and remain so if padded out to d digits (or fewer) with leading zeros.

Subcase 1.2 ($5^d \mid n$ and n is odd). By Lemma 32(ii) we have $2^{d+2} \mid (n-1)h(n)$. For $d = 1, 2$, Lemma 32(iii) implies that both $n - 1$ and $h(n)$ are divisible by 4; these give the solutions 5 and 25 respectively. For $d = 3$ both are divisible by 8, which gives the solution 625. For $d > 3$, Lemma 32(iv,v) imply that one is divisible by 2^{d-1} and the other (necessarily) by 8.

Subcase 1.2.1 ($5^d \mid n$, $2^{d-1} \mid (n-1)$, and $d \geq 4$). The system

$$\begin{aligned} 2^{d-1} &\mid n - 1 \\ 5^d &\mid n \end{aligned} \tag{20}$$

has solutions $q(d) \in [0, 5 \cdot 10^{d-1})$ and $q(d) + 5 \cdot 10^{d-1} \in [5 \cdot 10^{d-1}, 10^d)$. The first few of these are

$$\begin{aligned}(q(d) : d \geq 1) &= (0, 25, -, \mathit{0625}, 40625, 390625, 2890625, 12890625 \ldots) \\ (q(d) + 5 \cdot 10^{d-1} : d \geq 1) &= (5, -, 625, 5625, 90625, 890625, 7890625, 62890625, \ldots).\end{aligned}$$

Dashes represent early terms that are not H-morphic and, as above, terms that need a leading zero to have the right number of digits are shown in italic. The terms $q(1) = 0$, $q(1) + 5 = 5$, $q(2) = 25$, and $q(3) + 500 = 625$ are shown above to be H-morphic, even though $8 \nmid h(n)$. On the other hand, $q(2) + 50 = 75$ and $q(3) = 125$ are not.

Subcase 1.2.2 ($5^d \mid n$, $2^{d-1} \mid h(n)$, and $d \geq 4$). Lemma 32(iv) implies that $8 \mid (n-1)$, so n is odd and we have

$$\begin{aligned}2^{d-1} &\mid n - n_1(d) \\ 5^d &\mid n.\end{aligned} \tag{21}$$

This has two solutions in $[0, 10^d)$, namely $q'(d) \in [0, 5 \cdot 10^{d-1})$, and $q'(d) + 5 \cdot 10^{d-1} \in [5 \cdot 10^{d-1}, 10^d)$. The first few terms of these sequences are

$$\begin{aligned}(q'(d) : d \geq 1) &= (-, 25, -, \mathit{0625}, 15625, 265625, 2265625, 47265625, \ldots), \\ (q'(d) + 5 \cdot 10^{d-1} : d \geq 1) &= (5, -, 625, 5625, 65625, 765625, 7265625, 97265625, \ldots).\end{aligned}$$

For $d > 4$, $q(d)$ ends in 0625, while $q'(d)$ ends in 5625. We conclude that $q(d) \neq q'(d)$ for $d > 4$.

Case 2 ($5^d \mid n - 1$). We consider the parity of n.

Subcase 2.1 ($5^d \mid n - 1$, n even). In this case 2^{d+2} divides $nh(n)$. For $d = 1$ there are no solutions. For $d \geq 2$, Lemma 32(iii) requires both n and $h(n)$ to be divisible by 4, while Lemma 32(iv) says that one of them is not divisible by 8, so that the other must be divisible by 2^d.

Subcase 2.1.1 ($5^d \mid n - 1$, $2^d \mid n$, and $d \geq 2$). By Lemma 32(iii), we have that $4 \mid h(n)$. We obtain the system

$$\begin{aligned}2^d &\mid n \\ 5^d &\mid n - 1.\end{aligned} \tag{22}$$

This has a unique solution $r(d)$ in $(0, 10^d)$ which is always automorphic [2]. The first few terms are

$$(r(d) : d \geq 1) = (-, 76, 376, 9376, \mathit{09376}, 109376, 7109376, 87109376, \ldots).$$

Note that while 6 is a solution to (22) for $d = 1$, $h(6) = 58$ is not divisible by 4, and 6 is not H-morphic.

Subcase 2.1.2 ($5^d \mid n-1$, $2^d \mid h(n)$, and $d \geq 2$). We obtain the system

$$\begin{aligned} 2^d &\mid n - n_0(d) \\ 5^d &\mid n - 1. \end{aligned} \tag{23}$$

This has unique solutions $r'(d)$ in $(0, 10^d)$, the first few of which are

$$(r'(d) : d \geq 1) = (-, 76, 876, 1876, 71876, 171876, 1171876, \mathit{01171876}, \ldots).$$

Again, $r(2) = r'(2)$ but, for $d > 2$, $r(d)$ ends in 376 while $r'(d)$ ends in 876; we conclude that for $d \geq 3$ the sequences $(r(d))$ and $(r'(d))$ are disjoint.

Subcase 2.2 ($5^d \mid n-1$, $n-1$ even). In this case 2^{d+2} divides $(n-1)h(n)$. For $d=1$ the only solution is $n=1$. For $d=2$, by Lemma 32(iii) both $n-1$ and $h(n)$ must be divisible by 4, and again $n=1$ is the only solution.

For $d \geq 3$, by Lemma 32(iv) both $n-1$ and $h(n)$ must be divisible by 8. For $d=3$ this gives $n=1$ yet again. For $d > 3$, we apply Lemma 32(v) to show that one of $n-1$ and $h(n)$ is divisible by 8, the other by 2^{d-1}.

Subcase 2.2.1 ($5^d \mid n-1$, $2^{d-1} \mid n-1$, and $d \geq 4$). We get solutions $s(d) := 5 \cdot 10^{d-1} + 1$: the first few values are

$$(s(d) : d \geq 1) = (-, -, -, 5001, 50001, 500001, \ldots, 5 \cdot 10^{d-1}, \ldots ; .)$$

We note that 6, 51 and 501 are not H-morphic.

Subcase 2.2.2 ($5^d \mid n-1$, $2^{d-1} \mid h(n)$, and $d \geq 4$). We have

$$\begin{aligned} 2^{d-1} &\mid n - n_1(d) \\ 5^d &\mid n - 1, \end{aligned} \tag{24}$$

which has a unique solution $t(d) \in [0, 5 \cdot 10^{d-1})$ and another solution in $[5 \cdot 10^{d-1}, 10^d)$, both H-morphic for $d > 4$. (For $d = 2, 3, 4$ we get $t(d) = 1$, and $51, 501$ are not H-morphic.) The first few values are

$$(t(d) : d \geq 1) = (1, \mathit{01}, \mathit{001}, \mathit{0001}, 25001, 375001, 4375001, 34375001, \ldots)$$

$$(t(d) + 5 \cdot 10^{d-1} : d \geq 1) = (-, -, -, 5001, 75001, 875001, 9375001, 84375001, \ldots)$$

□

Remark 34. The initial elements (less than 100,000) of the sequence of decimal H-morphic numbers are $(0, 1, 5, 25$, 76, 376, 500, 625, 876, 1876, 2500, 5001, 5625, 9376, 15625 ,25001, 40625, 50001, 62500, 65625, 71876, 75001, 90625, $\ldots)$. This sequence has been added to the OEIS as A301912.

Remark 35. For $d > 4$, the last four digits of $p(d)$, $q(d)$, $q'(d)$, $r(d)$, $r'(d)$, $s(d)$, and $t(d)$ are 2500, 0625, 5625, 9376, 1876, 0001, and 5001 respectively. Thus, for $d > 4$, the dth terms of the ten sequences of Proposition 33 are distinct. We may have equalities within a sequence (e.g, $p(5) = p(6)$); and we may also have equalities between differently-indexed terms of related sequences: e.g., $q(4) = q(3) + 5 \cdot 10^{d-1}$.

Corollary 36. *For $d > 4$, there are at most 10 and at least 5 decimal H-morphic numbers with d digits.*

Proof. The maximum follows from the theorem above; every d-digit H-morphic number is the d-th element of one of the ten subsequences. None of these subsequences ever has a d-th element larger than 10^d, but some may be less than 10^d, in which case that subsequence has no d-digit term.

This cannot happen for $s(d)$, $q(d) + 5 \cdot 10^d$, $q'(d) + 5 \cdot 10^d$, or $t(d) + 5 \cdot 10^d$. Furthermore, adding the residues from (20,22), we get

$$\begin{aligned} 2^{d-1} &\mid q(d) + r(d) - 1 \\ 5^d &\mid q(d) + r(d) - 1, \end{aligned} \tag{25}$$

so that $5 \cdot 10^{d-1}$ always divides $q(d) + r(d) - 1$. As $0 < q(d), r(d)$, at least one of $\{q(d), r(d)\}$ must be greater than $25 \cdot 10^{d-2}$, and *a fortiori* must have d digits. □

Remark 37. The maximum is first attained when $d = 7$, and all six of $p(7) = 4062500$, $q(7) = 2890625$, $q'(7) = 2265625$, $r(7) = 7109376$, $r'(7) = 1171876$, and $t(7) = 4375001$ are greater than 10^6.

The minimum is first attained when $d = 168$, and $p(168) \approx 0.2896 \times 10^{167}$, $q'(168) \approx 0.0695 \times 10^{167}$, $r(168) \approx 0.1197 \times 10^{167}$, $r'(168) \approx 0.4093 \times 10^{167}$, $t(168) \approx 0.1892 \times 10^{167}$, while $q(168) \approx 4.880 \times 10^{167}$.

Remark 38. Recall that $\hat{0} + \hat{1} = -3$ in $\mathbb{Q}_2$; and (for $d > 1$), $n_0(d) + n_1(d) = -3$ in $\mathbb{Z}/2^d\mathbb{Z}$. Taking a linear combination of the residues from (19,20,21), we obtain

$$\begin{aligned} 2^{d-1} &\mid (p(d) + q'(d) + 3q(d)) \\ 5^d &\mid (p(d) + q'(d) + 3q(d)). \end{aligned} \tag{26}$$

Hence, $p(d) + q'(d) + 3q(d)$ is positive and divisible by $5 \cdot 10^{d-1}$, so that at least one of $p(d)$, $q(d)$, and $q'(d)$ is larger than 10^{d-1}. Thus, if for some $d > 3$ there are only five d-digit H-morphic numbers, they are

$$\{q(d), q(d) + 5 \cdot 10^d, q'(d) + 5 \cdot 10^d, s(d), t(d) + 5 \cdot 10^d\}.$$

Remark 39. Pickover [8, page 158], defines *cake* numbers to be those of the form $K(n) = (n^2 + n + 2)/2 = T(n) + 1$, the number of pieces into which a pancake may be cut with n cuts. (The sequence appears in OEIS as A000124, though the name is used for a different sequence.) Pickover conjectures, on the basis of a computer search, that no decimal K-morphic numbers exist. Applying the methods of this paper, a d-digit number is K-morphic (base B) if and only if $2B^d \mid k(n)$ where $k(n) := n^2 - n + 2$.

By coincidence, the discriminant of k (like that of h) is -7. The polynomial $k(n)$ is identically zero (and its derivative is always nonzero) on $\mathbb{Z}/2\mathbb{Z}$. It follows that, like h, $k(n)$ has zeros modulo any power of 2. Moreover, $k(n)$ has a nonliftable zero (this time, 4) in $\mathbb{Z}/7\mathbb{Z}$; has liftable zeros modulo any power of any odd prime that is congruent to 1,2 or 4 (mod 7); and has no zeros modulo any other prime power. Finally, $k(n)$ is even for every n, so the extra factor of 2 is automatically provided. We conclude that the sequences of K-morphic numbers base $2, 4, 8, 11, 16, 22, 23, \ldots$ are infinite, while 4 is the unique base-7 K-morphic number. However, $k(n)$ has no zeros modulo any power of 5, and hence no decimal numbers are K-morphic.

5 Relations between sequences

Proposition 40. *In any base, every T-morphic number is automorphic; and the converse also holds in any odd base.*

Proof. This follows immediately from Fairbairn's observation [2] that, in base B, a d-digit number n is automorphic if and only if $B^d \mid n(n-1)$.

□

Proposition 41. *For any base not divisible by 3, all T-morphic numbers are P-morphic.*

Proof. By Lemma 10(ii), we have $3 \mid n(n-1)(2n+5)$. If 3 does not divide B, then

$$2B^d \mid n(n-1) \Rightarrow 6B^d \mid n(n-1)(2n+5),$$

and the result follows.

□

When the base is divisible by 3, the result may fail. For instance, $T(4) = 10 = 14_6$, and $P(4) = 30 = 50_6$; so 4 is T-morphic but not P-morphic (base 6).

Combining these yields a stronger result.

Corollary 42. *If $B \equiv 1$ or 5 (mod 6), every automorphic number (base B) is P-morphic.*

There is an analogous result for H-morphic numbers.

Proposition 43.

(i) *In any odd base, every automorphic number is H-morphic;*

(ii) *In any even base, every automorphic number of two or more digits is H-morphic;*

(iii) *If $4 \mid B$, every automorphic number is H-morphic.*

Proof. We always have $4 \mid n(n-1)(n^2+3n+4)$; if B is odd,

$$B^d \mid n(n-1)(n^2+3n+4) \Rightarrow 4B^d \mid n(n-1)(n^2+3n+4).$$

If B is even, a d-digit automorphic number n is congruent to 0 or 1 (mod 2^d); in particular, for $d \geq 2$, $n \equiv 0$ or 1 (mod 4). Then $4 \mid n^2+3n+4$, so again $4B^d \mid n(n-1)(n^2+3n+4)$. A similar argument applies when $4 \mid B$.

□

The exception is nonvacuous; in particular, 6 is automorphic but not H-morphic (base 10). However, no other base-10 number has this property.

Finally, combining the above with remarks 22 and 35, we find that

$$\underline{A093534} \cap \underline{A301912} = \underline{A067270}.$$

Proposition 44. *A decimal number n is T-morphic if and only n is P-morphic and H-morphic.*

We ask whether this is true in other bases.

6 Conclusion

We have generalized the methods of Fairbairn [2] and used them to characterize T-morphic mumbers to any base, and decimal and prime-power-base P- and H-morphic numbers. For a prime power base, the sequences obtained are fairly straightforward; for composite bases, the sequence is typically the union of several subsequences, which may themselves be more or less regular.

We have also derived various inclusions between the sequences of automorphic, T-morphic, P-morphic, and H-morphic numbers, mostly base-dependent.

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