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Alternating Sums of the Reciprocal Fibonacci Numbers

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Abstract

In this paper, we investigate the alternating sums of the reciprocal Fibonacci numbers $\sum_{k=n}^{mn}(-1)^k/F_{ak+b}$, where $a \in \{1,2,3\}$ and $b < a$. The integer parts of the reciprocals of these sums are expressed explicitly in terms of the Fibonacci numbers.

1 Introduction

For an integer $n \geq 0$, the *Fibonacci number* F_n is defined recurrently by $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$.

Recently, Ohtsuka and Nakamura [1] studied the infinite sums of the reciprocal Fibonacci numbers, and established the following result, where $\lfloor \cdot \rfloor$ denotes the floor function.

Theorem 1. *For all $n \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even;} \\ F_{n-2} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

More recently, Wang and Wen [4] strengthened Theorem 1 to the finite sum case.

Theorem 2. *If $m \geq 3$ and $n \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even;} \\ F_{n-2} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

In this article, we focus on the alternating sums of the reciprocal Fibonacci numbers

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{ak+b}},$$

where $a \in \{1, 2, 3\}$ and $b < a$. By evaluating the integer parts of these sums, we obtain several interesting families of identities concerning the Fibonacci numbers.

2 Results for $a = 1$

We first introduce several well-known results, which will be used throughout the article. The detailed proofs can be found in, for example, [3, Thm. 7, p. 9] and [2].

Lemma 3. *For any positive intergers m and n, we have*

$$F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1}.$$

Lemma 4. *For all $n \geq 1$, we have*

$$F_{2n+1} = F_{n+1} F_{n+2} - F_{n-1} F_n.$$

Lemma 5. *Let a, b, c, d be positive integers with $a + b = c + d$ and $b \geq \max\{c, d\}$. Then*

$$F_a F_b - F_c F_d = (-1)^{a+1} F_{b-c} F_{b-d}.$$

For the sake of argument, we present four auxiliary functions

$$\begin{aligned} f_1(n) &= \frac{1}{F_{n+1}} - \frac{(-1)^n}{F_n} - \frac{1}{F_{n+2}}, \\ f_2(n) &= \frac{1}{F_{n+1} - 1} - \frac{(-1)^n}{F_n} - \frac{1}{F_{n+2} - 1}, \\ f_3(n) &= \frac{-1}{F_{n+1} + 1} - \frac{(-1)^n}{F_n} + \frac{1}{F_{n+2} + 1}, \\ f_4(n) &= \frac{-1}{F_{n+1}} - \frac{(-1)^n}{F_n} + \frac{1}{F_{n+2}}. \end{aligned}$$

It is clear that $f_i(n)$ $(1 \leq i \leq 4)$ is positive if n is odd, and negative otherwise.

Lemma 6. *If $n \geq 2$ is even, then*

$$f_1(n) + f_1(n+1) < 0.$$

Proof. Since n is even, it is straightforward to see

$$\begin{aligned}
f_1(n) + f_1(n+1) &= \frac{2}{F_{n+1}} - \frac{1}{F_n} - \frac{1}{F_{n+3}} \\
&= \frac{(2F_n - F_{n+1}) F_{n+3} - F_n F_{n+1}}{F_n F_{n+1} F_{n+3}} \\
&= \frac{F_{n-2} F_{n+3} - F_n F_{n+1}}{F_n F_{n+1} F_{n+3}} \\
&= \frac{-2}{F_n F_{n+1} F_{n+3}} \\
&< 0,
\end{aligned}$$

where the last equality follows from Lemma 5 and the fact that n is even. □

Lemma 7. *For all $n \geq 2$, we have*

$$f_2(n) + f_2(n+1) > 0.$$

Proof. The statement is clearly true if n is odd. Thus, we focus on the case where n is even. It follows from the definition of $f_2(n)$ and Lemma 5 that

$$\begin{aligned}
f_2(n) + f_2(n+1) &= \left(\frac{1}{F_{n+1} - 1} - \frac{1}{F_{n+3} - 1}\right) - \left(\frac{1}{F_n} - \frac{1}{F_{n+1}}\right) \\
&= \frac{F_{n+2}}{(F_{n+1} - 1)(F_{n+3} - 1)} - \frac{F_{n-1}}{F_n F_{n+1}} \\
&= \frac{F_{n+1}(F_n F_{n+2} - F_{n-1} F_{n+3}) + F_{n-1}(F_{n+1} + F_{n+3} - 1)}{F_n F_{n+1}(F_{n+1} - 1)(F_{n+3} - 1)} \\
&= \frac{-2F_{n+1} + F_{n-1}(2F_{n+1} + F_{n+2} - 1)}{F_n F_{n+1}(F_{n+1} - 1)(F_{n+3} - 1)} \\
&= \frac{2(F_{n-1} - 1) F_{n+1} + F_{n-1}(F_{n+2} - 1)}{F_n F_{n+1}(F_{n+1} - 1)(F_{n+3} - 1)} \\
&> 0,
\end{aligned}$$

which completes the proof. □

Lemma 8. *For all $n \geq 2$, we have*

$$\frac{F_{n+2}}{(F_{n+1}-1)(F_{n+3}-1)} - \frac{F_{n-1}}{F_n F_{n+1}} - \frac{1}{F_{2n+1}-1} \geq 0.$$

Proof. Applying Lemma 3, it is easy to see that, for $n \geq 2$,

$$F_{2n+1} - 1 - 2F_n F_{n+1} = F_n^2 + F_{n+1}^2 - 2F_n F_{n+1} - 1 = (F_{n+1} - F_n)^2 - 1 \geq 0,$$

from which we derive the conclusion that

$$\frac{1}{F_{2n+1}-1} \leq \frac{1}{2F_n F_{n+1}}.$$

Therefore, we have

$$\begin{aligned}
&\frac{F_{n+2}}{(F_{n+1}-1)(F_{n+3}-1)} - \frac{F_{n-1}}{F_n F_{n+1}} - \frac{1}{F_{2n+1}-1} \\
&\qquad \geq \frac{F_{n+2}}{(F_{n+1}-1)(F_{n+3}-1)} - \frac{F_{n-1}}{F_n F_{n+1}} - \frac{1}{2F_n F_{n+1}} \\
&\qquad = \frac{F_{n+2}}{(F_{n+1}-1)(F_{n+3}-1)} - \frac{2F_{n-1}+1}{2F_n F_{n+1}},
\end{aligned}$$

whose numerator is

$$\psi(n) := 2F_n F_{n+1} F_{n+2} - (2F_{n-1}+1)(F_{n+1}-1)(F_{n+3}-1).$$

Applying Lemma 5 repeatedly and the fact $F_{n+3} = 3F_{n+1} - F_{n-1}$, we can obtain

$$\begin{aligned}
\psi(n) &= 2F_{n+1}(F_n F_{n+2} - F_{n-1}F_{n+3}) + 2F_{n-1}F_{n+1} + 2F_{n-1}F_{n+3} - F_{n+1}F_{n+3} \\
&\quad + (F_{n+1} + F_{n+3}) - 2F_{n-1} - 1 \\
&= \left((-1)^{n+1} + 1\right) 4F_{n+1} + 2F_{n-1}F_{n+1} + (2F_{n-1} - F_{n+1})F_{n+3} - 3F_{n-1} - 1 \\
&= \left((-1)^{n+1} + 1\right) 4F_{n+1} + F_{n-1}(2F_{n+1} - F_{n+2}) + (F_{n-1}F_{n+2} - F_{n-2}F_{n+3}) \\
&\quad - 3F_{n-1} - 1 \\
&= \left((-1)^{n+1} + 1\right) 4F_{n+1} + F_{n-1}^2 - 3F_{n-1} - 1 + (-1)^n 3.
\end{aligned}$$

If n is even, we have $\psi(n) = (F_{n-1}-1)(F_{n-1}-2) \geq 0$. If n is odd, we have

$$\psi(n) = (F_{n-1}+1)(F_{n-1}+4) + 8(F_n - 1) > 0.$$

Therefore, $\psi(n) \geq 0$ always holds. This completes the proof. $\square$

Lemma 9. *If $n \geq 2$ and $m \geq 2$, then*

$$f_2(n) + f_2(n+1) + f_2(mn) + \frac{1}{F_{mn+2}-1} > 0.$$

Proof. If mn is odd, then the result follows from Lemma 7 and the fact $f_2(mn) > 0$. So we assume that mn is even. Now we have

$$f_2(mn) + \frac{1}{F_{mn+2}-1} = \frac{1}{F_{mn+1}-1} - \frac{1}{F_{mn}} = \frac{-(F_{mn-1}-1)}{F_{mn}(F_{mn+1}-1)} > \frac{-1}{F_{mn+1}-1}.$$

From the proof of Lemma 7 we know that whether n is even or odd,

$$f_2(n) + f_2(n+1) \geq \frac{F_{n+2}}{(F_{n+1}-1)(F_{n+3}-1)} - \frac{F_{n-1}}{F_n F_{n+1}}.$$

Therefore,

$$\begin{aligned} f_2(n) + f_2(n+1) + f_2(mn) + \frac{1}{F_{mn+2}-1} &> \frac{F_{n+2}}{(F_{n+1}-1)(F_{n+3}-1)} - \frac{F_{n-1}}{F_n F_{n+1}} - \frac{1}{F_{mn+1}-1} \\ &\geq \frac{F_{n+2}}{(F_{n+1}-1)(F_{n+3}-1)} - \frac{F_{n-1}}{F_n F_{n+1}} - \frac{1}{F_{2n+1}-1} \\ &\geq 0, \end{aligned}$$

where the last inequality follows from Lemma 8. □

Employing the fact $2(F_{2n+2}+1) \geq (F_{n+1}+1)(F_{n+3}+1)$ and similar arguments in the proof of Lemma 8, we have the following result, whose proof is omitted here.

Lemma 10. *If $n \geq 5$ is odd, then*

$$f_3(n) + f_3(n+1) > \frac{1}{F_{2n+2}+1}.$$

Now we establish two properties about $f_4(n)$.

Lemma 11. *For $n \geq 1$, we have*

$$f_4(n) + f_4(n+1) < 0.$$

Proof. If n is even, the result follows from the definition of $f_4(n)$. Next we consider the case where n is odd. Applying the argument in the proof of Lemma 6, we can easily deduce that

$$f_4(n) + f_4(n+1) = \frac{-2}{F_{n+1}} + \frac{1}{F_n} + \frac{1}{F_{n+3}} = \frac{-2}{F_n F_{n+1} F_{n+3}} < 0.$$

This completes the proof. □

Lemma 12. *If $n \geq 1$ and $m \geq 2$, then*

$$f_4(n) + f_4(n+1) + f_4(mn) < 0.$$

Proof. If mn is even, the result follows from Lemma 11 and the fact $f_4(mn) < 0$. So we assume that mn is odd, which implies that $m \geq 3$ and n is odd. Since mn is odd, we have

$$f_4(mn) = \frac{-1}{F_{mn+1}} + \frac{1}{F_{mn}} + \frac{1}{F_{mn+2}} < \frac{1}{F_{mn}} \leq \frac{1}{F_{3n}}.$$

Now we have

$$f_4(n) + f_4(n+1) + f_4(mn) < \frac{-2}{F_n F_{n+1} F_{n+3}} + \frac{1}{F_{3n}}.$$

To complete the proof, we only need to show that $2F_{3n} > F_n F_{n+1} F_{n+3}$.

It follows from Lemma 3 that $F_{2n+2} = F_{n-1}F_{n+2} + F_n F_{n+3}$, which implies $F_n F_{n+1} F_{n+3} < F_{n+1} F_{2n+2}$. Furthermore, employing Lemma 3 again, we can conclude that

$$\begin{aligned}
F_{n+1}F_{2n+2} &= (F_{n-1} + F_n)(F_{2n} + F_{2n+1}) \\
&= (F_{n-1}F_{2n} + F_n F_{2n+1}) + F_{n-1}F_{2n+1} + F_n F_{2n} \\
&= F_{3n} + F_{n-1}F_{2n+1} + F_{n+1}F_{2n} - F_{n-1}F_{2n} \\
&= F_{3n} + (F_{n-1}F_{2n-1} + F_{n+1}F_{2n}) \\
&< 2F_{3n},
\end{aligned}$$

which completes the proof. □

Theorem 13. *If $n \geq 4$ and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n+1} - 1, & \text{if } n \text{ is even;} \\ -F_{n+1} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We first consider the case where n is even. It follows from Lemma 6 that

$$\sum_{k=n}^{mn-1} f_1(k) < 0.$$

It is clear that mn is even, which ensures that

$$f_1(mn) + \frac{1}{F_{mn+2}} < 0.$$

With the help of $f_1(n)$ and the above two inequalities, we can obtain

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_k} = \frac{1}{F_{n+1}} - \left(\frac{1}{F_{mn+2}} + f_1(mn) \right) - \sum_{k=n}^{mn-1} f_1(k) > \frac{1}{F_{n+1}}.$$

Applying Lemma 7 and Lemma 9, we have

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{(-1)^k}{F_k} &= \frac{1}{F_{n+1}-1} - \left(f_2(n) + f_2(n+1) + f_2(mn) + \frac{1}{F_{mn+2}-1} \right) - \sum_{k=n+2}^{mn-1} f_2(k) \\
&< \frac{1}{F_{n+1}-1}.
\end{aligned}$$

Therefore, we obtain

$$\frac{1}{F_{n+1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_k} < \frac{1}{F_{n+1}-1},$$

which shows that the statement is true when n is even.

We now turn to consider the case where $n \geq 5$ is odd. If mn is odd, it is easy to see that

$$f_3(mn) - \frac{1}{F_{mn+2}+1} > 0.$$

Lemma 10 tells us that $f_3(n) + f_3(n+1) > 0$. Therefore,

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_k} = \frac{-1}{F_{n+1}+1} - \sum_{k=n}^{mn-1} f_3(k) - \left(f_3(mn) - \frac{1}{F_{mn+2}+1} \right) < \frac{-1}{F_{n+1}+1}.$$

If mn is even, employing Lemma 10 again, we can deduce

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{(-1)^k}{F_k} &= \frac{-1}{F_{n+1}+1} - \sum_{k=n+2}^{mn} f_3(k) - \left(f_3(n) + f_3(n+1) - \frac{1}{F_{mn+2}+1} \right) \\
&\leq \frac{-1}{F_{n+1}+1} - \sum_{k=n+2}^{mn} f_3(k) - \left(f_3(n) + f_3(n+1) - \frac{1}{F_{2n+2}+1} \right) \\
&< \frac{-1}{F_{n+1}+1}.
\end{aligned}$$

Now we can conclude that if $n \geq 5$ is odd, then

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_k} < \frac{-1}{F_{n+1}+1}.$$

If mn is even, then Lemma 11 implies that

$$\sum_{k=n}^{mn} f_4(k) < 0.$$

If mn is odd, invoking Lemma 11 and Lemma 12, we can get

$$\sum_{k=n}^{mn} f_4(k) = \sum_{k=n+2}^{mn-1} f_4(k) + (f_4(n) + f_4(n+1) + f_4(mn)) < 0.$$

Thus, we always have

$$\sum_{k=n}^{mn} f_4(k) < 0,$$

from which we obtain

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_k} = \frac{-1}{F_{n+1}} + \frac{1}{F_{mn+2}} - \sum_{k=n}^{mn} f_4(k) > \frac{-1}{F_{n+1}}.$$

Therefore, we arrive at

$$\frac{-1}{F_{n+1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_k} < \frac{-1}{F_{n+1}+1},$$

which shows that the result holds for odd n. □

3 Results for $a = 2$

We first introduce the following notations

$$\begin{aligned}
g_1(n) &= \frac{1}{F_{2n-2}+F_{2n}} - \frac{(-1)^n}{F_{2n}} - \frac{1}{F_{2n}+F_{2n+2}}, \\
g_2(n) &= \frac{1}{F_{2n-2}+F_{2n}-1} - \frac{(-1)^n}{F_{2n}} - \frac{1}{F_{2n}+F_{2n+2}-1}, \\
g_3(n) &= \frac{1}{F_{2n-2}+F_{2n}+1} - \frac{(-1)^n}{F_{2n}} - \frac{1}{F_{2n}+F_{2n+2}+1}, \\
g_4(n) &= \frac{-1}{F_{2n-2}+F_{2n}} - \frac{(-1)^n}{F_{2n}} + \frac{1}{F_{2n}+F_{2n+2}}, \\
g_5(n) &= \frac{-1}{F_{2n-2}+F_{2n}+1} - \frac{(-1)^n}{F_{2n}} + \frac{1}{F_{2n}+F_{2n+2}+1}.
\end{aligned}$$

It is routine to check that for $1 \le i \le 5$, $g_i(n)$ is positive if n is odd, and negative otherwise.

Lemma 14. *If $n \ge 1$, then $g_1(n) + g_1(n+1) > 0$ and*

$$g_1(n) + g_1(n+1) > g_1(n+2) + g_1(n+3).$$

Proof. If n is odd, we have

$$\begin{aligned} g_1(n)+g_1(n+1) &= \left(\frac{1}{F_{2n-2}+F_{2n}}-\frac{1}{F_{2n+2}+F_{2n+4}}\right)+\left(\frac{1}{F_{2n}}-\frac{1}{F_{2n+2}}\right) \\ &= \frac{5F_{2n+1}}{(F_{2n-2}+F_{2n})(F_{2n+2}+F_{2n+4})}+\frac{F_{2n+1}}{F_{2n}F_{2n+2}} \\ &> 0. \end{aligned}$$

Applying the easily checked fact

$$\begin{aligned} \frac{F_{2n+1}}{F_{2n-2}+F_{2n}} &> \frac{F_{2n+5}}{F_{2n+6}+F_{2n+8}}, \\ \frac{F_{2n+1}}{F_{2n}F_{2n+2}} &> \frac{F_{2n+5}}{F_{2n+4}F_{2n+6}}, \end{aligned}$$

we can conclude that $g_1(n)+g_1(n+1) > g_1(n+2)+g_1(n+3)$.

Now we consider the case where n is even. Doing some elementary manipulations and using Lemma 5, we have

$$\begin{aligned} g_1(n)+g_1(n+1) &= \left(\frac{1}{F_{2n-2}+F_{2n}}-\frac{1}{F_{2n}}\right)+\left(\frac{1}{F_{2n+2}}-\frac{1}{F_{2n+2}+F_{2n+4}}\right) \\ &= \frac{F_{2n-2}(F_{2n}F_{2n+4}-F_{2n+2}^2)+(F_{2n}^2-F_{2n-2}F_{2n+2})F_{2n+4}}{F_{2n}F_{2n+2}(F_{2n-2}+F_{2n})(F_{2n+2}+F_{2n+4})} \\ &= \frac{F_{2n+4}-F_{2n-2}}{F_{2n}F_{2n+2}(F_{2n-2}+F_{2n})(F_{2n+2}+F_{2n+4})} \\ &= \frac{4F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2}+F_{2n})(F_{2n+2}+F_{2n+4})} \\ &> 0. \end{aligned}$$

Applying the above identity, we see that

$$\frac{g_1(n)+g_1(n+1)}{g_1(n+2)+g_1(n+3)} = \frac{F_{2n+1}F_{2n+4}F_{2n+6}}{F_{2n}F_{2n+2}F_{2n+5}} \cdot \frac{F_{2n+6}+F_{2n+8}}{F_{2n-2}+F_{2n}} > 1.$$

Thus, $g_1(n)+g_1(n+1) > g_1(n+2)+g_1(n+3)$ also holds. $\square$

Lemma 15. *For $n \geq 1$, we have*

$$F_{6n+2} > F_{2n}(F_{2n-2}+F_{2n})(F_{2n+2}+F_{2n+4}).$$

Proof. It follows from Lemma 5 that

$$\begin{aligned} F_{2n-1}F_{2n+3}-F_{2n-2}F_{2n+4} &= 5, \\ F_{2n-1}F_{2n+1}-F_{2n}^2 &= 1, \\ F_{2n+1}F_{2n+3}-F_{2n}F_{2n+4} &= 2. \end{aligned}$$

Thus, $F_{2n-1}F_{2n+3} > F_{2n-2}F_{2n+4}$, $F_{2n-1}F_{2n+1} > F_{2n}^2$, and $F_{2n+1}F_{2n+3} > F_{2n}F_{2n+4}$.

Employing Lemma 3 repeatedly and the above three inequalities, we have

$$\begin{aligned}
F_{6n+2} &= F_{2n}F_{4n+1} + F_{2n+1}F_{4n+2} \\
&= F_{2n}(F_{2n-2}F_{2n+2} + F_{2n-1}F_{2n+3}) + F_{2n+1}(F_{2n-1}F_{2n+2} + F_{2n}F_{2n+3}) \\
&> F_{2n-2}F_{2n}F_{2n+2} + F_{2n-2}F_{2n+4}F_{2n} + F_{2n}^2F_{2n+2} + F_{2n}F_{2n+4}F_{2n} \\
&= F_{2n}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}),
\end{aligned}$$

which completes the proof. □

Lemma 16. *If $n \geq 1$ and $m \geq 3$, then*

$$g_1(n) + g_1(n+1) + g_1(mn) > 0.$$

Proof. If mn is odd, then the result follows from Lemma 14 and the fact $g_1(mn) > 0$. Thus we focus on the case where mn is even. For $k \geq 1$,

$$\begin{aligned}
\frac{1}{F_{2k-2} + F_{2k}} - \frac{1}{F_{2k}} &= -\frac{F_{2k-2}}{(F_{2k-2} + F_{2k})F_{2k}} \\
&= -\frac{F_{2k-2}}{F_{2k-2}F_{2k} + F_{2k}^2} \\
&> -\frac{F_{2k-2}}{F_{2k-2}F_{2k+2}} \\
&= -\frac{1}{F_{2k+2}},
\end{aligned}$$

where the inequality follows from $F_{2k}^2 - F_{2k-2}F_{2k+2} = 1$. Since mn is even, employing the above inequality, we have

$$g_1(mn) > -\frac{1}{F_{2mn+2}} - \frac{1}{F_{2mn} + F_{2mn+2}} > -\frac{2}{F_{2mn+2}} \geq -\frac{2}{F_{6n+2}}.$$

From the proof of Lemma 14 we know that whether n is even or odd, we always have

$$g_1(n) + g_1(n+1) \geq \frac{4F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})}.$$

Therefore,

$$\begin{aligned}
g_1(n) + g_1(n+1) + g_1(mn) &> \frac{4F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})} - \frac{2}{F_{6n+2}} \\
&> \frac{2}{F_{2n}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})} - \frac{2}{F_{6n+2}} \\
&> 0,
\end{aligned}$$

where the last inequality follows from Lemma 15. □

Lemma 17. *If $n > 0$, then*

$$2F_{4n}(F_{4n} + F_{4n+2}) > F_{2n+2}F_{4n+3}(F_{2n-2} + F_{2n}).$$

Proof. It suffices to show that $2F_{4n}^2 > F_{2n-2}F_{2n+2}F_{4n+3}$ and $2F_{4n}F_{4n+2} > F_{2n}F_{2n+2}F_{4n+3}$. These two inequalities can be proved using similar arguments, so we only prove the first one.

Applying Lemma 5 repeatedly and Lemma 3, we can obtain

$$\begin{aligned} 2F_{4n}^2 &= 2F_{4n-3}F_{4n+3} - 8 \\ &= 2(F_{2n-2}^2 + F_{2n-1}^2)F_{4n+3} - 8 \\ &> (F_{2n-2}F_{2n-1} + 2F_{2n-1}^2)F_{4n+3} - 8 \\ &= F_{2n-1}F_{2n+1}F_{4n+3} - 8 \\ &= (F_{2n-2}F_{2n+2} + 2)F_{4n+3} - 8 \\ &> F_{2n-2}F_{2n+2}F_{4n+3}. \end{aligned}$$

The proof is completed. □

Lemma 18. *For all $n \geq 2$, we have*

$$g_2(n) + g_2(n+1) + g_2(2n) > 0.$$

Proof. It is straightforward to verify that $F_{2n-2} + F_{2n} + F_{2n+2} + F_{2n+4} = 3(F_{2n} + F_{2n+2})$. Applying Lemma 5 repeatedly, we get

$$\begin{aligned} (F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}) &= F_{2n-2}F_{2n+2} + F_{2n-2}F_{2n+4} + F_{2n}F_{2n+2} + F_{2n}F_{2n+4} \\ &= F_{2n-2}F_{2n+2} + (F_{2n}F_{2n+2} - 3) + F_{2n}F_{2n+2} \\ &\quad + F_{2n}(2F_{2n+2} + F_{2n+1}) \\ &= (F_{2n-2}F_{2n+2} - F_{2n}^2) + (F_{2n}^2 + F_{2n}F_{2n+1}) \\ &\quad + 4F_{2n}F_{2n+2} - 3 \\ &= 5F_{2n}F_{2n+2} - 4. \end{aligned}$$

It follows from the definition of $g_2(n)$ and the above two equations that

$$\begin{aligned} g_2(n) + g_2(n+1) &\geq \left(\frac{1}{F_{2n-2} + F_{2n} - 1} - \frac{1}{F_{2n+2} + F_{2n+4} - 1}\right) - \left(\frac{1}{F_{2n}} - \frac{1}{F_{2n+2}}\right) \\ &= \frac{5F_{2n+1}}{(F_{2n-2} + F_{2n} - 1)(F_{2n+2} + F_{2n+4} - 1)} - \frac{F_{2n+1}}{F_{2n}F_{2n+2}} \\ &= \frac{3(F_{2n} + F_{2n+2} + 1)F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n} - 1)(F_{2n+2} + F_{2n+4} - 1)} \\ &> \frac{1}{F_{2n+2}(F_{2n-2} + F_{2n} - 1)}, \end{aligned}$$

where the last inequality follows from $3F_n > F_{n+2}$.

It is routine to show

$$\begin{aligned} 2(F_{4n+2} - F_{4n-2}) &= 2(2F_{4n} + F_{4n-1} - F_{4n-2}) \\ &= 3F_{4n} + F_{4n} + 2F_{4n-3} \\ &> 3F_{4n} + (2F_{4n-2} + F_{4n-3}) + F_{4n-2} \\ &> 3(F_{4n-2} + F_{4n}), \end{aligned}$$

which means

$$F_{4n+2} - F_{4n-2} > \frac{3}{2}(F_{4n-2} + F_{4n}).$$

Employing the above inequality, we can deduce that

$$\begin{aligned} g_2(2n) &= \frac{F_{4n+2} - F_{4n-2}}{(F_{4n-2} + F_{4n} - 1)(F_{4n} + F_{4n+2} - 1)} - \frac{1}{F_{4n}} \\ &> \frac{3}{2(F_{4n} + F_{4n+2} - 1)} - \frac{1}{F_{4n}} \\ &= \frac{-F_{4n+3} + 2}{2F_{4n}(F_{4n} + F_{4n+2} - 1)} \\ &> -\frac{F_{4n+3}}{2F_{4n}(F_{4n} + F_{4n+2} - 1)}. \end{aligned}$$

Now we conclude that

$$g_2(n) + g_2(n+1) + g_2(2n) > \frac{1}{F_{2n+2}(F_{2n-2} + F_{2n} - 1)} - \frac{F_{4n+3}}{2F_{4n}(F_{4n} + F_{4n+2} - 1)} > 0,$$

where the last inequality follows from Lemma 17. □

Applying the argument in the proof of Lemma 18, it can be readily seen the following property of $g_3(n)$, whose proof is omitted here.

Lemma 19. *If $n \geq 2$ is even, we have*

$$g_3(n) + g_3(n+1) < 0.$$

Imitating the proof of Lemma 14 and Lemma 16 respectively, we can easily get the following results on $g_4(n)$.

Lemma 20. *For $n \geq 1$, we have*

$$g_4(n) + g_4(n+1) < 0.$$

Lemma 21. *If $n \geq 1$ and $m \geq 2$, then*

$$g_4(n) + g_4(n+1) + g_4(mn) < 0.$$

Lemma 22. *If $n \geq 1$ is odd, we have*

$$g_5(n) + g_5(n+1) > \frac{1}{F_{4n} + F_{4n+2} + 1}.$$

Proof. It is easy to see that the result is true for $n = 1$, thus we assume that $n \geq 3$. From the proof of Lemma 18, we can easily obtain that if $n \geq 3$ is odd, then

$$\begin{aligned} g_5(n) + g_5(n+1) &= \frac{3(F_{2n} + F_{2n+2} - 1)F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n} + 1)(F_{2n+2} + F_{2n+4} + 1)} \\ &> \frac{1}{F_{2n+2}(F_{2n-2} + F_{2n} + 1)}. \end{aligned}$$

Employing Lemma 3 repeatedly, it is easy to see that

$$\begin{aligned} F_{2n+2}(F_{2n-2} + F_{2n} + 1) &< F_{2n-2}F_{2n+3} + F_{2n}F_{2n+3} + F_{2n+2} \\ &= F_{4n} - F_{2n-3}F_{2n+2} + F_{4n+2} - F_{2n-1}F_{2n+2} + F_{2n+2} \\ &< F_{4n} + F_{4n+2}. \end{aligned}$$

Combining the above two inequalities yields the desired result. □

Lemma 23. *For $n \geq 2$, we have*

$$F_{4n-2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}) > F_{4n}(F_{4n-2} + F_{4n}).$$

Proof. We first consider the right-hand side. Applying $F_{4n}^2 - F_{4n-1}F_{4n+1} = -1$, we have

$$F_{4n}(F_{4n-2} + F_{4n}) = F_{4n-2}F_{4n} + F_{4n}^2 = F_{4n-2}F_{4n} + F_{4n-1}F_{4n+1} - 1 = F_{8n-1} - 1.$$

For the left-hand side, we have that if $n \geq 2$, then

$$\begin{aligned} (F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}) &= F_{2n-2}F_{2n+2} + F_{2n}F_{2n+2} + F_{2n-2}F_{2n+4} + F_{2n}F_{2n+4} \\ &> (F_{2n-2}F_{2n+1} + F_{2n-1}F_{2n+2}) + (F_{2n-2}F_{2n+3} \\ &\quad + F_{2n-1}F_{2n+4}) + F_{2n-2}F_{2n+4} \\ &> F_{4n} + F_{4n+2} + 2. \end{aligned}$$

Therefore, using the fact $F_{4n-2}F_{4n+2} - F_{4n-1}F_{4n+1} = -2$, we have

$$\begin{aligned} F_{4n-2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4}) &> F_{4n-2}F_{4n} + F_{4n-2}F_{4n+2} + 2 \\ &= F_{4n-2}F_{4n} + F_{4n-1}F_{4n+1} \\ &= F_{8n-1}. \end{aligned}$$

Thus the left-hand side is greater than the right-hand side. □

Theorem 24. *If $n \geq 2$ is even and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} \right)^{-1} \right\rfloor = \begin{cases} F_{2n-2} + F_{2n} - 1, & \text{if } m = 2; \\ F_{2n-2} + F_{2n}, & \text{if } m > 2. \end{cases}$$

Proof. We first consider the case where $m = 2$. From Lemma 14 we know that

$$\sum_{k=n}^{2n-1} g_1(k) < \frac{4F_{2n+1}}{F_{2n}F_{2n+2}(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})} \cdot \frac{n}{2} < \frac{1}{(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})}.$$

In addition,

$$g_1(2n) + \frac{1}{F_{4n} + F_{4n+2}} = \frac{1}{F_{4n-2} + F_{4n}} - \frac{1}{F_{4n}} = \frac{-F_{4n-2}}{F_{4n}(F_{4n-2} + F_{4n})}.$$

Therefore, invoking Lemma 23, we have

$$\sum_{k=n}^{2n} g_1(k) + \frac{1}{F_{4n} + F_{4n+2}} < \frac{1}{(F_{2n-2} + F_{2n})(F_{2n+2} + F_{2n+4})} - \frac{F_{4n-2}}{F_{4n}(F_{4n-2} + F_{4n})} < 0.$$

Now with the help of $g_1(n)$, we can obtain

$$\sum_{k=n}^{2n} \frac{(-1)^k}{F_{2k}} = \frac{1}{F_{2n-2} + F_{2n}} - \frac{1}{F_{4n} + F_{4n+2}} - \sum_{k=n}^{2n} g_1(k) > \frac{1}{F_{2n-2} + F_{2n}}.$$

From the proof of Lemma 18, we know that $g_2(n) + g_2(n + 1) > 0$. Moreover, applying Lemma 18, we can deduce

$$\sum_{k=n}^{2n} g_2(k) = g_2(n) + g_2(n + 1) + g_2(2n) + \sum_{k=n+2}^{2n-1} g_2(k) > 0.$$

Therefore,

$$\sum_{k=n}^{2n} \frac{(-1)^k}{F_{2k}} = \frac{1}{F_{2n-2} + F_{2n} - 1} - \frac{1}{F_{4n} + F_{4n+2} - 1} - \sum_{k=n}^{2n} g_2(k) < \frac{1}{F_{2n-2} + F_{2n} - 1}.$$

We now conclude that

$$\frac{1}{F_{2n-2} + F_{2n}} < \sum_{k=n}^{2n} \frac{(-1)^k}{F_{2k}} < \frac{1}{F_{2n-2} + F_{2n} - 1},$$

which shows that the statement for $m = 2$ is true.

Next we turn to consider the case where $m > 2$. First, employing Lemma 14 and Lemma 16, we see that

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} < \frac{1}{F_{2n-2}+F_{2n}} - (g_1(n)+g_1(n+1)+g_1(mn)) - \sum_{k=n+2}^{mn-1} g_1(k) < \frac{1}{F_{2n-2}+F_{2n}}.$$

We write the sum in terms of $g_3(n)$ as

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} &= \frac{1}{F_{2n-2}+F_{2n}+1} - \sum_{k=n}^{mn-1} g_3(k) - \left(g_3(mn) + \frac{1}{F_{2mn}+F_{2mn+2}+1}\right) \\
&= \frac{1}{F_{2n-2}+F_{2n}+1} - \sum_{k=n}^{mn-1} g_3(k) - \left(\frac{1}{F_{2mn-2}+F_{2mn}+1} - \frac{1}{F_{2mn}}\right) \\
&> \frac{1}{F_{2n-2}+F_{2n}+1},
\end{aligned}$$

where the last inequality follows from Lemma 19. Now we get

$$\frac{1}{F_{2n-2}+F_{2n}+1} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} < \frac{1}{F_{2n-2}+F_{2n}},$$

which yields the desired identity. □

Theorem 25. *If $n \geq 1$ is odd and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}} \right)^{-1} \right\rfloor = -F_{2n-2} - F_{2n} - 1.$$

Proof. If mn is even, it follows from Lemma 20 that

$$\sum_{k=n}^{mn} g_4(k) < 0.$$

If mn is odd, then Lemma 20 and Lemma 21 ensure that

$$\sum_{k=n}^{mn} g_4(k) = \sum_{k=n+2}^{mn-1} g_4(k) + (g_4(n)+g_4(n+1)+g_4(mn)) < 0.$$

Therefore, we always have

$$\sum_{k=n}^{mn} g_4(k) < 0.$$

With the help of $g_4(n)$, we have

$$\sum_{k=n}^{mn}\frac{(-1)^k}{F_{2k}}=\frac{-1}{F_{2n-2}+F_{2n}}+\frac{1}{F_{2mn-2}+F_{2mn}}-\sum_{k=n}^{mn}g_4(k)>\frac{-1}{F_{2n-2}+F_{2n}}.$$

From Lemma 22 we know that if n is odd, then $g_5(n)+g_5(n+1)>0$. Now we claim that

$$\sum_{k=n}^{mn}g_5(k)>\frac{1}{F_{2mn}+F_{2mn+2}+1}.$$

If mn is even, employing Lemma 22, we obtain

$$\begin{aligned}\sum_{k=n}^{mn}g_5(k)-\frac{1}{F_{2mn}+F_{2mn+2}+1} &\geq \sum_{k=n}^{mn}g_5(k)-\frac{1}{F_{4n}+F_{4n+2}+1}\\ &\geq g_5(n)+g_5(n+1)-\frac{1}{F_{4n}+F_{4n+2}+1}\\ &> 0.\end{aligned}$$

If mn is odd, then

$$\begin{aligned}\sum_{k=n}^{mn}g_5(k)-\frac{1}{F_{2mn}+F_{2mn+2}+1} &= \sum_{k=n}^{mn-1}g_5(k)+\left(g_5(mn)-\frac{1}{F_{2mn}+F_{2mn+2}+1}\right)\\ &> -\frac{1}{F_{2mn-2}+F_{2mn}+1}+\frac{1}{F_{2mn}}\\ &> 0.\end{aligned}$$

Therefore, we have

$$\sum_{k=n}^{mn}\frac{(-1)^k}{F_{2k}}=\frac{-1}{F_{2n-2}+F_{2n}+1}+\frac{1}{F_{2mn-2}+F_{2mn}+1}-\sum_{k=n}^{mn}g_5(k)<\frac{-1}{F_{2n-2}+F_{2n}+1}.$$

Now we can conclude that

$$\frac{-1}{F_{2n-2}+F_{2n}}<\sum_{k=n}^{mn}\frac{(-1)^k}{F_{2k}}<\frac{-1}{F_{2n-2}+F_{2n}+1},$$

from which the desired result follows. □

Similarly, we can prove the following results.

Theorem 26. *If $n\geq 4$ is even and $m\geq 2$, then*

$$\left\lfloor\left(\sum_{k=n}^{mn}\frac{(-1)^k}{F_{2k-1}}\right)^{-1}\right\rfloor=F_{2n-3}+F_{2n-1}-1.$$

Theorem 27. *If $n \geq 3$ is odd and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}} \right)^{-1} \right\rfloor = \begin{cases} -F_{2n-3} - F_{2n-1} - 1, & \text{if } m = 2; \\ -F_{2n-3} - F_{2n-1}, & \text{if } m > 2. \end{cases}$$

4 Results for $a = 3$

We first introduce the following notations:

$$\begin{aligned} s_1(n) &= \frac{1}{2F_{3n-1}} - \frac{(-1)^n}{F_{3n}} - \frac{1}{2F_{3n+2}}, \\ s_2(n) &= \frac{1}{2F_{3n-1} - 1} - \frac{(-1)^n}{F_{3n}} - \frac{1}{2F_{3n+2} - 1}, \\ s_3(n) &= \frac{-1}{2F_{3n-1}} - \frac{(-1)^n}{F_{3n}} + \frac{1}{2F_{3n+2}}, \\ s_4(n) &= \frac{-1}{2F_{3n-1} + 1} - \frac{(-1)^n}{F_{3n}} + \frac{1}{2F_{3n+2} + 1}. \end{aligned}$$

It is easy to see that for each i, $s_i(n)$ is positive if n is odd, and negative otherwise.

Lemma 28. *If $n \geq 2$ is even, then*

$$s_1(n) + s_1(n+1) < 0.$$

Proof. Since n is even, applying Lemma 5 twice, we have

$$\begin{aligned} s_1(n) + s_1(n+1) &= \left(\frac{1}{2F_{3n-1}} - \frac{1}{2F_{3n+5}} \right) - \left(\frac{1}{F_{3n}} - \frac{1}{F_{3n+3}} \right) \\ &= \frac{2F_{3n+2}}{F_{3n-1}F_{3n+5}} - \frac{2F_{3n+1}}{F_{3n}F_{3n+3}} \\ &= 2 \cdot \frac{F_{3n}F_{3n+2}F_{3n+3} - F_{3n-1}F_{3n+1}F_{3n+5}}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} \\ &= 2 \cdot \frac{F_{3n}F_{3n+2}F_{3n+3} - (F_{3n}^2 + 1)F_{3n+5}}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} \\ &= 2 \cdot \frac{F_{3n}(F_{3n+2}F_{3n+3} - F_{3n}F_{3n+5}) - F_{3n+5}}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} \\ &= 2 \cdot \frac{2F_{3n} - F_{3n+5}}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} \\ &< 0, \end{aligned}$$

which completes the proof. ◻

Lemma 29. *For all $n \geq 1$, we have*

$$s_2(n) + s_2(n+1) > 0.$$

Proof. It is clear that the result holds if n is odd. In the rest, we assume that n is even. Applying the analysis in the proof of Lemma 28, we can easily obtain

$$\begin{aligned}
s_2(n) + s_2(n+1) &= \left(\frac{1}{2F_{3n-1}-1} - \frac{1}{2F_{3n+5}-1}\right) - \left(\frac{1}{F_{3n}} - \frac{1}{F_{3n+3}}\right) \\
&= \frac{8F_{3n+2}}{(2F_{3n-1}-1)(2F_{3n+5}-1)} - \frac{2F_{3n+1}}{F_{3n}F_{3n+3}} \\
&= \frac{8(F_{3n}F_{3n+2}F_{3n+3} - F_{3n-1}F_{3n+1}F_{3n+5}) + 2F_{3n+1}(2F_{3n-1}+2F_{3n+5}-1)}{(2F_{3n-1}-1)(2F_{3n+5}-1)F_{3n}F_{3n+3}} \\
&= \frac{16F_{3n} - 8F_{3n+5} + 2F_{3n+1}(2F_{3n-1}+2F_{3n+5}-1)}{(2F_{3n-1}-1)(2F_{3n+5}-1)F_{3n}F_{3n+3}} \\
&> \frac{4F_{3n+1}F_{3n+5} - 8F_{3n+5}}{(2F_{3n-1}-1)(2F_{3n+5}-1)F_{3n}F_{3n+3}} \\
&> 0.
\end{aligned}$$

The proof is completed. $\square$

Lemma 30. *If $n \geq 1$ and $m \geq 2$, then*

$$s_2(n) + s_2(n+1) + s_2(mn) > 0.$$

Proof. If mn is odd, then the result follows from Lemma 29 and the fact $s_2(mn) > 0$. So we assume that mn is even. Now it is clear that

$$s_2(mn) = \frac{1}{2F_{3mn-1}-1} - \frac{1}{F_{3mn}} - \frac{1}{2F_{3mn+2}-1} > -\frac{1}{F_{3mn}} \geq -\frac{1}{F_{6n}}.$$

If n is odd, we have

$$s_2(n) + s_2(n+1) \quad > \quad \frac{1}{F_{3n}} - \frac{1}{F_{3n+3}} = \frac{2F_{3n+1}}{F_{3n}F_{3n+3}} > \frac{2}{F_{3n}F_{3n+3}}.$$

If n is even, then from Lemma 29 we know that

$$\begin{aligned}
s_2(n) + s_2(n+1) \quad &> \quad \frac{4F_{3n+1}F_{3n+5} - 8F_{3n+5}}{(2F_{3n-1}-1)(2F_{3n+5}-1)F_{3n}F_{3n+3}} \\
&= \quad \frac{4F_{3n+5}(2F_{3n-1}+F_{3n-2}-2)}{(2F_{3n-1}-1)(2F_{3n+5}-1)F_{3n}F_{3n+3}} \\
&> \quad \frac{2}{F_{3n}F_{3n+3}}.
\end{aligned}$$

Now we can derive the conclusion that

$$s_2(n) + s_2(n+1) + s_2(mn) > \frac{2}{F_{3n}F_{3n+3}} - \frac{1}{F_{6n}} \geq 0,$$

where the last inequality follows from

$$2F_{6n} = F_{3n}(2F_{3n-1} + 2F_{3n+1}) > F_{3n}(F_{3n} + 2F_{3n+1}) = F_{3n}F_{3n+3}.$$

This completes the proof. □

Lemma 31. *For all $n \geq 1$,*

$$s_3(n) + s_3(n+1) < 0.$$

Proof. The result clearly holds when n is even. If n is odd, applying similar analysis in the proof of Lemma 28, we can easily derive

$$s_3(n) + s_3(n+1) = 2 \cdot \frac{2F_{3n} - F_{3n+5}}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} < 0,$$

which completes the proof. □

Lemma 32. *If $n \geq 1$ and $m \geq 2$, then*

$$s_3(n) + s_3(n+1) + s_3(mn) < 0.$$

Proof. If mn is even, then the result follows from Lemma 31 and the fact $s_3(mn) < 0$. Now we assume that mn is odd, which implies that n is odd and $m \geq 3$. First we have

$$s_3(mn) = \frac{-1}{2F_{3mn-1}} + \frac{1}{F_{3mn}} + \frac{1}{2F_{3mn+2}} < \frac{1}{F_{3mn}} \leq \frac{1}{F_{9n}}.$$

Moreover, from the proof of Lemma 31 we know

$$s_3(n) + s_3(n+1) = -\frac{2(F_{3n+5} - 2F_{3n})}{F_{3n-1}F_{3n}F_{3n+3}F_{3n+5}} < -\frac{1}{F_{3n-1}F_{3n}F_{3n+3}}.$$

Now we arrive at

$$s_3(n) + s_3(n+1) + s_3(mn) < -\frac{1}{F_{3n-1}F_{3n}F_{3n+3}} + \frac{1}{F_{9n}} < 0,$$

where the last inequality follows from

$$F_{9n} = F_{3n-2}F_{6n+1} + F_{3n-1}F_{6n+2} > F_{3n-1}(F_{3n-1}F_{3n+2} + F_{3n}F_{3n+3}) > F_{3n-1}F_{3n}F_{3n+3}.$$

The proof is completed. □

Lemma 33. *If $n \geq 1$ is odd, then*

$$s_4(n) + s_4(n+1) > \frac{1}{2F_{6n+2}+1}.$$

Proof. It is easy to check that the result holds for $n = 1$, so we assume that $n \geq 3$. Applying the similar analysis in the proof of Lemma 28, we have that, for $n \geq 3$,

$$\begin{aligned}
s_4(n) + s_4(n+1) &= -\left(\frac{1}{2F_{3n-1}+1} - \frac{1}{2F_{3n+5}+1}\right) + \left(\frac{1}{F_{3n}} - \frac{1}{F_{3n+3}}\right) \\
&= -\frac{2F_{3n+5} - 2F_{3n-1}}{(2F_{3n-1}+1)(2F_{3n+5}+1)} + \frac{2F_{3n+1}}{F_{3n}F_{3n+3}} \\
&> -\frac{F_{3n+5} - F_{3n-1}}{(2F_{3n-1}+1)F_{3n+5}} + \frac{2F_{3n+1}}{F_{3n}F_{3n+3}} \\
&= -\frac{4F_{3n+2}}{(2F_{3n-1}+1)F_{3n+5}} + \frac{2F_{3n+1}}{F_{3n}F_{3n+3}} \\
&= \frac{4(F_{3n-1}F_{3n+1}F_{3n+5} - F_{3n}F_{3n+2}F_{3n+3}) + 2F_{3n+1}F_{3n+5}}{(2F_{3n-1}+1)F_{3n}F_{3n+3}F_{3n+5}} \\
&= \frac{4(2F_{3n} - F_{3n+5}) + 2F_{3n+1}F_{3n+5}}{(2F_{3n-1}+1)F_{3n}F_{3n+3}F_{3n+5}} \\
&> \frac{(2F_{3n+1} - 4)F_{3n+5}}{(2F_{3n-1}+1)F_{3n}F_{3n+3}F_{3n+5}} \\
&> \frac{1}{F_{3n}F_{3n+3}}.
\end{aligned}$$

In addition, we have

$$F_{2n+2} = F_{n-1}F_{n+2} + F_nF_{n+3} > F_nF_{n+3}.$$

Combining the above two inequalities together yields the desired result. $\square$

Theorem 34. *If $n \geq 1$ and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} \right)^{-1} \right\rfloor = \begin{cases} 2F_{3n-1} - 1, & \text{if } n \text{ is even;} \\ -2F_{3n-1} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We first consider the case where n is even. With the help of $s_1(n)$, we have

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} &= \frac{1}{2F_{3n-1}} - \sum_{k=n}^{mn-1} s_1(k) - \left(s_1(mn) + \frac{1}{2F_{3mn+2}}\right) \\ &= \frac{1}{2F_{3n-1}} - \sum_{k=n}^{mn-1} s_1(k) - \left(\frac{1}{2F_{3mn+2}} - \frac{1}{F_{3mn}}\right) \\ &> \frac{1}{2F_{3n-1}} - \sum_{k=n}^{mn-1} s_1(k) \\ &> \frac{1}{2F_{3n-1}},\end{aligned}$$

where the last inequality follows from Lemma 28.

Employing Lemma 29 and Lemma 30, we can deduce that

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} &= \frac{1}{2F_{3n-1}-1} - \frac{1}{2F_{3mn+2}-1} - \sum_{k=n+2}^{mn-1} s_2(k) - (s_2(n) + s_2(n+1) + s_2(mn)) \\ &< \frac{1}{2F_{3n-1}-1}.\end{aligned}$$

Therefore, we obtain

$$\frac{1}{2F_{3n-1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} < \frac{1}{2F_{3n-1}-1},$$

which shows that the statement is true when n is even.

We now turn to consider the case where n is odd. If m is even, applying Lemma 31 and Lemma 33, we can deduce that

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} = \frac{-1}{2F_{3n-1}} + \frac{1}{2F_{3mn+2}} - \sum_{k=n}^{mn} s_3(k) > \frac{-1}{2F_{3n-1}},$$

and

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} &= \frac{-1}{2F_{3n-1}+1} - \sum_{k=n+2}^{mn} s_4(k) - \left(s_4(n) + s_4(n+1) - \frac{1}{2F_{3mn+2}+1}\right) \\ &\leq \frac{-1}{2F_{3n-1}+1} - \sum_{k=n+2}^{mn} s_4(k) - \left(s_4(n) + s_4(n+1) - \frac{1}{2F_{6n+2}+1}\right) \\ &< \frac{-1}{2F_{3n-1}+1}.\end{aligned}$$

Thus, if n is odd and m is even, we have

$$\frac{-1}{2F_{3n-1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} < \frac{-1}{2F_{3n-1}+1}.$$

If m is odd, then Lemma 31 and Lemma 32 implies that

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} = \frac{-1}{2F_{3n-1}} + \frac{1}{2F_{3mn+2}} - \sum_{k=n+2}^{mn-1} s_3(k) - (s_3(n) + s_3(n+1) + s_3(mn)) > \frac{-1}{2F_{3n-1}}.$$

And it follows from Lemma 33 that

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} &= \frac{-1}{2F_{3n-1}+1} - \sum_{k=n}^{mn-1} s_4(k) - \left(s_4(mn) - \frac{1}{2F_{3mn+2}+1}\right) \\
&= \frac{-1}{2F_{3n-1}+1} - \sum_{k=n}^{mn-1} s_4(k) - \left(\frac{1}{F_{3mn}} - \frac{1}{2F_{3mn-1}+1}\right) \\
&< \frac{-1}{2F_{3n-1}+1}.
\end{aligned}$$

Thus, if n and m are both odd, then

$$\frac{-1}{2F_{3n-1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k}} < \frac{-1}{2F_{3n-1}+1}$$

also holds. Hence, the statement is true when n is odd. $\square$

Theorem 35. *If $n \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} \right)^{-1} \right\rfloor = \begin{cases} 2F_{3n} - 1, & \text{if } n \text{ is even;} \\ -2F_{3n} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 36. *If $n \geq 1$ and $m \geq 3$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} \right)^{-1} \right\rfloor = \begin{cases} 2F_{3n}, & \text{if } n \text{ is even;} \\ -2F_{3n}, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 37. *If $n \geq 1$ and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+2}} \right)^{-1} \right\rfloor = \begin{cases} 2F_{3n+1} - 1, & \text{if } n \text{ is even;} \\ -2F_{3n+1} - 1, & \text{if } n \text{ is odd.} \end{cases}$$

Remark 38. We will prove Theorem 35 and Theorem 36 in detail in the next section. The proof of Theorem 37 is very similar to that of Theorem 34, thus omitted here.

5 Proof of Theorem 35 and Theorem 36

We begin with introducing the following auxiliary functions:

$$
\begin{aligned}
t_1(n) &= \frac{1}{2F_{3n}} - \frac{(-1)^n}{F_{3n+1}} - \frac{1}{2F_{3n+3}}, \\
t_2(n) &= \frac{1}{2F_{3n}-1} - \frac{(-1)^n}{F_{3n+1}} - \frac{1}{2F_{3n+3}-1}, \\
t_3(n) &= \frac{1}{2F_{3n}+1} - \frac{(-1)^n}{F_{3n+1}} - \frac{1}{2F_{3n+3}+1}, \\
t_4(n) &= \frac{-1}{2F_{3n}} - \frac{(-1)^n}{F_{3n+1}} + \frac{1}{2F_{3n+3}}, \\
t_5(n) &= \frac{-1}{2F_{3n}+1} - \frac{(-1)^n}{F_{3n+1}} + \frac{1}{2F_{3n+3}+1}, \\
t_6(n) &= \frac{-1}{2F_{3n}-1} - \frac{(-1)^n}{F_{3n+1}} + \frac{1}{2F_{3n+3}-1}.
\end{aligned}
$$

It is straightforward to check that each $t_i(n)$ is positive if n is odd, and negative otherwise.

Lemma 39. *For all $n \geq 1$, we have $t_1(n) + t_1(n+1) > 0$ and*

$$t_1(n) + t_1(n+1) > t_1(n+2) + t_1(n+3).$$

Proof. If n is odd, we have

$$t_1(n) + t_1(n+1) = \left(\frac{1}{2F_{3n}} - \frac{1}{2F_{3n+6}}\right) + \left(\frac{1}{F_{3n+1}} - \frac{1}{F_{3n+4}}\right) = \frac{2F_{3n+3}}{F_{3n}F_{3n+6}} + \frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}} > 0.$$

Since

$$
\begin{aligned}
\frac{F_{3n+3}}{F_{3n}F_{3n+6}} &> \frac{F_{3n+9}}{F_{3n+6}F_{3n+12}}, \\
\frac{F_{3n+2}}{F_{3n+1}F_{3n+4}} &> \frac{F_{3n+8}}{F_{3n+7}F_{3n+10}},
\end{aligned}
$$

we can conclude that $t_1(n) + t_1(n+1) > t_1(n+2) + t_1(n+3)$.

Now we consider the case where n is even. Applying Lemma 5 repeatedly, we have

$$\begin{aligned}
t_1(n)+t_1(n+1) &= \left(\frac{1}{2F_{3n}}-\frac{1}{2F_{3n+6}}\right)-\left(\frac{1}{F_{3n+1}}-\frac{1}{F_{3n+4}}\right)\\
&= \frac{2F_{3n+3}}{F_{3n}F_{3n+6}}-\frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}}\\
&= 2\cdot\frac{F_{3n+1}F_{3n+3}F_{3n+4}-F_{3n}F_{3n+2}F_{3n+6}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}\\
&= 2\cdot\frac{F_{3n+1}F_{3n+2}F_{3n+3}+F_{3n+1}F_{3n+3}^2}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}\\
&\quad -2\cdot\frac{F_{3n}F_{3n+2}F_{3n+4}+F_{3n}F_{3n+2}F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}\\
&= 2\cdot\frac{F_{3n+2}(F_{3n+1}F_{3n+3}-F_{3n}F_{3n+4})+F_{3n+1}(F_{3n+1}F_{3n+5}-1)}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}\\
&\quad -2\cdot\frac{F_{3n}F_{3n+2}F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}\\
&= 2\cdot\frac{2F_{3n+2}+F_{3n+5}(F_{3n+1}^2-F_{3n}F_{3n+2})-F_{3n+1}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}\\
&= 2\cdot\frac{2F_{3n+2}+F_{3n+5}-F_{3n+1}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}\\
&= 2\cdot\frac{F_{3n}+F_{3n+2}+F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}}\\
&> 0.
\end{aligned}$$

In addition, it is easy to see that $F_{3n}+F_{3n+2}+F_{3n+5}=3F_{3n}+3F_{3n+1}+F_{3n+4}$, thus

$$t_1(n)+t_1(n+1)=\frac{6}{F_{3n+1}F_{3n+4}F_{3n+6}}+\frac{6}{F_{3n}F_{3n+4}F_{3n+6}}+\frac{2}{F_{3n}F_{3n+1}F_{3n+6}},$$

which decreases as n grows. □

Lemma 40. *For all $n\geq 1$, we have*

$$2F_{3n+3}>F_nF_{n+1}F_{n+6}.$$

Proof. Applying Lemma 3 repeatedly, we obtain

$$\begin{aligned}
F_{3n+3} &= F_nF_{2n+2} + F_{n+1}F_{2n+3} \\
&= F_n(F_nF_{n+1} + F_{n+1}F_{n+2}) + F_{n+1}(F_nF_{n+2} + F_{n+1}F_{n+3}) \\
&= F_nF_{n+1}(F_n + 2F_{n+2}) + F_{n+1}^2(F_n + 2F_{n+1}) \\
&= F_nF_{n+1}(F_n + F_{n+1} + 2F_{n+2}) + 2F_{n+1}^3 \\
&> F_nF_{n+1}(3F_{n+2} + 2F_{n+1}) \\
&= F_nF_{n+1}F_{n+5}.
\end{aligned}$$

Therefore,

$$2F_{3n+3} - F_nF_{n+1}F_{n+6} > 2F_nF_{n+1}F_{n+5} - F_nF_{n+1}F_{n+6} = F_nF_{n+1}(2F_{n+5} - F_{n+6}) > 0,$$

which completes the proof. □

Lemma 41. *If $n \geq 1$ and $m \geq 3$, then*

$$t_1(n) + t_1(n+1) + t_1(mn) > 0.$$

Proof. If mn is odd, then the result follows from Lemma 39 and the fact $t_1(mn) > 0$. Now we assume that mn is even. It follows from Lemma 5 that $F_{3mn}F_{3mn+1} = F_{3mn-2}F_{3mn+3} + 2$, from which we get

$$\begin{aligned}
t_1(mn) &= \frac{1}{2F_{3mn}} - \frac{1}{F_{3mn+1}} - \frac{1}{2F_{3mn+3}} \\
&= -\frac{F_{3mn-2}}{2(F_{3mn-2}F_{3mn+3} + 2)} - \frac{1}{2F_{3mn+3}} \\
&> -\frac{F_{3mn-2}}{2F_{3mn-2}F_{3mn+3}} - \frac{1}{2F_{3mn+3}} \\
&= -\frac{1}{F_{3mn+3}} \\
&\geq -\frac{1}{F_{9n+3}}.
\end{aligned}$$

On the other hand, it follows from the proof of Lemma 39 that

$$t_1(n) + t_1(n+1) > \frac{2}{F_{3n}F_{3n+1}F_{3n+6}}.$$

Now we arrive at

$$t_1(n) + t_1(n+1) + t_1(mn) > \frac{2}{F_{3n}F_{3n+1}F_{3n+6}} - \frac{1}{F_{9n+3}} > 0,$$

where the last inequality follows from Lemma 40. □

Lemma 42. *For all $n \geq 2$, we have*

$$F_{2n}F_{2n+1} - F_{n+1}F_{n+4}F_{2n-2} < 0.$$

Proof. It follows from Lemma 4 and Lemma 5 respectively that

$$\begin{aligned} F_{n+2}F_{n+3} - F_nF_{n+1} &= F_{2n+3}, \\ F_{n+1}F_{n+4} - F_{n+2}F_{n+3} &= (-1)^n, \end{aligned}$$

from which we can deduce that

$$F_{n+1}F_{n+4} = F_nF_{n+1} + F_{2n+3} + (-1)^n > F_{2n+3} + 2.$$

Therefore,

$$\begin{aligned} F_{2n}F_{2n+1} - F_{n+1}F_{n+4}F_{2n-2} &< F_{2n}F_{2n+1} - (F_{2n+3} + 2)F_{2n-2} \\ &= (F_{2n}F_{2n+1} - F_{2n-2}F_{2n+3}) - 2F_{2n-2} \\ &= 2 - 2F_{2n-2} \\ &\leq 0, \end{aligned}$$

where the last equality follows from Lemma 5. □

Lemma 43. *If $n \geq 2$ is even, then*

$$\sum_{k=n}^{2n} t_1(k) + \frac{1}{2F_{6n+3}} < 0.$$

Proof. From the proof of Lemma 39 we know that if n is even, then

$$t_1(n) + t_1(n+1) = 2 \cdot \frac{F_{3n} + F_{3n+2} + F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}} < \frac{2F_{3n+6}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}} = \frac{2}{F_{3n}F_{3n+1}F_{3n+4}}.$$

Applying Lemma 39 again and the above inequality, we have

$$\begin{aligned} \sum_{k=n}^{2n} t_1(k) + \frac{1}{2F_{6n+3}} &= \sum_{k=n}^{2n-1} t_1(k) + \left(t_1(2n) + \frac{1}{2F_{6n+3}}\right) \\ &< \frac{2}{F_{3n}F_{3n+1}F_{3n+4}} \cdot \frac{n}{2} + \left(\frac{1}{2F_{6n}} - \frac{1}{F_{6n+1}}\right) \\ &= \frac{n}{F_{3n}F_{3n+1}F_{3n+4}} - \frac{F_{6n-2}}{2F_{6n}F_{6n+1}} \\ &< \frac{1}{2F_{3n+1}F_{3n+4}} - \frac{F_{6n-2}}{2F_{6n}F_{6n+1}} \\ &< 0, \end{aligned}$$

where the last inequality follows from Lemma 42. □

Lemma 44. *For all $n \geq 1$, we have*

$$t_2(n) + t_2(n+1) > 0.$$

Proof. It is easy to see that the result is true when n is odd. So we assume that n is even. It follows from the definition of $t_2(n)$ that

$$\begin{aligned}
t_2(n) + t_2(n+1) &= \left(\frac{1}{2F_{3n}-1} - \frac{1}{2F_{3n+6}-1}\right) - \frac{1}{F_{3n+1}} + \frac{1}{F_{3n+4}} \\
&= \frac{2F_{3n+6} - 2F_{3n}}{(2F_{3n}-1)(2F_{3n+6}-1)} - \frac{1}{F_{3n+1}} + \frac{1}{F_{3n+4}} \\
&> \frac{F_{3n+6} - F_{3n}}{2F_{3n}F_{3n+6}} - \frac{1}{F_{3n+1}} + \frac{1}{F_{3n+4}} \\
&= \frac{1}{2F_{3n}} - \frac{1}{F_{3n+1}} + \frac{1}{F_{3n+4}} - \frac{1}{2F_{3n+6}} \\
&= t_1(n) + t_1(n+1) \\
&> 0,
\end{aligned}$$

where the last inequality follows from the proof of Lemma 39. □

Lemma 45. *If $n \geq 1$ and $m \geq 2$, then*

$$t_2(n) + t_2(n+1) + t_2(mn) > 0.$$

Proof. If mn is odd, then the result follows from Lemma 44 and the fact $t_2(mn) > 0$. Thus we assume that mn is even in the rest. Applying the argument in the proof of Lemma 44 and Lemma 41, we can easily obtain

$$t_2(mn) > t_1(mn) > -\frac{1}{F_{3mn+3}} \geq -\frac{1}{F_{6n+3}}.$$

If n is odd, we have

$$t_2(n) + t_2(n+1) > \frac{1}{F_{3n+1}} - \frac{1}{F_{3n+4}} = \frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}} > \frac{2}{(2F_{3n}-1)F_{3n+4}}.$$

If n is even, then from the proof of Lemma 44 and Lemma 39 we know that

$$\begin{aligned} t_2(n)+t_2(n+1) &> \frac{F_{3n+6}-F_{3n}}{(2F_{3n}-1)F_{3n+6}} - \frac{1}{F_{3n+1}} + \frac{1}{F_{3n+4}} \\ &= \frac{4F_{3n+3}}{(2F_{3n}-1)F_{3n+6}} - \frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}} \\ &= \frac{4(F_{3n+1}F_{3n+3}F_{3n+4} - F_{3n}F_{3n+2}F_{3n+6}) + 2F_{3n+2}F_{3n+6}}{(2F_{3n}-1)F_{3n+1}F_{3n+4}F_{3n+6}} \\ &> \frac{2F_{3n+2}F_{3n+6}}{(2F_{3n}-1)F_{3n+1}F_{3n+4}F_{3n+6}} \\ &> \frac{2}{(2F_{3n}-1)F_{3n+4}}. \end{aligned}$$

Therefore, we always have

$$t_2(n)+t_2(n+1) > \frac{2}{(2F_{3n}-1)F_{3n+4}},$$

from which we get

$$t_2(n)+t_2(n+1)+t_2(mn) > \frac{2}{(2F_{3n}-1)F_{3n+4}} - \frac{1}{F_{6n+3}} > 0,$$

where the last inequality follows from the fact $F_{6n+3} = F_{3n-1}F_{3n+3} + F_{3n}F_{3n+4}$. $\square$

Lemma 46. *If $n \geq 2$ is even, then*

$$t_3(n)+t_3(n+1) < 0.$$

Proof. Applying the analysis in the proof of Lemma 39, we can deduce that

$$\begin{aligned}
t_3(n)+t_3(n+1) &= \left(\frac{1}{2F_{3n}+1}-\frac{1}{2F_{3n+6}+1}\right)-\left(\frac{1}{F_{3n+1}}-\frac{1}{F_{3n+4}}\right)\\
&= \frac{2F_{3n+6}-2F_{3n}}{(2F_{3n}+1)(2F_{3n+6}+1)}-\frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}}\\
&< \frac{F_{3n+6}-F_{3n}}{(2F_{3n}+1)F_{3n+6}}-\frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}}\\
&= \frac{4F_{3n+3}}{(2F_{3n}+1)F_{3n+6}}-\frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}}\\
&= \frac{4(F_{3n+1}F_{3n+3}F_{3n+4}-F_{3n}F_{3n+2}F_{3n+6})-2F_{3n+2}F_{3n+6}}{(2F_{3n}+1)F_{3n+1}F_{3n+4}F_{3n+6}}\\
&= \frac{4(F_{3n}+F_{3n+2}+F_{3n+5})-2F_{3n+2}F_{3n+6}}{(2F_{3n}+1)F_{3n+1}F_{3n+4}F_{3n+6}}\\
&< \frac{4F_{3n+6}-2F_{3n+2}F_{3n+6}}{(2F_{3n}+1)F_{3n+1}F_{3n+4}F_{3n+6}}\\
&< 0.
\end{aligned}$$

The proof is completed. □

Lemma 47. *If $n \geq 1$ is odd, then*

$$t_4(n)+t_4(n+1) > \frac{1}{2F_{9n+3}}.$$

Proof. Applying similar arguments in the proof of Lemma 39, we obtain that if n is odd,

$$t_4(n)+t_4(n+1) = 2\cdot\frac{F_{3n}+F_{3n+2}+F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}} > \frac{2}{F_{3n}F_{3n+1}F_{3n+6}}.$$

It follows from Lemma 40 that

$$\frac{1}{F_{3n}F_{3n+1}F_{3n+6}} > \frac{1}{2F_{9n+3}}.$$

Combining the above two inequalities yields the desired result. □

Lemma 48. *For all $n \geq 2$, we have*

$$\sum_{k=n}^{2n} t_4(k) < \frac{1}{2F_{6n+3}}.$$

Proof. If n is even, it is easy to see that $t_4(n)+t_4(n+1)<0$. Thus,

$$\sum_{k=n}^{2n} t_4(k) = \sum_{k=n}^{2n-1} t_4(k) + t_4(2n) < 0 < \frac{1}{2F_{6n+3}}.$$

If n is odd,

$$t_4(n)+t_4(n+1) = 2\cdot\frac{F_{3n}+F_{3n+2}+F_{3n+5}}{F_{3n}F_{3n+1}F_{3n+4}F_{3n+6}} < \frac{2}{F_{3n}F_{3n+1}F_{3n+4}},$$

which implies that

$$\sum_{k=n}^{2n} t_4(k) < \frac{2}{F_{3n}F_{3n+1}F_{3n+4}}\cdot\frac{n}{2} < \frac{1}{2F_{3n+1}F_{3n+4}} < \frac{1}{2F_{6n+3}},$$

where the last inequality follows from that for $n\geq 1$,

$$F_{3n+1}F_{3n+4} = F_{3n-1}F_{3n+4} + F_{3n}F_{3n+4} > F_{3n-1}F_{3n+3} + F_{3n}F_{3n+4} = F_{6n+3}.$$

This completes the proof. □

Lemma 49. *If $n\geq 1$ is odd, then*

$$t_5(n)+t_5(n+1) > \frac{1}{2F_{6n+3}+1}.$$

Proof. Imitating the proof of Lemma 46, we can easily obtain

$$\begin{aligned} t_5(n)+t_5(n+1) &> \frac{4(F_{3n}+F_{3n+2}+F_{3n+5})+2F_{3n+2}F_{3n+6}}{(2F_{3n}+1)F_{3n+1}F_{3n+4}F_{3n+6}} \\ &> \frac{2F_{3n+2}F_{3n+6}}{(2F_{3n}+1)F_{3n+1}F_{3n+4}F_{3n+6}} \\ &> \frac{1}{F_{3n+1}F_{3n+4}}. \end{aligned}$$

In addition, we have

$$2F_{2n+3} = 2F_{n-1}F_{n+3} + 2F_nF_{n+4} > F_{n+1}F_{n+4}.$$

Combining the above two inequalities together yields the desired result. □

Lemma 50. *For all $n\geq 1$, we have*

$$t_6(n)+t_6(n+1) < 0.$$

Proof. It is clear that the result holds if n is even. Now we assume that n is odd. Applying the telescoping technique in the proof of Lemma 45 and the similar analysis in the proof of Lemma 39, we obtain

$$\begin{aligned} t_6(n)+t_6(n+1) &< -\left\{\frac{4F_{3n+3}}{(2F_{3n}-1)F_{3n+6}}-\frac{2F_{3n+2}}{F_{3n+1}F_{3n+4}}\right\} \\ &= -\frac{4(F_{3n+1}F_{3n+3}F_{3n+4}-F_{3n}F_{3n+2}F_{3n+6})+2F_{3n+2}F_{3n+6}}{(2F_{3n}-1)F_{3n+1}F_{3n+4}F_{3n+6}} \\ &= -\frac{-4(F_{3n}+F_{3n+2}+F_{3n+5})+2F_{3n+2}F_{3n+6}}{(2F_{3n}-1)F_{3n+1}F_{3n+4}F_{3n+6}} \\ &= \frac{4(F_{3n}+F_{3n+2}+F_{3n+5})-2F_{3n+2}F_{3n+6}}{(2F_{3n}-1)F_{3n+1}F_{3n+4}F_{3n+6}} \\ &< \frac{4F_{3n+6}-2F_{3n+2}F_{3n+6}}{(2F_{3n}-1)F_{3n+1}F_{3n+4}F_{3n+6}} \\ &< 0, \end{aligned}$$

which completes the proof. $\square$

Lemma 51. *If $n \geq 1$ and $m \geq 2$, we have*

$$t_6(n)+t_6(n+1)+t_6(mn)<0.$$

Proof. If mn is even, then the result follows from Lemma 50 and the fact $t_6(mn)<0$, so we assume that mn is odd in the rest. Now we have

$$\begin{aligned} t_6(mn) &= \frac{-1}{2F_{3mn}-1}+\frac{1}{F_{3mn+1}}+\frac{1}{2F_{3mn+3}-1} \\ &= \frac{-4F_{3mn+1}}{(2F_{3mn}-1)(2F_{3mn+3}-1)}+\frac{1}{F_{3mn+1}} \\ &< \frac{-2F_{3mn+1}}{(2F_{3mn}-1)F_{3mn+3}}+\frac{1}{F_{3mn+1}} \\ &= \frac{-2F_{3mn+1}^2+2F_{3mn}F_{3mn+3}-F_{3mn+3}}{(2F_{3mn}-1)F_{3mn+1}F_{3mn+3}} \\ &= \frac{-2(F_{3mn+1}^2-F_{3mn}F_{3mn+2})+2F_{3mn}F_{3mn+1}-F_{3mn+1}-F_{3mn+2}}{(2F_{3mn}-1)F_{3mn+1}F_{3mn+3}} \\ &= \frac{2+(2F_{3mn}-1)F_{3mn+1}-F_{3mn+2}}{(2F_{3mn}-1)F_{3mn+1}F_{3mn+3}} \\ &< \frac{(2F_{3mn}-1)F_{3mn+1}}{(2F_{3mn}-1)F_{3mn+1}F_{3mn+3}} \\ &= \frac{1}{F_{3mn+3}}. \end{aligned}$$

Since mn is odd, we must have that n is odd and $m \geq 3$. Therefore,

$$t_6(mn) < \frac{1}{F_{9n+3}}.$$

It follows from the proof of Lemma 50 that if n is odd,

$$t_6(n) + t_6(n+1) < \frac{4 - 2F_{3n+2}}{(2F_{3n} - 1)F_{3n+1}F_{3n+4}} < \frac{2 - F_{3n+2}}{F_{3n}F_{3n+1}F_{3n+4}} < -\frac{1}{F_{3n}F_{3n+1}F_{3n+4}}.$$

Now we arrive at

$$t_6(n) + t_6(n+1) + t_6(mn) < \frac{1}{F_{9n+3}} - \frac{1}{F_{3n}F_{3n+1}F_{3n+4}}.$$

Employing Lemma 3, we easily see that $F_{3n+3} > F_{2n}F_{n+4}$ and $F_{2n} > F_nF_{n+1}$, which implies

$$F_{9n+3} > F_{3n}F_{3n+1}F_{3n+4}.$$

Therefore,

$$t_6(n) + t_6(n+1) + t_6(mn) < \frac{1}{F_{9n+3}} - \frac{1}{F_{3n}F_{3n+1}F_{3n+4}} < 0.$$

The proof is completed. □

Proof of Theorem 35. We first consider the case where n is even. Applying Lemma 43, we have

$$\sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} = \frac{1}{2F_{3n}} - \frac{1}{2F_{6n+3}} - \sum_{k=n}^{2n} t_1(k) > \frac{1}{2F_{3n}}.$$

It follows from Lemma 44 and Lemma 45 that

$$\sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} = \frac{1}{2F_{3n} - 1} - \frac{1}{2F_{6n+3} - 1} - (t_2(n) + t_2(n+1) + t_2(2n)) - \sum_{k=n+2}^{2n-1} t_2(k) < \frac{1}{2F_{3n} - 1}.$$

Therefore, we have

$$\frac{1}{2F_{3n}} < \sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} < \frac{1}{2F_{3n} - 1},$$

which means that the result holds when n is even.

We now turn to consider the case where n is odd. From Lemma 48, we know that

$$\sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} = \frac{-1}{2F_{3n}} + \frac{1}{2F_{6n+3}} - \sum_{k=n}^{2n} t_4(k) > \frac{-1}{2F_{3n}}.$$

With the help of Lemma 49, it is easy to see that

$$\sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} = \frac{-1}{2F_{3n}+1} - \left(t_5(n) + t_5(n+1) - \frac{1}{2F_{6n+3}+1}\right) - \sum_{k=n+2}^{2n} t_5(k) < \frac{-1}{2F_{3n}+1}.$$

Thus, we obtain

$$\frac{-1}{2F_{3n}} < \sum_{k=n}^{2n} \frac{(-1)^k}{F_{3k+1}} < \frac{-1}{2F_{3n}+1},$$

which yields the desired identity. $\square$

Proof of Theorem 36. We first consider the case where n is even. Applying Lemma 39 and Lemma 41, we see

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} = \frac{1}{2F_{3n}} - \frac{1}{2F_{3mn+3}} - \sum_{k=n+2}^{mn-1} t_1(k) - (t_1(n) + t_1(n+1) + t_1(mn)) < \frac{1}{2F_{3n}}.$$

It follows from Lemma 46 that

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} &= \frac{1}{2F_{3n}+1} - \sum_{k=n}^{mn-1} t_3(k) - \left(t_3(mn) + \frac{1}{2F_{3mn+3}+1}\right) \\
&= \frac{1}{2F_{3n}+1} - \sum_{k=n}^{mn-1} t_3(k) - \left(\frac{1}{2F_{3mn}+1} - \frac{1}{F_{3mn+1}}\right) \\
&> \frac{1}{2F_{3n}+1}.
\end{aligned}$$

Therefore, we obtain

$$\frac{1}{2F_{3n}+1} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} < \frac{1}{2F_{3n}},$$

which shows that the statement is true when n is even.

Now we turn to consider the case where n is odd. Lemma 47 tells us that

$$t_4(n) + t_4(n+1) > 0.$$

Hence if mn is odd,

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} &= \frac{-1}{2F_{3n}} - \sum_{k=n}^{mn-1} t_4(k) - \left(t_4(mn) - \frac{1}{2F_{3mn+3}}\right) \\
&= \frac{-1}{2F_{3n}} - \sum_{k=n}^{mn-1} t_4(k) - \left(\frac{1}{F_{3mn+1}} - \frac{1}{2F_{3mn}}\right) \\
&< \frac{-1}{2F_{3n}}.
\end{aligned}$$

And it follows from Lemma 50 and Lemma 51 that

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} &= \frac{-1}{2F_{3n}-1} + \frac{1}{2F_{3mn+3}-1} - \sum_{k=n+2}^{mn-1} t_6(k) - (t_6(n) + t_6(n+1) + t_6(mn)) \\ &> \frac{-1}{2F_{3n}-1}.\end{aligned}$$

Therefore, we have

$$\frac{-1}{2F_{3n}-1} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} < \frac{-1}{2F_{3n}}.$$

If mn is even, then Lemma 47 implies that

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} &= \frac{-1}{2F_{3n}} - \sum_{k=n+2}^{mn} t_4(k) - \left(t_4(n) + t_4(n+1) - \frac{1}{2F_{3mn+3}}\right) \\ &< \frac{-1}{2F_{3n}} - \sum_{k=n+2}^{mn} t_4(k) - \left(t_4(n) + t_4(n+1) - \frac{1}{2F_{9n+3}}\right) \\ &< \frac{-1}{2F_{3n}},\end{aligned}$$

and from Lemma 50 we obtain

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} = \frac{-1}{2F_{3n}-1} + \frac{1}{2F_{3mn+3}-1} - \sum_{k=n}^{mn} t_6(k) > \frac{-1}{2F_{3n}-1}.$$

Hence, we also have

$$\frac{-1}{2F_{3n}-1} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{3k+1}} < \frac{-1}{2F_{3n}},$$

which yields the desired identity. □

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