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On the Periodicity Problem for Residual r-Fubini Sequences

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Abstract

For any positive integer r, the r-Fubini number with parameter n, denoted by $F_{n,r}$, is equal to the number of ways that the elements of a set with $n+r$ elements can be weakly ordered such that the r least elements are in distinct orders. In this article we focus on the sequence of residues of the r-Fubini numbers modulo an arbitrary positive integer s and show that this sequence is periodic and then, exhibit how to calculate its period length.

1 Introduction

The Fubini numbers (also known as the *ordered Bell numbers*) form an integer sequence in which the nth term counts the number of weak orderings of a set with n elements. Weak ordering means that the elements can be ordered, allowing ties. Cayley [2] studied the Fubini numbers as the number of a certain kind of trees with $n+1$ terminal nodes. The Fubini numbers can also be defined as the sum of the *Stirling numbers of the second kind*, $\left\{ {n \atop k} \right\}$, which counts the number of partitions of an n-element set into k non-empty subsets. The sequence of residues of the Fubini numbers modulo a positive integer s was studied by Poonen [6]. He showed that this sequence is periodic and calculated the period length for each positive integer s.

The r-Stirling numbers of the second kind are defined as an extension to the Stirling numbers of the second kind, in which the first r elements contained in distinct subsets. Similarly the r-Fubini numbers, which are denoted by $F_{n,r}$, are defined as the number of ways which the elements of a set with $n+r$ elements can be weakly ordered such that the first r elements are in distinct places. Consider the sequence of remainders of $F_{n,r}$ modulo an arbitrary number $s \in \mathbb{N}$ in which r is fixed, which is denoted by $A_{r,s}$. One can study the periodicity problem for this sequence. Mező [4] investigated this problem for $s = 10$. In this article $\omega(A_{r,s})$, the period of $A_{r,s}$, is computed for any positive integer s. Based on the fundamental theorem of arithmetic, $\omega(A_{r,p})$ is calculated for powers of odd primes p^m. The cases $s = 2^m$ are studied separately. Therefore if $s = 2^m p_1^{m_1} p_1^{m_1} \cdots p_k^{m_k}$ is the prime factorization, then the $\omega(A_{r,s})$ is equal to the least common multiple of $\omega(A_{r,p_i^{m_i}})$s and $\omega(A_{r,2^m})$, for $i = 1, 2, \ldots, k$.

Section 2 contains the basic definitions and relations. The length of the periods in the case of odd prime powers are computed in the Section 3. The similar results about the 2 powers are stated in the Section 4. The last section contains the final theorem which presents the conclusion of the article.

2 Basic concepts

Let $\left\{ {n \atop k} \right\}$ be the Stirling number of the second kind with the parameters n and k and let $\left\{ {n \atop k} \right\}_r$ be the r-Stirling number of the second kind with parameters n and k. It is clear that $n \geq k \geq r$. Fubini numbers are computed as follows [4]:

$$F_n = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}.$$

In a similar way we can evaluate the r-Fubini number $F_{n,r}$ by

$$F_{n,r} = \sum_{k=0}^{n} (k+r)! \left\{ {n+r \atop k+r} \right\}_r .$$

There are simple relations and formulae about $\left\{ {n \atop k} \right\}_r$ which are listed below. One can find a proof of them in [1, 4, 5] and [3, Thm. 4.5.1, p. 158].

$$\left\{ {n \atop m} \right\}_r = \left\{ {n \atop m} \right\}_{r-1} - (r-1)\left\{ {n-1 \atop m} \right\}_{r-1}, 1 \le r \le n \tag{1}$$

$$\left\{ {n \atop m} \right\}_1 = \left\{ {n \atop m} \right\} \tag{2}$$

$$\left\{ {n+r \atop r} \right\}_r = r^n \tag{3}$$

$$\left\{ {n+r \atop r+1} \right\}_r = (r+1)^n - r^n \tag{4}$$

$$\left\{ {n \atop m} \right\} = \frac{1}{m!}\sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j}j^n \tag{5}$$

$$\left\{ {n \atop m} \right\}_r = \frac{1}{m!}\sum_{j=r}^{m}(-1)^{m-j}\binom{m}{j}j^{n-(r-1)}\left(\frac{(j-1)!}{(j-r)!}\right). \tag{6}$$

By $\varphi(n)$ we indicate the number of positive integer numbers less than n and co-prime to it. It is known as Euler's totient function. The value of $\varphi(n)$ can be computed via the following relation [3, Example 4.7.3, p. 167]:

$$\varphi(n) = n\prod_{p|n}(1-\frac{1}{p}).$$

3 The r-Fubini residues modulo prime powers

Let p be a prime number greater than 2 and m be a positive integer. If $(F_{n,r})$ denotes the sequence of r-Fubini numbers for a fixed positive integer r, we indicate by $A_{r,q} = (F_{n,r} (\bmod\ q))$, for $n \in \mathbb{N}$, the sequence of residues of the r-Fubini numbers modulo the positive integer q. In this section we try to compute the period length of the sequence $A_{r,q}$ when $q = p^m$. This length is denoted by $\omega(A_{r,q})$.

Proposition 1. *Let p be an odd prime and let $q = p^m$, $m \in \mathbb{N}$. If $q \le r$, then $\omega(A_{r,q}) = 1$.*

Proof. The proof is very simple. Since $p \le r$, we can deduce that $p \mid (k+r)!$, for $k \ge 0$, and by the relation $F_{n,r} = \sum_{k=0}^{n}(k+r)!\left\{ {n+r \atop k+r} \right\}_r$, we have $p \mid F_{n,r}$. Therefore $\omega(A_{r,p}) = 1$. □

As pointed out in the above proposition, it is sufficient to investigate the period length in the cases of $q > r$.

Lemma 2. *Let p be an odd prime and $r, m \in \mathbb{N}$ with $p \ge r+1$. Then*

$$p^m - r \ge m.$$

Proof. For $m = 1$ the result is obvious. Suppose the inequality holds for any $m \geq 2$. Since $p(p+m) > 2(p+m) > 2p+m$, we have

$$p^2 + pm - p \geq p + m. \tag{7}$$

Since $p - 1 \geq r$, the induction hypothesis can be reformulated to $p^m \geq p - 1 + m$. Multiplication by p results $p^{m+1} \geq p^2 + pm - p$. By (7) we have $p^{m+1} \geq p + (m+1) - 1$. $\square$

Theorem 3. *Let p be an odd prime and $q = p^m$. After the $(m-1)$th term the sequence $A_{r,q}$ has a period with length $\omega(A_{r,q}) = \varphi(q)$. In other words, $F_{n+\varphi(q),r} \equiv F_{n,r} \pmod{q}$, for $n \geq m - 1$.*

Proof. If $n \geq q - r - 1$ we can write

$$\begin{aligned}
F_{n+\varphi(q),r} - F_{n,r} &= \sum_{k=0}^{n+\varphi(q)} (k+r)! \left\{ {n+\varphi(q)+r \atop k+r} \right\}_r - \sum_{k=0}^{n} (k+r)! \left\{ {n+r \atop k+r} \right\}_r \\
&\equiv \sum_{k=0}^{q-r-1} (k+r)! \left(\left\{ {n+\varphi(q)+r \atop k+r} \right\}_r - \left\{ {n+r \atop k+r} \right\}_r \right) \\
&\equiv \sum_{k=0}^{q-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} \left(\frac{(j-1)!}{(j-r)!} \right) \left(j^{\varphi(q)} - 1 \right) \pmod{q}.
\end{aligned}$$

If $j = cp$, $c \in \mathbb{N}$, then $j^{n+1} = (cp)^{q-r+h}$, for some $h \geq 0$, so from Lemma 2 it follows that $j^{n+1} \equiv 0 \pmod{q}$. If $\gcd(j, q) = 1$, by Euler's theorem $j^{\varphi(q)} - 1 \equiv 0 \pmod{q}$, so the right hand side of the above congruence relation vanished and we have

$$F_{n+\varphi(q),r} \equiv F_{n,r} \pmod{q}, \text{ for } n \geq q - r - 1. \tag{8}$$

If $m - 1 \leq n < q - r - 1$ then

$$\begin{aligned}
F_{n+\varphi(q),r} - F_{n,r} &\equiv \sum_{k=0}^{q-r-1} (k+r)! \left(\left\{ {n+\varphi(q)+r \atop k+r} \right\}_r - \left\{ {n+r \atop k+r} \right\}_r \right) \\
&\quad - \sum_{k=n+\varphi(q)+1}^{q-r-1} (k+r)! \left\{ {n+\varphi(q)+r \atop k+r} \right\}_r + \sum_{k=n+1}^{q-r-1} (k+r)! \left\{ {n+r \atop k+r} \right\}_r \\
&\equiv \sum_{k=0}^{q-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} \left(\frac{(j-1)!}{(j-r)!} \right) \left(j^{\varphi(q)} - 1 \right) \\
&\quad - \sum_{k=n+\varphi(q)+1}^{q-r-1} (k+r)! \left\{ {n+\varphi(q)+r \atop k+r} \right\}_r + \sum_{k=n+1}^{q-r-1} (k+r)! \left\{ {n+r \atop k+r} \right\}_r \pmod{q}.
\end{aligned}$$

Since $n \geq m-1$, in the indices where $j = cp$, $c \in \mathbb{N}$, we have $j^{n+1} = (cp)^{m+h}$, for some $h \geq 0$, and it is deduced that $j^{n+1} \equiv 0 \pmod{q}$. When $\gcd(j,q) = 1$, again $j^{\varphi(q)} - 1 \equiv 0 \pmod{q}$ by Euler's theorem. In the sums $\sum_{k=n+1}^{q-r-1}(k+r)!\left\{{n+r \atop k+r}\right\}_r$ and $\sum_{k=n+\varphi(q)+1}^{q-r-1}(k+r)!\left\{{n+\varphi(q)+r \atop k+r}\right\}_r$ the upper parameter of the r-Stirling number is less than the lower one, and therefore these two sums are equal to zero. So

$$F_{n+\varphi(q),r} - F_{n,r} \equiv \sum_{k=0}^{q-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} \left(\frac{(j-1)!}{(j-r)!} \right) (j^{\varphi(q)} - 1) \equiv 0 \pmod{q},$$

and therefore

$$F_{n+\varphi(q),r} \equiv F_{n,r} \pmod{q} \text{ for } m-1 \leq n < q-r-1. \tag{9}$$

Combining results (8) and (9) gives $F_{n+\varphi(q),r} \equiv F_{n,r} \pmod{q}$, for $n \geq m-1$. ☐

4 The r-Fubini residues modulo powers of 2

As in many other computations in number theory, the case of $p = 2$ has its own difficulties that require special attention. In the case of powers of 2, initially we calculate the residues of 2-Fubini numbers and then use the results in the case of the r-Fubini numbers. We classify the sequence of remainders of 2-Fubini numbers modulo 2^m, $m \geq 7$, in Theorem 6 and then, work on remainders of the r-Fubini numbers modulo 2^m, $m \geq 7$ in Theorem 9. The special cases will be proved in Theorems 4, 7 and 8. The trivial cases in which $2^m \leq r$ with period length 1 are omitted.

Theorem 4. *If $3 \leq m \leq 6$, then after the $(m-1)$th term the sequence $A_{2,2^m}$ has a period with length $\omega(A_{2,2^m}) = 2$.*

Proof. By using the formula $F_{n,2} = \sum_{k=0}^{n}(k+2)!\left\{{n+2 \atop k+2}\right\}_2$ we prove that $F_{n+2,2} - F_{n,2} \equiv 0 \pmod{64}$. Then $F_{n+2,2} - F_{n,2} \equiv 0 \pmod{2^m}$ for $3 \leq m \leq 5$.

$$\begin{aligned} F_{n+2,2} - F_{n,2} &= \sum_{k=0}^{n+2}(k+2)!\left\{{n+4 \atop k+2}\right\}_2 - \sum_{k=0}^{n}(k+2)!\left\{{n+2 \atop k+2}\right\}_2 \\ &\equiv \sum_{k=0}^{5}(k+2)! \left(\left\{{n+4 \atop k+2}\right\}_2 - \left\{{n+2 \atop k+2}\right\}_2 \right) \\ &\equiv \sum_{k=0}^{5}\sum_{j=2}^{k+2}(-1)^{k+2-j}\binom{k+2}{j} j^{n+1}(j^2-1)(j-1) \pmod{64}. \end{aligned}$$

In the case $m = 6$ then $n \geq 5$, so if j is even, then $j^{n+1} = (2c)^{6+h}$, for some $h \geq 0$ and therefore $64 \mid j^{n+1}$. For odd j we have $\gcd(j, 64) = 1$, so by Euler's theorem we have

$j^{32} \equiv 1 \pmod{64}$, and therefore $j^{n+1+32} \equiv j^{n+1} \pmod{64}$. This implies that

$$\begin{aligned}
F_{n+2,2} - F_{n,2} &\equiv \sum_{k=0}^{5} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} (-1)^{k+2-(2l+1)} \binom{k+2}{2l+1} (2l+1)^{n+1} \left((2l+1)^2 - 1\right) \times 2l \\
&\equiv 16 \sum_{k=0}^{5} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \left(\frac{l(l+1)}{2}\right) l \pmod{64}.
\end{aligned}$$

Enumerating the last summation for $2 \le n \le 33$ shows that it is divisible by 64 and because of periodicity of remainders of j^{n+1} modulo 64, the result follows. □

Analogous to Lemma 2, it can be easily deduced by induction, showing that for each positive integer $m > 1$ we have

$$2^m - 2 \ge m \ . \tag{10}$$

This can be shown by using the relation $2^{m+1} \ge 2m + 4 > m + 3$, for $m > 1$. The following lemma provides a simple but essential relation used in the next theorem. Its proof is provided in Appendix A.

Lemma 5. *For $m \ge 7$ and $5 \le i \le 2^{m-6}$ we have $2^{m-6} - i \mid 2^{i-5}\binom{2^{m-6}-1}{i}$.*

Theorem 6. *If $m \ge 7$, after the $(m-1)$th term, the sequence $A_{2,2^m}$ has a period with length $\omega(A_{2,2^m}) = 2^{m-6}$.*

Proof. In the case of $n \ge 2^m - 3$, from (10) we can deduce that $n \ge 2^m - 3 \ge m - 1$. So we have

$$\begin{aligned}
F_{n+2^{m-6},2} - F_{n,2} &\equiv \sum_{k=0}^{n+2^{m-6}} (k+2)! \left\{ {n + 2^{m-6} + 2 \atop k+2} \right\}_2 - \sum_{k=0}^{n} (k+2)! \left\{ {n+2 \atop k+2} \right\}_2 \\
&\equiv \sum_{k=0}^{2^m-3} (k+2)! \left(\left\{ {n + 2^{m-6} + 2 \atop k+2} \right\}_2 - \left\{ {n+2 \atop k+2} \right\}_2 \right) \\
&\equiv \sum_{k=0}^{2^m-3} \sum_{j=2}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1} (j^{2^{m-6}} - 1)(j-1) \pmod{2^m}.
\end{aligned}$$

When j is even, then $j^{n+1} = (2c)^{2^m-2+h}$, for some $h \ge 0$. So by (10), $2^m \mid j^{n+1}$. For odd j we have

$$F_{n+2^{m-6},2} - F_{n,2} \equiv \sum_{k=0}^{2^m-3} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} (-1)^{k+2-(2l+1)} \binom{k+2}{2l+1} (2l+1)^{n+1} ((2l+1)^{2^{m-6}} - 1) \times 2l$$
$$\equiv 2^{m-4} \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \left(\frac{(2l+1)^{2^{m-6}} - 1}{2^{m-5}} \right) l$$
$$\equiv 2^{m-4} \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \sum_{i=1}^{2^{m-6}} l^i 2^{i-1} \left(\frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \right)$$
$$\times l \pmod{2^m}.$$

The last expression contains $m-4$ factors of 2, so it is sufficient to prove that the last summation is divisible by 16. This summation is denoted by $\mathcal{S}$. Simplify the summation $\sum_{i=1}^{2^{m-6}} l^i 2^{i-1} \frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!}$ and using Lemma 5 gives

$$\sum_{i=1}^{2^{m-6}} l^i 2^{i-1} \left(\frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \right) \equiv \sum_{i=1}^{4} l^i 2^{i-1} \left(\frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \right) \equiv l + l^2(2^{m-6}-1)$$
$$+ \frac{l^3 \times 2(2^{m-6}-1)(2^{m-6}-2)}{3} + \frac{l^4(2^{m-6}-1)(2^{m-6}-2)(2^{m-6}-3)}{3} \pmod{16}.$$

Assume $m \geq 10$ (the case $7 \leq m \leq 9$ is studied at the end of the proof). So $16 \mid 2^{m-6}$. Let $3a = 2(2^{m-6}-1)(2^{m-6}-2)$ and $3b = (2^{m-6}-1)(2^{m-6}-2)(2^{m-6}-3)$. Then $3a \equiv 4 \pmod{16}$ and $3b \equiv -6 \pmod{16}$. Therefore $a \equiv -4 \pmod{16}$ and $b \equiv -2 \pmod{16}$. So the proof continues as follows:

$$\mathcal{S} \equiv \sum_{k=0}^{2^m-3} (-1)^{k+1} \left(\sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} (l - l^2 - 4l^3 - 2l^4) l \right) \pmod{16}$$
$$\mathcal{S} \equiv \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \left(\frac{l(l+1)}{2} \right) (-2l^2 - 2l + 1) l \pmod{8}.$$

Let $P(l)$ and $A(k,r,n)$ be the remainder of $\frac{1}{2}(2l+1)^{n+1}(l(l+1))(-2l^2-2l+1)l$ and $\sum_{l=-\infty}^{\infty} \binom{k+2}{2l+r} P(l)$ divided by 8, respectively. By Pascal's identity, we have $\binom{k+2}{2l+r} = \binom{k+1}{2l+r} + \binom{k+1}{2l+r-1}$ and therefore

$$\sum_{l=-\infty}^{\infty} \binom{k+2}{2l+r} P(l) = \sum_{l=-\infty}^{\infty} \binom{k+1}{2l+r} P(l) + \sum_{l=-\infty}^{\infty} \binom{k+1}{2l+r-1} P(l),$$

so

$$A(k,r,n) = A(k-1,r,n) + A(k-1,r-1,n). \tag{11}$$

We can write

$$A(k,r+32,n) \equiv \sum_{l=-\infty}^{\infty} \binom{k+2}{2l+r+32} P(l) \pmod 8.$$

The sequence $(P(l))_{l=-\infty}^{\infty}$ has period 16, so $P(l+16) = P(l)$. Set $l' = l+16$, then

$$A(k,r+32,n) \equiv \sum_{l'=-\infty}^{\infty} \binom{k+2}{2l'+r} P(l') \equiv A(k,r,n) \pmod 8. \tag{12}$$

Since $\gcd(2l+1,16) = 1$, Euler's theorem implies $(2l+1)^8 \equiv 1 \pmod{16}$ and therefore $(2l+1)^{n+1+8} \equiv (2l+1)^{n+1} \pmod{16}$. The quantity $A(6,r,n)$ vanishes for $1 \le r \le 32$ and $9 \le n \le 24$, by enumeration, then by (11) and (12), we deduce that

$$A(k,r,n) = 0, \text{ for } k \ge 6. \tag{13}$$

Therefore

$$\begin{aligned} A(k,1,n) &\equiv \sum_{l=-\infty}^{\infty} \binom{k+2}{2l+1} (2l+1)^{n+1} \left(\frac{l(l+1)}{2}\right) (-2l^2-2l+1)l \\ &\equiv \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \left(\frac{l(l+1)}{2}\right) (-2l^2-2l+1)l \equiv 0 \pmod 8, \end{aligned}$$

for $k \ge 6$. If $1 \le k \le 5$, $9 \le n \le 24$ and $1 \le r \le 32$ we have $\sum_{k=1}^{5} (-1)^{k+1} A(k,r,n) \equiv 0 \pmod 8$. The period length of $A(k,r,n)$ with respect to r and n implies that

$$\sum_{k=1}^{5} (-1)^{k+1} A(k,1,n) \equiv 0 \pmod 8, \text{ for } n \ge 9.$$

Combining this with (13) we have

$$\mathcal{S} \equiv \sum_{k=1}^{2^m-3} (-1)^{k+1} A(k,1,n) \equiv 0 \pmod 8, \text{ for } n \ge 0.$$

So the result follows in the case of $n \geq 2^m - 3$. If $m-1 \leq n < 2^m - 3$ we can write

$$
\begin{aligned}
F_{n+2^{m-6},2} - F_{n,2} &= \sum_{k=0}^{n+2^{m-6}} (k+2)! \left\{ {n+2^{m-6}+2 \atop k+2} \right\}_2 - \sum_{k=0}^{n} (k+2)! \left\{ {n+2 \atop k+2} \right\}_2 \\
&= \sum_{k=0}^{2^m-3} (k+2)! \left(\left\{ {n+2^{m-6}+2 \atop k+2} \right\}_2 - \left\{ {n+2 \atop k+2} \right\}_2 \right) \\
&- \sum_{k=n+2^{m-6}+1}^{2^m-3} (k+2)! \left\{ {n+2^{m-6}+2 \atop k+2} \right\}_2 + \sum_{k=n+1}^{2^m-3} (k+2)! \left\{ {n+2 \atop k+2} \right\}_2 \\
&\equiv \sum_{k=0}^{2^m-3} \sum_{j=1}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1} (j^{2^{m-6}} - 1)(j-1) \pmod{2^m}.
\end{aligned}
$$

When j is even, then $j^{n+1} = (2c)^{m+h}$, for some $h \geq 0$, so $2^m \mid j^{n+1}$. Since $m \geq 10$, for odd j we have

$$
\begin{aligned}
&\sum_{k=0}^{2^m-3} \sum_{j=1}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1} (j^{2^{m-6}} - 1)(j-1) \\
&\equiv 2^{m-4} \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} (l - l^2 - 4l^3 - 2l^4) l \pmod{2^m}.
\end{aligned}
$$

The last summation is exactly the $\mathcal{S}$ and the proof will be similar as above. Combine with the previous case we have the following congruence relation

$$
F_{n+2^{m-6},2} \equiv F_{n,2} \pmod{2^m}, \text{ for } m \geq 10. \tag{14}
$$

In the case where $7 \leq m \leq 9$, the remainder value of the sum

$$
\sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \left(\sum_{i=1}^{4} l^i 2^{i-1} \frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \right) l
$$

modulo 16 is computed for $m-1 \leq n \leq m+14$. Divisibility of all these values by 16 implies that the recent sum is divisible by 16, and therefore

$$
F_{n+2^{m-6},2} \equiv F_{n,2} \pmod{2^m}, \text{ for } 7 \leq m \leq 9. \tag{15}
$$

Summing up the congruence relations (14) and (15) gives

$$
\omega(A_{2,2^m}) = 2^{m-6}, \text{ for } m \geq 7.
$$

$\square$

Theorem 7. *For $m = 1$ and $m = 2$, the sequence $A_{r,2^m}$ is periodic from the first term and the period length is $\omega(A_{r,2^m}) = 1$.*

Proof. The proof of this theorem is divided into three cases. For $r = 2$ we have

$$\begin{aligned}
F_{n+1,2} - F_{n,2} &= \sum_{k=0}^{n+1}(k+2)!\left\{{n+3 \atop k+2}\right\}_2 - \sum_{k=0}^{n}(k+2)!\left\{{n+2 \atop k+2}\right\}_2 \\
&\equiv 2\left(\left\{{n+3 \atop 2}\right\}_2 - \left\{{n+2 \atop 2}\right\}_2\right) + 6\left(\left\{{n+3 \atop 3}\right\}_2 - \left\{{n+2 \atop 3}\right\}_2\right) \pmod 4 \\
&= 2\left(2^{n+1} - 2^n\right) + 6\left(3^{n+1} - 2^{n+1} - (3^n - 2^n)\right) \\
&= 2^{n+1} + 6(2 \times 3^n - 2^n) = 4(2^{n-1} + 3^{n+1} - 3 \times 2^{n-1}) \\
&= 4(3^{n+1} - 2^n) \equiv 0 \pmod 4.
\end{aligned}$$

So we can deduce that $\omega(A_{2,4}) = 1$ and obviously $\omega(A_{2,2}) = 1$.

For $r = 3$ we can write

$$\begin{aligned}
F_{n+1,3} - F_{n,3} =& \sum_{k=0}^{n+1}(k+3)!\left\{{n+4 \atop k+3}\right\}_3 - \sum_{k=0}^{n}(k+3)!\left\{{n+3 \atop k+3}\right\}_3 \\
&\equiv 6\left(\left\{{n+4 \atop 3}\right\}_3 - \left\{{n+3 \atop 3}\right\}_3\right) \pmod 4 \\
=& 6\left(3^{n+1} - 3^n\right) = 6 \times 2 \times 3^n = 4 \times 3^{n+1} \equiv 0 \pmod 4.
\end{aligned}$$

Therefore we have $\omega(A_{3,4}) = 1$ and $\omega(A_{3,2}) = 1$.

Finally if $r \geq 4$, let $r = 4 + h$, for some $h \geq 0$, then

$$F_{n+1,r} - F_{n,r} = \sum_{k=0}^{n+1}(k+r)!\left\{{n+1+r \atop k+r}\right\}_r - \sum_{k=0}^{n}(k+r)!\left\{{n+r \atop k+r}\right\}_r.$$

Since $4 \mid (k+r)!$, for all $k \geq 0$, we can write $F_{n+1,r} - F_{n,r} \equiv 0 \pmod 4$. Therefore $\omega(A_{r,4}) = 1$ and $\omega(A_{r,2}) = 1$. □

Theorem 8. *If $3 \leq m \leq 6$, after the $(m-1)$th term, the sequence $A_{r,2^m}$ has a period with length $\omega(A_{r,2^m}) = 2$.*

Proof. The proof of this theorem is similar to the proof of Theorem 4. It is enough to prove the theorem for $m = 6$; then the result follows for $m = 3, 4$ and 5. Since $n \geq m - 1$, then for $m=6$ we have $n \geq 5$. For $3 \leq r \leq 7$ we have

$$\begin{aligned}
F_{n+2,r} - F_{n,r} &= \sum_{k=0}^{n+2} (k+r)! \left\{ {n+2+r \atop k+r} \right\}_r - \sum_{k=0}^{n} (k+r)! \left\{ {n+r \atop k+r} \right\}_r \\
&\equiv \sum_{k=0}^{7-r} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^2-1) \left(\frac{(j-1)!}{(j-r)!} \right) \\
&\equiv \sum_{k=0}^{7-r} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^2-1)(j-1) \left(\frac{(j-2)!}{(j-r)!} \right) \pmod{64}.
\end{aligned}$$

When j is even, then $j^{n+1} = (2c)^{6+h}$, for some $h \geq 0$, and so $64 \mid j^{n+1}$. For odd j we have $\gcd(j,64)=1$ and Euler's theorem gives $j^{32} \equiv 1 \pmod{64}$. Therefore $j^{n+1+32} \equiv j^{n+1} \pmod{64}$, and we can write

$$\begin{aligned}
&\sum_{k=0}^{7-r} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^2-1)(j-1) \frac{(j-2)!}{(j-r)!} \\
&\equiv \sum_{k=0}^{7-r} \sum_{l=\lfloor r/2 \rfloor}^{\lfloor (k+r-1)/2 \rfloor} (-1)^{k+r-(2l+1)} \binom{k+r}{2l+1} (2l+1)^{n+1} ((2l+1)^2-1) \times 2l \left(\frac{(2l-1)!}{(2l+1-r)!} \right) \\
&\equiv 16 \sum_{k=0}^{7-r} (-1)^{k+r-1} \sum_{l=\lfloor r/2 \rfloor}^{\lfloor (k+r-1)/2 \rfloor} \binom{k+r}{2l+1} (2l+1)^{n+1} \left(\frac{l(l+1)}{2} \right) l \left(\frac{(2l-1)!}{(2l+1-r)!} \right) \pmod{64}.
\end{aligned}$$

By computation we see that the recent summation is divisible by 4, for $2 \leq n \leq 33$. So the proof for $3 \leq r \leq 7$ is completed.

If $r \geq 8$, since $64 \mid 8!$, then $64 \mid (k+r)!$, and

$$F_{n+2,r} - F_{n,r} = \sum_{k=0}^{n+2} (k+r)! \left\{ {n+2+r \atop k+r} \right\}_r - \sum_{k=0}^{n} (k+r)! \left\{ {n+r \atop k+r} \right\}_r \equiv 0 \pmod{64},$$

so $\omega(A_{r,2^6}) = 2$, for $r \geq 8$, and the proof is completed. □

Theorem 9. *If $m \geq 7$, after the $(m-1)$th term, the sequence $A_{r,2^m}$ has a period with length $\omega(A_{r,2^m}) = 2^{m-6}$.*

Proof. The proof of this theorem is similar to the proof of Theorem 6. In the case of

$n \geq 2^m - r - 1$ and $r \geq 8$ we have

$$\begin{aligned}
F_{n+2^{m-6},r} - F_{n,r} &= \sum_{k=0}^{n+2^{m-6}} (k+r)! {n+2^{m-6}+r \brace k+r}_r - \sum_{k=0}^{n} (k+r)! {n+r \brace k+r}_r \\
&\equiv \sum_{k=0}^{2^m-r-1} (k+r)! \left({n+2^{m-6}+r \brace k+r}_r - {n+r \brace k+r}_r \right) \\
&\equiv \sum_{k=0}^{2^m-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^{2^{m-6}} - 1) \left(\frac{(j-1)!}{(j-r)!} \right) \pmod{2^m}.
\end{aligned}$$

In the case of $2^m > r > 2^m - m$, since $m \geq 7$ this implies that $r > 2^m - m \geq 2^{m-1}$, so

$$2^m \mid (2^{m-1})! \mid (k+r)!, \text{ for each } k \geq 0.$$

Therefore both summations in the above first equation are zero modulo 2^m and in this case $\omega(A_{r,2^m}) = 2^{m-6}$. When $r \leq 2^m - m$, if j is even then $j^{n+1} = (2c)^{2^m-r+h}$, for some $h \geq 0$. So $2^m \mid j^{n+1}$. For odd j we have $(j, 2^{m-5}) = 1$, and $2^{m-5} \mid j^{2^{m-6}} - 1$ by Euler's theorem. Since $r \geq 8$ we can write $\frac{(j-1)!}{(j-r)!} = \left(\frac{(j-8)!}{(j-r)!} \right) \prod_{i=1}^{7} (j-i)$. Therefore $32 \mid \frac{(j-1)!}{(j-r)!}$ and $2^m \mid (j^{2^{m-6}} - 1) \left(\frac{(j-1)!}{(j-r)!} \right)$.

In the case of $m - 1 \leq n < 2^m - r - 1$ and $r \geq 8$ we have

$$\begin{aligned}
F_{n+2^{m-6},r} - F_{n,r} &= \sum_{k=0}^{n+2^{m-6}} (k+r)! {n+2^{m-6}+r \brace k+r}_r - \sum_{k=0}^{n} (k+r)! {n+r \brace k+r}_r \\
&\equiv \sum_{k=0}^{2^m-r-1} (k+r)! \left({n+2^{m-6}+r \brace k+r}_r - {n+r \brace k+r}_r \right) \\
&\quad - \sum_{k=n+2^{m-6}+1}^{2^m-r-1} (k+r)! {n+2^{m-6}+r \brace k+r} + \sum_{k=n+1}^{2^m-r-1} (k+r)! {n+r \brace k+r} \pmod{2^m} \\
&= \sum_{k=0}^{2^m-r-1} (k+r)! \left({n+2^{m-6}+r \brace k+r}_r - {n+r \brace k+r}_r \right) + 0,
\end{aligned}$$

and the proof proceeds as in the previous case. In the case of $3 \leq r \leq 7$ one can deduce similarly to the proof of Theorem 6 that

$$F_{n+2^{m-6},r} - F_{n,r} \equiv \sum_{k=0}^{2^m-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^{2^{m-6}} - 1) \left(\frac{(j-1)!}{(j-r)!} \right) \pmod{2^m}.$$

Exactly the same as Theorem 6, the terms with even j vanish and only the terms with odd

j remain. So we have

$$F_{n+2^{m-6},r} - F_{n,r} \equiv \sum_{k=0}^{2^m-r-1} \sum_{l=\lfloor r/2\rfloor}^{\lfloor (k+r-1)/2\rfloor} (-1)^{k+r-(2l+1)} \binom{k+r}{2l+1} (2l+1)^{n+1}((2l+1)^{2^{m-6}}-1)$$
$$\times \left(\frac{((2l+1)-1)!}{((2l+1)-r)!}\right)$$
$$\equiv 2^{m-5} \sum_{k=0}^{2^m-r-1} (-1)^{k+r-1} \sum_{l=\lfloor r/2\rfloor}^{\lfloor (k+r-1)/2\rfloor} \binom{k+r}{2l+1} (2l+1)^{n+1}$$
$$\times \left(\sum_{i=1}^{2^{m-6}} l^i 2^{i-1} \left(\frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!}\right)\right) \left(\frac{(2l)!}{(2l-r+1)!}\right) \pmod{2^m}.$$

Since $\gcd(2l+1, 16) = 1$, Euler's theorem shows that $(2l+1)^{n+1+8} \equiv (2l+1)^{n+1} \pmod{16}$. If $m \geq 10$, we have

$$F_{n+2^{m-6},r} - F_{n,r} \equiv 2^{m-4} \sum_{k=0}^{2^m-r-1} (-1)^{k+r-1} \sum_{l=\lfloor r/2\rfloor}^{\lfloor (k+r-1)/2\rfloor} \binom{k+r}{2l+1} (2l+1)^{n+1} \left(\frac{l(l+1)}{2}\right)$$
$$\times (-2l^2-2l+1) \left(\frac{(2l)!}{(2l-r+1)!}\right) \pmod{2^m}.$$

Therefore it is sufficient to compute the above summation (without factor 2^{m-4}) for $3 \leq r \leq 7$ and $9 \leq n \leq 16$ to show that it is divisible by 16.

For $7 \leq m \leq 9$ we evaluate the sum

$$\sum_{k=0}^{2^m-r-1} (-1)^{k+r-1} \sum_{l=\lfloor r/2\rfloor}^{\lfloor (k+r-1)/2\rfloor} \binom{k+r}{2l+1} (2l+1)^{n+1} \sum_{i=1}^{2^{m-6}} l^i 2^{i-1} \frac{(2^{m-6}-1)!}{i!(2^{m-6}-i)!} \left(\frac{(2l)!}{(2l-r+1)!}\right)$$

for $m-1 \leq n \leq m+6$ to show that it is divisible by 32. Then it follows that $\omega(A_{r,2^m}) = 2^{m-6}$, for all $m \geq 7$. $\square$

5 The conclusion

We now state the final theorem, which shows how to compute $\omega(A_{r,s})$ for any $s \in \mathbb{N}$.

Theorem 10. *Let $s \in \mathbb{N}$ and $s > 1$ with the prime factorization $s = 2^m p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ and let $D = \{p_i^{m_i} \mid p_i^{m_i} > r, 1 \leq i \leq k\}$. Define $E = \{m_i - 1 \mid p_i^{m_i} \in D\}$, $F = \{\varphi(p_i^{m_i}) \mid p_i^{m_i} \in D\}$ and $a = \max(E \cup \{m-1\})$ and let b be the least common multiple* (lcm) *of the elements of F. Then*

$$\omega(A_{r,s}) = \begin{cases} b, & \text{if } 0 \leq m \leq 2 \text{ or } 2^m \leq r; \\ \mathrm{lcm}(2,b), & \text{if } 3 \leq m \leq 6 \text{ and } 2^m > r \text{ ;} \\ \mathrm{lcm}(2^{m-6},b), & \text{if } m \geq 7 \text{ and } 2^m > r, \end{cases} \tag{16}$$

and periodicity of the sequence $A_{r,s}$ is seen after the a-th term.

Proof. Let l be the right hand side of (16). For each $d \in D \cup \{2^m\}$, $\omega(A_{r,d}) \mid l$ and for each $p_j^{m_j} \notin D$ such that $1 \leq j \leq k$, we have $1 = \omega(A_{r,p_j^{m_j}}) \mid l$, so

$$\begin{aligned} F_{n+l,r} &\equiv F_{n,r} \pmod{2^m} \\ F_{n+l,r} &\equiv F_{n,r} \pmod{p_i^{m_i}}, \text{ for } i = 1, 2, \ldots, k. \end{aligned}$$

Since $\gcd(2^m, p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k}) = 1$, the multiplication of all above congruence relations gives the required result. $\square$

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A Proof of Lemma 5

After simplifying the lemma's relation we have

$$\frac{2^{i-5}\binom{2^{m-6}-1}{i}}{2^{m-6}-i} = \frac{2^{i-5}(2^{m-6}-1)(2^{m-6}-2)\cdots(2^{m-6}-i+1)}{i!}. \tag{17}$$

It is sufficient to show that the right hand side of (17) is integer. We know that $\binom{2^{m-6}}{i} \in \mathbb{N}$, i.e.,

$$i! \mid 2^{m-6}(2^{m-6}-1)\cdots(2^{m-6}-i+1).$$

If O_i denotes the product of the odd factors of $i!$, since $(O_i, 2^{m-6}) = 1$, then $O_i \mid (2^{m-6}-1)\cdots(2^{m-6}-i+1)$. So in (17) we only need to prove that

$$\nu_2(2^{i-5}(2^{m-6}-1)(2^{m-6}-2)\cdots(2^{m-6}-i+1)) \geq \nu_2(i!),$$

where by $\nu_2(x)$ we mean that $2^{\nu_2(x)} \mid x$, but $2^{\nu_2(x)+1} \nmid x$. Let $A = \nu_2((2^{m-6}-1)(2^{m-6}-2)\cdots(2^{m-6}-i+1))$ and $B = \nu_2(i!)$. Let e be the unique integer such that $2^e \leq i < 2^{e+1}$. So

$$A = \sum_{k=1}^{e} \lfloor \frac{i-1}{2^k} \rfloor, \quad B = \sum_{k=1}^{e} \lfloor \frac{i}{2^k} \rfloor. \tag{18}$$

If we show that

$$B - A \leq e \tag{19}$$

then the lemma is concluded if it is proved that

$$i + A \geq B + 5. \tag{20}$$

It can easily be shown that $B = \nu_2(i!)$ and $A = \nu_2((i-1)!)$, so $B - A = \nu_2(i)$. Since $2^e \leq i < 2^{e+1}$, therefore $\nu_2(i) \leq e$ and (19) follows. For $e = 2$, integer possibilities for inequality (20) are as follows:

i	A	B
5	3	3
6	3	4
7	4	4

For $e \geq 3$ one can deduce by simple induction that

$$2^e \geq e + 5,$$

so $i \geq 2^e \geq e + 5$. Add $B - e$ to these inequalities and use (19) demonstrates (20) for $i \geq 8$.

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