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On the Sum of the Reciprocals of the Middle Prime Factors of an Integer

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Abstract

We consider the arithmetical function $p^{(\beta)}(n) := p_{\max(1,\lfloor \beta k \rfloor)}$ for a given fixed number $\beta \in (0,1)$, where $p_1 < p_2 < \cdots < p_k$ are the prime factors of n. We provide an estimate for the sum of the reciprocals of $p^{(\beta)}(n)$ for $n \leq x$, which improves and generalizes an earlier result of De Koninck and Luca.

1 Introduction

Given an integer $n \geq 2$, let $P(n)$ denote its largest prime factor and let $P(1) = 1$. At the end of the 1970's and early 1980's, many papers focused on estimating the global behavior of the sum of the reciprocals of $P(n)$ for $n \leq x$. For the highlights, see the papers of Erdős and Ivić [7] and [8]. The best estimate was obtained in 1986 by Erdős, Ivić, and Pomerance [5, Thm. 1], as they proved that

$$\sum_{n \leq x} \frac{1}{P(n)} = x \int_2^x \rho\left(\frac{\log x}{\log t}\right) t^{-2} \mathrm{d}t \left(1 + O\left(\sqrt{\frac{\log_2 x}{\log x}}\right)\right) \qquad (x \to \infty),$$

where $\rho(u)$ is the Dickman function and $\log_k x$ denotes the k-th iterate of log evaluated at x. Here and in what follows, we shall assume that the input x in such an expression is sufficiently large so that the iterated logarithms are real and positive. For any integer $k \geq 2$, letting $P_k(n)$ stand for the k-th largest prime factor with multiplicity of the integer n, De Koninck [2, Thm. 2] proved that there exists a constant c_k such that

$$\sum_{\substack{n\le x\\ \Omega(n)\ge k}} \frac{1}{P_k(n)} = c_k \frac{x\left(\log_2 x\right)^{k-2}}{\log x}\left(1+O\left(\frac{1}{\log_2 x}\right)\right) \qquad (x\to\infty),$$

where $\Omega(n)$ stands for the number of prime factors of n counting multiplicities.

During the 1984 Oberwolfach Conference on Analytic Number Theory, Erdős asked De Koninck if he had thought of estimating the sum of the reciprocals of the middle prime factors of the positive integers $n\le x$. Given an integer $n\ge 2$, write it as $n=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$, where $p_1<p_2<\cdots<p_k$ are its distinct prime factors, and the a_i are positive integers. Denote the number of distinct prime factors of n by $\omega(n)$, so that $\omega(n)=k$, and let $p^{(m)}(n):=p_{\left\lfloor\frac{\omega(n)+1}{2}\right\rfloor}$ denote its middle prime factor. De Koninck and Luca [4] proved that, as $x\to\infty$,

$$\sum_{1<n\le x}\frac{1}{p^{(m)}(n)} = \frac{x}{\log x}\exp\left(\left(\sqrt{2}+o(1)\right)\sqrt{\log_2 x\log_3 x}\right). \tag{1}$$

Expanding the main ideas of the proof of the upper bound given by De Koninck and Luca [4], our goal here is to improve and generalize equation (1). For an integer $n\ge 2$ and a fixed real number $\beta\in(0,1)$, we denote by $p^{(\beta)}(n)=p_{\max(1,\lfloor\beta k\rfloor)}$ the *β-positioned prime factor* of n, where $p_1<p_2<\cdots<p_k$ are its prime factors. As De Koninck and Luca did with the middle prime factor, we obtain an estimate for the sum of the reciprocals of the β-positioned prime factors of the integers $n\le x$.

Theorem 1. *There exist four constants α_1, α_2, α_3, and α_4 such that, as $x\to\infty$,*

$$\sum_{1<n\le x}\frac{1}{p^{(\beta)}(n)} = \frac{x}{\log x}\exp\left(\frac{\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}}{\beta^\beta\left(1-\beta\right)^{1-2\beta}}\left(G\left(x,\beta\right)+O\left(\frac{1}{\log_3^2 x}\right)\right)\right), \tag{2}$$

where $G(x,\beta)=1+\alpha_1\frac{\log_4 x}{\log_3 x}+\alpha_2\frac{1}{\log_3 x}+\alpha_3\frac{\log_4^2 x}{\log_3^2 x}+\alpha_4\frac{\log_4 x}{\log_3^2 x}$. In particular,

$$\alpha_1=\frac{-\beta\left(2-\beta\right)}{1-\beta}, \qquad \alpha_3=\frac{2-\beta}{2}\alpha_1,$$

$$\alpha_2=\beta\left(\log\beta-\frac{3-2\beta}{1-\beta}\log\left(1-\beta\right)-\frac{1}{1-\beta}\right), \qquad \alpha_4=\left(2-\beta\right)\alpha_2-\alpha_1+\frac{\beta}{1-\beta}.$$

By setting $\beta=1/2$, the following corollary shows that (2) is an improvement over equation (1).

Corollary 2. *As $x \to \infty$,*

$$\sum_{1<n\le x} \frac{1}{p^{(m)}(n)} = \frac{x}{\log x} \exp\left(\sqrt{2\log_2 x \log_3 x}\left(1 + c_1\frac{\log_4 x}{\log_3 x} + \frac{c_2}{\log_3 x} + c_3\frac{\log_4^2 x}{\log_3^2 x} + c_4\frac{\log_4 x}{\log_3^2 x}\right)\right) \times \exp\left(O\left(\sqrt{\frac{\log_2 x}{\log_3^3 x}}\right)\right),$$

where $c_1 = \frac{-3}{2}$, $c_2 = \frac{3}{2}\log 2 - 1$, $c_3 = \frac{-9}{8}$, and $c_4 = 1 + \frac{9}{4}\log 2$.

2 Preliminary results

Throughout this paper, p and q always stand for prime numbers, $\beta \in (0,1)$ is a fixed real number, and x is a large number. Our goal is to estimate

$$\sum_{1<n\le x} \frac{1}{p^{(\beta)}(n)} = \sum_{p\le x} \frac{1}{p}\#\left\{n \le x : p^{(\beta)}(n) = p\right\} = \sum_{p\le x}\frac{1}{p}\sum_{k\ge 1} \#\mathcal{N}_{p,k}(x), \tag{3}$$

where $\mathcal{N}_{p,k}(x) := \left\{n \le x : p^{(\beta)}(n) = p, \omega(n) = k\right\}$. Note that, given any x, the sum over k is finite since the integer k must satisfy $k \le \left\lfloor \frac{\log x}{\log 2} \right\rfloor$. Moreover, it is possible that $\#\mathcal{N}_{p,k}(x) = 0$ for some primes p and integers k. Hence, the integers k and prime numbers p are dependent. We shall see that the main contribution to equation (3) is reached when the prime numbers p and the integers k are in particular sets. Let

$$\begin{aligned}
\mathcal{N}_1(x) &:= \{n \le x : \Omega(n) > 10\log_2 x\};\\
\mathcal{N}_2(x) &:= \{n \le x : p_\beta(n) > \log x\};\\
\mathcal{N}_3(x) &:= \{n \le x : \omega(n) \in \{1, 2, \ldots, M\}\};\\
\mathcal{N}_4(x) &:= \{n \le x\} \setminus (\mathcal{N}_1(x) \cup \mathcal{N}_2(x) \cup \mathcal{N}_3(x)),
\end{aligned}$$

where $M := \left\lceil \max\left(\frac{2}{\beta}, \frac{2}{1-\beta}\right)\right\rceil$. We first show that

$$\sum_{n\in\mathcal{N}_i(x)} \frac{1}{p^{(\beta)}(n)} \ll \frac{x(\log_2 x)^{M-1}}{\log x} \qquad \text{for } i = 1, 2, 3. \tag{4}$$

By [9, Lemma 13], it follows that

$$\sum_{n\in\mathcal{N}_1(x)} \frac{1}{p^{(\beta)}(n)} \ll \#\mathcal{N}_1(x) = \sum_{\substack{n\le x\\ \Omega(n)>10\log_2 x}} 1 \ll x\log x\,\frac{10\log_2 x}{2^{10\log_2 x}} \ll \frac{x}{(\log x)^5}. \tag{5}$$

For the integers $n \in \mathcal{N}_2(x)$, we have

$$\sum_{n\in\mathcal{N}_2(x)} \frac{1}{p^{(\beta)}(n)} \leq \sum_{n\leq x} \frac{1}{\log x} \leq \frac{x}{\log x}. \tag{6}$$

Finally,

$$\sum_{n\in\mathcal{N}_3(x)} \frac{1}{p^{(\beta)}(n)} \ll \#\mathcal{N}_3(x) \ll \frac{x\left(\log_2 x\right)^{M-1}}{\log x}, \tag{7}$$

by the Hardy-Ramanujan inequality (see Lemma 4). Hence, combining the bounds (5), (6) and (7), the upper bound (4) follows.

For each integer $n \in \mathcal{N}_{p,k}(x) \cap \mathcal{N}_4(x)$, we write $\omega(n) = k = \frac{1}{\beta}k_0 + \delta$, where $k_0 = \lfloor \beta k \rfloor$, so that $\delta \in \left[0, \frac{1}{\beta}\right)$ is fixed. Note that $k \in [M+1, 10\log_2 x]$ and that $p \in [2, \log x]$. Let us write $n = ap^\alpha b$, where $a \geq 2$, $P(a) < p$, $\omega(a) = k_0 - 1$, $1 \leq \alpha \leq 10\log_2 x$, $p(b) > p$, and $\omega(b) = \left(\frac{1}{\beta} - 1\right)k_0 + \delta$, where $P(n)$ and $p(n)$ denote respectively the largest and the smallest prime factors of n. It follows from the bounds (5) and (7) that

$$\sum_{n\in\mathcal{N}_4(x)} \frac{1}{p^{(\beta)}(n)} = \sum_{p\in[3,\log x]} \frac{1}{p} \sum_{k\in[M+1,10\log_2 x]} \#\left(\mathcal{N}_{p,k}(x) \cap \mathcal{N}_4(x)\right) + O\left(\frac{x\left(\log_2 x\right)^{M-1}}{\log x}\right), \tag{8}$$

where $p \geq 3$ comes from the fact that $a \geq 2$. Note that the β-positioned prime factors of some integers n that are in the sets $\mathcal{N}_i(x)$ for $i = 1, 2, 3$ are counted multiple times on the right-hand side of (8), but that their contribution is taken into consideration by the error term of equation (8). The objective is now to estimate the main term of equation (8). For this, three preliminary results will be useful.

Lemma 3 (Alladi [1], Theorem 6)**.** *Given a positive integer λ, let*

$$\omega_\lambda(x,y) := \#\left\{n \leq x : p(n) \geq y, \omega(n) = \lambda\right\}$$

and

$$g\left(s,y,z\right) := \prod_p \left(1 + \frac{z}{p^s - 1}\right)\left(1 - \frac{1}{p^s}\right)^z \prod_{p<y}\left(1 + \frac{z}{p^s-1}\right)^{-1}$$

for each $z \in \mathbb{C}$. Then, for any $r > 0$, in the range $2 \leq y \leq \exp\left(\left(\log x\right)^{2/5}\right)$, $\Re(s) > \frac{1}{2}$ and $\lambda \leq r\log_2 x$, we have

$$\omega_\lambda\left(x,y\right) = \frac{x}{\log x}\frac{g\left(1,y,\mu\right)}{\Gamma\left(1+\mu\right)}\frac{\left(\log_2 x\right)^{\lambda-1}}{(\lambda-1)!} + O\left(\frac{x\left(\log_2 x\right)^{\lambda-1}\left(\log y\right)^{-\mu}\left(\log_2 y\right)^2\lambda}{(\lambda-1)!\log x\left(\log_2 x\right)^2}\right) \quad (x \to \infty),$$

where $\mu := \frac{\lambda-1}{\log_2 x}$.

Lemma 3 and the following lemmas are used to estimate the sum over b for the integers $n = ap^\alpha b$.

Lemma 4 (Hardy-Ramanujan inequality)**.** *For any integer $\lambda \geq 1$, define*

$$\Pi_\lambda(x) = \#\left\{n \leq x : \omega(n) = \lambda\right\}.$$

There exist positive constants c and x_0 such that, uniformly for $1 \leq \lambda \leq 10 \log_2 x$,

$$\Pi_\lambda(x) \leq c \frac{x}{\log x} \frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}$$

for all $x > x_0$.

The next lemma follows from the proof of Theorem 1 in Erdős and Tenenbaum [6]. It will be used to obtain an estimate for the sum over a of the integers $n = ap^\alpha b \leq x$.

Lemma 5 (Erdős and Tenenbaum)**.** *Let $\epsilon > 0$. For every prime number $p \geq 5$, define the function $\rho = \rho\left(k_0 - 1, p\right)$ as the unique solution to $\sum\limits_{q<p} \frac{\rho}{q-1+\rho} = k_0 - 1$ for $1 \leq k_0 - 1 \leq \pi(p) - 2$, and define the functions*

$$w(t) = \begin{cases} \Gamma\left(t+1\right) t^{-t} e^t, & \text{if } t > 0; \\ 1, & \text{if } t = 0; \end{cases}$$

and

$$F\left(z, p\right) = \prod_{q<p} \left(1 + \frac{z}{q-1}\right)$$

for $z \in \mathbb{C}$. Moreover, let $\mathcal{G} := \left\{a \in \mathbb{N} : \omega(a) = k_0 - 1, P(a) < p\right\}$. Then, uniformly for $1 \leq k_0 \leq p^{1-\epsilon}$, we have

$$\sum_{a \in \mathcal{G}} \frac{1}{a} = \frac{F\left(\rho, p\right)}{\rho^{k_0-1} w\left(k_0 - 1\right)} \left(1 + O\left(R^{-1}\right)\right),$$

where $R = \log\left(\frac{\log p}{\log(k_0+1)}\right)\left(1 + \log^+\left(\frac{k_0}{\log\left(\frac{\log p}{\log(k_0+1)}\right)}\right)\right)$ and $\log^+ x := \max\left(0, \log x\right)$.

In the following lemmas, we study three functions that are used to estimate the main term of the right-hand side of equation (8).

Lemma 6. *Let $B > 1$ and define the function $f : (0, \infty) \to (0, \infty)$ by $f(t) = \left(\frac{eB}{t}\right)^t$. The function f is concave and reaches its maximum when $t = B$.*

Definition 7. For $x > -\frac{1}{e}$, we define the *Lambert-W function* as the inverse of the real-valued function $h(y) = ye^y$, which is defined for $y > -1$, so that $W(x)e^{W(x)} = x$.

In particular, one can easily show that the Lambert-W function goes to ∞ as $x \to \infty$, that it is strictly increasing and that it goes to 0 as $x \to 0$. Using these facts, the following lemmas can be proved.

Lemma 8. *As $x \to \infty$,*

$$W(x) = \log x - \log_2 x + \frac{\log_2 x}{\log x} + O\left(\left(\frac{\log_2 x}{\log x}\right)^2\right).$$

Lemma 9. *As $x \to 0$,*

$$W(x) = x + O\left(x^2\right).$$

The third function will be useful in the evaluation of some sums.

Lemma 10. *Let $D > 0$ and $C \in \mathbb{R}$, and define the function $g : \left(e^C, \infty\right) \to (0, \infty)$ by $g(t) = \exp\left(\frac{D}{\beta}(\log t - C)^\beta - t\right)$. The function g is concave and reaches its maximum when*

$$\log t = \log t_0 := (1-\beta)\, W\left(\frac{D^{\frac{1}{1-\beta}}}{1-\beta} \exp\left(\frac{-C}{1-\beta}\right)\right) + C,$$

where W stands for the Lambert-W function.

Proof. Clearly,

$$\begin{aligned}
g'(t) = 0 \iff & \frac{D}{t\,(\log t - C)^{1-\beta}} - 1 = 0 \iff D = t\,(\log t - C)^{1-\beta} \\
\iff & D^{\frac{1}{1-\beta}} = t^{\frac{1}{1-\beta}}\,(\log t - C) \iff D^{\frac{1}{1-\beta}} = \exp\left(\frac{\log t}{1-\beta}\right)(\log t - C) \\
\iff & \frac{D^{\frac{1}{1-\beta}} \exp\left(\frac{-C}{1-\beta}\right)}{1-\beta} = \exp\left(\frac{\log t - C}{1-\beta}\right)\left(\frac{\log t - C}{1-\beta}\right).
\end{aligned}$$

Hence, by the definition of the Lambert-W function, it follows that this last equation is equivalent to $\frac{\log t - C}{1-\beta} = W\left(\frac{D^{\frac{1}{1-\beta}}}{1-\beta} \exp\left(\frac{-C}{1-\beta}\right)\right)$. Since $t > e^C$, this equation always has a unique solution. $\square$

3 Estimation of the main term

We will now consider the primes p belonging to the interval

$$I := \left[\exp\left(\left(\frac{\log_2 x}{(\log_3 x)^{\frac{4}{1-\beta}} \log_4 x}\right)^{1-\beta}\right), \exp\left((\log_2 x)^{1-\beta} \log_3 x\right)\right] \tag{9}$$

and the positive integers k belonging to the interval

$$J := \left[\frac{1}{4}\frac{(\log_2 x)^{1-\beta}(\log_3 x)^{\beta}}{\beta^{\beta}(1-\beta)^{1-3\beta}}, 2e\frac{(\log_2 x)^{1-\beta}(\log_3 x)^{\beta}}{\beta^{\beta}(1-\beta)^{1-\beta}}\right]. \tag{10}$$

We will show that the main contribution to the right-hand side of equation (8) comes from the primes $p \in I$ and integers $k \in J$. Note that $\#\mathcal{N}_{p,k}(x) \neq 0$ for any prime number $p \in I$ and integer $k \in J$. Let

$$\mathcal{A} = \mathcal{A}(k,p) := \{a \in \mathbb{N} : \omega(a) = k_0 - 1, P(a) < p \text{ and } \Omega(a) \leq 10\log_2 x\} \tag{11}$$

and

$$\mathcal{B} = \mathcal{B}(k,p) := \left\{b \in \mathbb{N} : \omega(b) = \left(\frac{1}{\beta} - 1\right)k_0 + \delta \text{ and } p(b) > p\right\}, \tag{12}$$

so that $n = ap^{\alpha}b \in \mathcal{N}_4(x)$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Hence, for $p \in I$ and $k \in J$, we have from the upper bound (5) that

$$\#(\mathcal{N}_{p,k}(x) \cap \mathcal{N}_4(x)) = \sum_{\substack{a \leq x \\ a \in \mathcal{A}}} \sum_{\alpha=1}^{\lfloor 10\log_2 x\rfloor} \sum_{\substack{b \leq \frac{x}{ap^{\alpha}} \\ b \in \mathcal{B}}} 1 + O\left(\frac{x}{(\log x)^5}\right). \tag{13}$$

Thus, it follows from equation (8) that

$$\sum_{\substack{n \in \mathcal{N}_4(x) \\ p^{(\beta)}(n) \in I \\ \omega(n) \in J}} \frac{1}{p^{(\beta)}(n)} = \sum_{p \in I} \frac{1}{p} \sum_{k \in J} \sum_{\substack{a \leq x \\ a \in \mathcal{A}}} \sum_{\alpha=1}^{\lfloor 10\log_2 x\rfloor} \sum_{\substack{b \leq \frac{x}{ap^{\alpha}} \\ b \in \mathcal{B}}} 1 + O\left(\frac{x(\log_2 x)^{M-1}}{\log x}\right). \tag{14}$$

It remains to estimate the sums in the right-hand side of (14). Let

$$\#\mathcal{N}'_{p,k}(x) := \sum_{\substack{a \leq x \\ a \in \mathcal{A}}} \sum_{\alpha=1}^{\lfloor 10\log_2 x\rfloor} \sum_{\substack{b \leq \frac{x}{ap^{\alpha}} \\ b \in \mathcal{B}}} 1.$$

Since $ap^{\alpha} = x^{o(1)}$, we obtain from Lemma 3 that

$$\sum_{\substack{b \leq \frac{x}{ap^{\alpha}} \\ b \in \mathcal{B}}} 1 = \upsilon_{\lambda}\left(\frac{x}{ap^{\alpha}}, p+2\right) = \frac{x}{ap^{\alpha}\log x}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}(1+o(1)) \qquad (x \to \infty),$$

where $\lambda = \left(\frac{1}{\beta} - 1\right)k_0 + \delta \geq 2$, because $k \geq M+1$. Hence, as $x \to \infty$,

$$\#\mathcal{N}'_{p,k}(x) \sim \frac{x}{p\log x}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}\sum_{\substack{a \leq x \\ a \in \mathcal{A}}} \frac{1}{a} \sum_{\alpha=1}^{\lfloor 10\log_2 x\rfloor} \frac{1}{p^{\alpha-1}} \sim \frac{x}{p\log x}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}\sum_{a \in \mathcal{A}} \frac{1}{a}, \tag{15}$$

because $a = x^{o(1)}$ for all $a \in \mathcal{A}$. Moreover, we have $\sum\limits_{a\in\mathcal{A}} \frac{1}{a} \sim \sum\limits_{a\in\mathcal{G}} \frac{1}{a}$ as $x \to \infty$, where the set $\mathcal{G}$ is defined as in Lemma 5. Indeed, we have

$$\sum_{a\in\mathcal{G}} \frac{1}{a} = \sum_{a\in\mathcal{A}} \frac{1}{a} + \sum_{\substack{a\in\mathcal{G}\\ \Omega(a)>10\log_2 x}} \frac{1}{a},$$

and Rankin's method (see, for example, [4, Chap. 9]) shows that $\sum\limits_{\substack{n>x\\ P(n)<y}} \frac{1}{n} \ll e^{-u/2}\log y$ for any $x \geq y \geq 2$, where $u = \frac{\log x}{\log y}$. Hence, it follows that

$$\sum_{\substack{a\in\mathcal{G}\\ \Omega(a)>10\log_2 x}} \frac{1}{a} \leq \sum_{\substack{a>2^{10\log_2 x}\\ P(a)<p}} \frac{1}{a} \ll \exp\left(-\frac{5\log_2 x\log 2}{\log p}\right)\log p.$$

We get from equation (15) that

$$\#\mathcal{N}'_{p,k}(x) \sim \frac{x}{p\log x}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}\sum_{a\in\mathcal{G}}\frac{1}{a}$$

as $x \to \infty$. By an explicit evaluation of $\sum\limits_{a\in\mathcal{G}} \frac{1}{a}$ using Lemma 5, we obtain

$$\#\mathcal{N}'_{p,k}(x) = \frac{x}{p\log x}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}\left(\frac{eA}{k_0}\right)^{k_0}\mathcal{R}, \tag{16}$$

where $A = A(x,p) = \log_2 p - \log_4 x - \log(1-\beta) + \frac{\log_4 x}{\log_3 x}$ and

$$\mathcal{R} = \mathcal{R}(x) = \exp\left(O\left(\frac{(\log_2 x)^{1-\beta}}{(\log_3 x)^{2-\beta}}\right)\right).$$

Indeed, following the proof of Erdős and Tenenbaum [6, Lemma 1], for any prime number $p \in I$ and integer $k \in J$, we have

$$\begin{aligned}\frac{1}{\rho} &= \frac{\log_2 p - \log_2 k_0 + \frac{\log_3 p}{\log k_0}}{k_0} + O\left(\frac{1}{k_0\log k_0}\right)\\ &= \frac{\log_2 p - \log_4 x - \log(1-\beta) + \frac{\log_4 x}{\log_3 x}}{k_0} + O\left(\frac{1}{k_0\log k_0}\right),\end{aligned} \tag{17}$$

and, by Erdős and Tenenbaum [6, Lemma 2], we obtain $F(\rho,p) = \exp\left(k_0 + O\left(\frac{\rho}{\log\rho}\right)\right)$.

Using the Stirling's formula to estimate $(\lambda-1)!$, it follows from equation (16) and the definition of λ in terms of k_0 that

$$\#\mathcal{N}'_{p,k}(x) = \frac{x}{p\log x}\left(\frac{e\left(\log_2 x\right)^{1-\beta} A^{\beta}}{\left(\frac{1}{\beta}-1\right)k_0}\right)^{\frac{k_0}{\beta}}\mathcal{R},$$

and, since $\frac{k_0}{\beta} = k + O(1)$, we obtain

$$\#\mathcal{N}'_{p,k}(x) = \frac{x}{p\log x}\left(\frac{eB}{k}\right)^{k}\mathcal{R}, \tag{18}$$

where $B = B(x,p) = \frac{(\log_2 x)^{1-\beta}A^{\beta}}{\beta\left(\frac{1}{\beta}-1\right)^{1-\beta}}$. Hence, combining equations (14) and (18), we have

$$\sum_{\substack{n\in\mathcal{N}_4(x)\\ p^{(\beta)}(n)\in I\\ \omega(n)\in J}} \frac{1}{p^{(\beta)}(n)} = \frac{x}{\log x}\mathcal{R}\sum_{p\in I}\frac{1}{p^2}\sum_{k\in J}\left(\frac{eB}{k}\right)^{k} + O\left(\frac{x\log_2 x\log_3 x}{\log x}\right).$$

We can conclude from Lemma 6 that

$$\sum_{\substack{n\in\mathcal{N}_4(x)\\ p^{(\beta)}(n)\in I\\ \omega(n)\in J}} \frac{1}{p^{(\beta)}(n)} = \frac{x}{\log x}\mathcal{R}\sum_{p\in I}\frac{1}{p^2}\exp\left(B\right) + O\left(\frac{x\log_2 x\log_3 x}{\log x}\right). \tag{19}$$

To estimate the sum over p, we note that

$$\sum_{p\in I}\frac{1}{p^2}\exp\left(B\right) = \sum_{p\in I}\frac{1}{p}\exp\left(B-\log p\right) = \sum_{p\in I}\frac{1}{p}g\left(\log p\right),$$

where g is the same function as the one defined in Lemma 10, $C = \log_4 x + \log\left(1-\beta\right) - \frac{\log_4 x}{\log_3 x}$ and $D = \left(\frac{\beta\log_2 x}{1-\beta}\right)^{1-\beta}$. Moreover, from this same lemma, we have

$$\sum_{p\in I}\frac{1}{p}g\left(\log p\right) \le g\left(t_0\right)\sum_{p\in I}\frac{1}{p} \ll g\left(t_0\right)\log_4 x, \tag{20}$$

where, in particular,

$$t_0 = \left(\frac{\beta}{\left(1-\beta\right)^2}\frac{\log_2 x}{\log_3 x}\right)^{1-\beta}\left(1+\left(2-\beta\right)\frac{\log_4 x}{\log_3 x} + O\left(\frac{1}{\log_3 x}\right)\right). \tag{21}$$

Lemma 10 also provides a lower bound for the sum over p. Indeed, let p_0 be the largest prime number in I such that $\log p_0 \leq t_0$. In particular, by Bertrand's postulate, one can conclude that the prime number p_0 satisfies $p_0 \in \left(\frac{1}{2}e^{t_0}, e^{t_0}\right]$. Thus, for every prime number $p \in \left(\frac{1}{\log_2 x}p_0, p_0 \log_2 x\right] \subset I$, there exist positive constants c_1, c_2, and c_3 such that

$$g\left(\log p\right) \geq g\left(\log p_0 - c_1\left(\log_3 x\right)\right) \geq g\left(t_0 - c_2\left(\log_3 x\right)\right) \geq g\left(t_0\right)\exp\left(-c_3\left(\left(\log_3 x\right)^{2-\beta}\right)\right)$$

for every $x > e^{e^e}$, so that

$$\begin{aligned}\sum_{p\in I}\frac{1}{p}g\left(\log p\right) &\geq g\left(t_0\right)\exp\left(-c_3\left(\log_3 x\right)^{2-\beta}\right)\sum_{p\in\left[\frac{p_0}{\log_2 x}, p_0\log_2 x\right]}\frac{1}{p}\\ &\gg g\left(t_0\right)\exp\left(-c_3\left(\log_3 x\right)^{2-\beta}\right)\left(\frac{\log_3 x}{\log_2 x}\right)^{1-\beta}\\ &\geq g\left(t_0\right)\exp\left(\left(\log_3 x\right)^{2-\beta}\right). \end{aligned} \tag{22}$$

Hence, from equation (19), the upper bound (20), and the lower bound (22), we have

$$\sum_{\substack{n\in\mathcal{N}_4(x)\\ p^{(\beta)}(n)\in I\\ \omega(n)\in J}}\frac{1}{p^{(\beta)}(n)} = \frac{x}{\log x}g\left(t_0\right)\mathcal{R}. \tag{23}$$

Estimates (4), (8), and (23) allow us to write

$$\sum_{n\leq x}\frac{1}{p^{(\beta)}(n)} = \frac{x}{\log x}g\left(t_0\right)\mathcal{R} + E(x), \tag{24}$$

where the error term $E(x)$ is defined by

$$E(x) = \sum_{\substack{n\in\mathcal{N}_4(x)\\ p^{(\beta)}(n)\notin I \text{ or } \omega(n)\notin J}}\frac{1}{p^{(\beta)}(n)}.$$

In particular, an explicit evaluation of $g\left(t_0\right)$ using equation (21) yields the main term on the right-hand side of (2). What is left to do is to obtain an upper bound for the error term.

4 Estimation of the error term

In this section, we show that the error term $E(x)$ satisfies $E(x) = o\left(\frac{x}{\log x}g\left(t_0\right)\right)$ as $x \to \infty$. We can proceed as in the proof of the upper bound given by De Koninck and Luca [4]. First, we have from upper bound (4) and equation (8) that

$$\begin{aligned}\sum_{n\le x}\frac{1}{p^{(\beta)}(n)}&=\sum_{n\in\mathcal{N}_4(x)}\frac{1}{p^{(\beta)}(n)}+O\left(\frac{x\,(\log_2 x)^{M-1}}{\log x}\right)\\&=\sum_{p\in[3,\log x]}\frac{1}{p}\sum_{k\in[M+1,10\log_2 x]}\#\left(\mathcal{N}_{p,k}(x)\cap\mathcal{N}_4(x)\right)+O\left(\frac{x\,(\log_2 x)^{M-1}}{\log x}\right).\end{aligned}\tag{25}$$

Moreover, by equation (13), we have

$$\#\left(\mathcal{N}_{p,k}(x)\cap\mathcal{N}_4(x)\right)\le\sum_{\substack{a\le x\\ a\in\mathcal{A}}}\sum_{\alpha=1}^{\lfloor 10\log_2 x\rfloor}\sum_{\substack{b\le\frac{x}{ap^{\alpha}}\\ b\in\mathcal{B}}}1.$$

Hence, we get from Lemma 4 that

$$\#\left(\mathcal{N}_{p,k}(x)\cap\mathcal{N}_4(x)\right)\ll\frac{x}{\log x}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}\sum_{\substack{a\le x\\ a\in\mathcal{A}}}\frac{1}{a}\sum_{\alpha=1}^{\lfloor 10\log_2 x\rfloor}\frac{1}{p^{\alpha}}\ll\frac{x}{p\log x}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}\sum_{a\in\mathcal{A}}\frac{1}{a}.\tag{26}$$

In light of the definition of the set $\mathcal{G}$, upper bound (26) yields

$$\#\left(\mathcal{N}_{p,k}(x)\cap\mathcal{N}_4(x)\right)\ll\frac{x}{p\log x}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}\sum_{a\in\mathcal{G}}\frac{1}{a}.\tag{27}$$

On the other hand, observe that $\sum_{a\in G}\frac{1}{a}=0$ if $p<p_{k_0}$ and that for $p\ge p_{k_0}$, we have

$$\begin{aligned}\sum_{a\in\mathcal{G}}\frac{1}{a}&\ll\sum_{\substack{p_1<p_2<\cdots<p_{k_0-1}<p\\ \alpha_i\ge1}}\frac{1}{p_1^{\alpha_1}\cdots p_{k_0-1}^{\alpha_{k_0-1}}}=\sum_{\substack{p_1<p\\ \alpha_1\ge1}}\frac{1}{p_1^{\alpha_1}}\sum_{\substack{p_1<p_2<p\\ \alpha_2\ge1}}\frac{1}{p_2^{\alpha_2}}\cdots\sum_{\substack{p_{k_0-2}<p_{k_0-1}<p\\ \alpha_{k_0-1}\ge1}}\frac{1}{p_{k_0-1}^{\alpha_{k_0-1}}}\\&\ll\sum_{p_1<p}\frac{1}{p_1-1}\sum_{p_1<p_2<p}\frac{1}{p_2-1}\cdots\sum_{p_{k_0-2}<p_{k_0-1}\le p}\frac{1}{p_{k_0-1}-1}\\&\le\frac{1}{(k_0-1)!}\left(\sum_{q\le p}\frac{1}{q-1}\right)^{k_0-1}\ll\frac{1}{(k_0-1)!}\left(\log_2 p+c\right)^{k_0-1}\end{aligned}$$

for some positive constant c, where the inequality comes from Mertens' estimate. Hence, it follows from upper bound (27) that

$$\#\left(\mathcal{N}_{p,k}(x)\cap\mathcal{N}_4(x)\right)\ll\frac{x}{p\log x}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}\frac{\left((1+o(1))\log_2 p\right)^{k_0-1}}{(k_0-1)!}.\tag{28}$$

Using bound (28) in equation (25) yields

$$\sum_{n\in\mathcal{N}_4(x)}\frac{1}{p^{(\beta)}(n)}\ll\frac{x}{\log x}\sum_{p\in[3,\log x]}\frac{1}{p^2}\sum_{k\in[M+1,10\log_2 x]}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}\frac{(\log_2 p+c)^{k_0-1}}{(k_0-1)!}+\frac{x\,(\log_2 x)^{M-1}}{\log x}. \tag{29}$$

Observe that we can assume that $k\geq\log_3 x$. Indeed, it follows from upper bound (29) that

$$\sum_{\substack{n\in\mathcal{N}_4(x)\\ \omega(n)<\log_3 x}}\frac{1}{p^{\beta}(n)}\ll\frac{x}{\log x}\left(\log_2 x\right)^{\log_3 x}\left((1+o(1))\log_2 x\right)^{\log_3 x}. \tag{30}$$

Hence, combining estimates (29) and (30), we obtain

$$\begin{aligned}\sum_{n\in\mathcal{N}_4(x)}\frac{1}{p^{(\beta)}(n)}\ll&\frac{x}{\log x}\sum_{p\in[3,\log x]}\frac{1}{p^2}\sum_{k\in[\log_3 x,10\log_2 x]}\frac{(\log_2 x)^{\lambda-1}}{(\lambda-1)!}\frac{(\log_2 p+c)^{k_0-1}}{(k_0-1)!}\\&+\frac{x}{\log x}\exp\left(2\left(\log_3 x\right)^2+o\left(\log_3 x\right)\right).\end{aligned}$$

By applying Stirling's formula to $(\lambda-1)!$, we then have

$$\begin{aligned}\sum_{n\in\mathcal{N}_4(x)}\frac{1}{p^{(\beta)}(n)}\ll&\frac{x}{\log x}\sum_{p\in[3,\log x]}\frac{1}{p^2}\sum_{k\in[\log_3 x,10\log_2 x]}\frac{1}{k}\left(\frac{e\log_2 x}{\lambda}\right)^{\lambda-1}\left(\frac{e\left(\log_2 p+c\right)}{k_0}\right)^{k_0-1}\\&+\frac{x}{\log x}\exp\left(2\left(\log_3 x\right)^2+o\left(\log_3 x\right)\right)\\\ll&\frac{x}{\log x}\sum_{p\in[3,\log x]}\frac{1}{p^2}\sum_{k\in[\log_3 x,10\log_2 x]}\left(\frac{e\log_2 x}{\lambda}\right)^{\lambda}\left(\frac{e\left(\log_2 p+c\right)}{k_0}\right)^{k_0}\\&+\frac{x}{\log x}\exp\left(2\left(\log_3 x\right)^2+o\left(\log_3 x\right)\right).\end{aligned}$$

Expressing λ and k_0 in terms of k, we get from this upper bound that

$$\begin{aligned}\sum_{n\in\mathcal{N}_4(x)}\frac{1}{p^{(\beta)}(n)}\ll&\frac{x\log_2 x}{\log x}\sum_{p\in[3,\log x]}\frac{1}{p^2}\sum_{k\in[\log_3 x,10\log_2 x]}\left(\frac{e\left(\log_2 x\right)^{1-\beta}\left(\log_2 p+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}k}\right)^{k}\\&+\frac{x}{\log x}\exp\left(2\left(\log_3 x\right)^2+o\left(\log_3 x\right)\right).\end{aligned} \tag{31}$$

Let K_1 and K_2 be the smallest and largest integers in the interval J respectively. Then, from Lemma 6, since

$$K_1=\frac{(1-\beta)^{\beta}}{4}\frac{\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-2\beta}}+O(1)$$

is smaller than the maximum of the function $\left(\frac{e(\log_2 x)^{1-\beta}(\log_3 x+c)^{\beta}}{\beta^{\beta}(1-\beta)^{1-\beta}t}\right)^t$, we obtain that

$$\sum_{k\in[\log_3 x,K_1]}\left(\frac{e\left(\log_2 x\right)^{1-\beta}\left(\log_2 p+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}k}\right)^k \ll K_1\left(\frac{e\left(\log_2 x\right)^{1-\beta}\left(\log_3 x+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}K_1}\right)^{K_1}$$
$$\ll\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}\left(\frac{4e}{\left(1-\beta\right)^{2\beta}}\right)^{K_1}\exp\left(o\left(K_1\right)\right).$$

Since $\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}\ll\exp\left(o\left(K_1\right)\right)$, it follows that

$$\sum_{k\in[\log_3 x,K_1]}\left(\frac{e\left(\log_2 x\right)^{1-\beta}\left(\log_2 p+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}k}\right)^k \ll\exp\left(K_1\left(1+\log 4-2\beta\log\left(1-\beta\right)+o(1)\right)\right)$$
$$\ll\exp\left(\left(\frac{3}{5}+o(1)\right)\frac{\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-2\beta}}\right),\quad (32)$$

where the last inequality comes from the fact that

$$K_1\left(1+\log 4-2\beta\log\left(1-\beta\right)\right)=\frac{1+\log 4-2\beta\log\left(1-\beta\right)}{4\left(1-\beta\right)^{-\beta}}\frac{\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-2\beta}}+O(1),$$

where the first fraction is strictly smaller than 3/5. Indeed, the function $F\left(\beta\right)$ defined for $\beta\in[0,1)$ by

$$F\left(\beta\right)=\frac{1+\log 4-2\beta\log\left(1-\beta\right)}{4\left(1-\beta\right)^{-\beta}}$$

is strictly decreasing and $F\left(0\right)=\frac{1+\log 4}{4}<\frac{3}{5}$. Hence, from upper bounds (31) and (32), we have that

$$\sum_{n\in\mathcal{N}_4(x)}\frac{1}{p^{(\beta)}(n)}\ll\frac{x\log_2 x}{\log x}\sum_{p\in[3,\log x]}\frac{1}{p^2}\sum_{k\in[K_1,10\log_2 x]}\left(\frac{e\left(\log_2 x\right)^{1-\beta}\left(\log_2 p+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}k}\right)^k$$
$$+\frac{x}{\log x}\exp\left(\left(\frac{3}{5}+o(1)\right)\frac{\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-2\beta}}\right).\quad (33)$$

Similarly, from Lemma 6, since

$$K_2=\frac{2e\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}+O(1)$$

is larger than the maximum of the function $\left(\frac{e(\log_2 x)^{1-\beta}(\log_3 x+c)^{\beta}}{\beta^{\beta}(1-\beta)^{1-\beta}t}\right)^{t}$, we obtain

$$\begin{aligned}\sum_{k\in[K_2,10\log_2 x]}\left(\frac{e\left(\log_2 x\right)^{1-\beta}\left(\log_2 p+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}k}\right)^{k} &\ll \sum_{k\in[K_2,10\log_2 x]}\left(\frac{e\left(\log_2 x\right)^{1-\beta}\left(\log_3 x+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}k}\right)^{k}\\ &\ll \log_2 x\left(\frac{e\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}\left(1+o(1)\right)}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}K_2}\right)^{K_2}\\ &\ll \log_2 x\left(\frac{1}{2}+o(1)\right)^{K_2} \ll \log_2 x. \qquad (34)\end{aligned}$$

It follows from estimates (33) and (34) that

$$\begin{aligned}\sum_{n\in\mathcal{N}_4(x)}\frac{1}{p^{(\beta)}(n)} \ll &\frac{x\log_2 x}{\log x}\sum_{p\in[3,\log x]}\frac{1}{p^2}\sum_{k\in J}\left(\frac{e\left(\log_2 x\right)^{1-\beta}\left(\log_2 p+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}k}\right)^{k}\\ &+\frac{x}{\log x}\exp\left(\left(\frac{3}{5}+o(1)\right)\frac{\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-2\beta}}\right).\end{aligned} \qquad (35)$$

Hence, from Lemma 6, we can conclude that

$$\begin{aligned}\sum_{n\in\mathcal{N}_4(x)}\frac{1}{p^{(\beta)}(n)} \ll &\frac{x\log_2 x}{\log x}\sum_{p\in[3,\log x]}\frac{1}{p^2}\exp\left(\frac{\left(\log_2 x\right)^{1-\beta}\left(\log_2 p+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}\right)\\ &+\frac{x}{\log x}\exp\left(\left(\frac{3}{5}+o(1)\right)\frac{\left(\log_2 x\right)^{1-\beta}\left(\log_3 x\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-2\beta}}\right).\end{aligned}$$

Let q_1 and q_2 be the smallest and largest prime numbers in the interval I respectively. Then, for $p\le q_1$, we have

$$\log_2 p\le\log_2 q_1=(1-\beta)\left(\log_3 x-\frac{4}{1-\beta}\log_4 x-\log_5 x\left(1+o(1)\right)\right)=:(1-\beta)\,\mathcal{C}(x),$$

so that

$$\frac{x\log_2 x}{\log x}\sum_{p\in[3,q_1]}\frac{1}{p^2}\exp\left(\frac{\left(\log_2 x\right)^{1-\beta}\left(\log_2 p+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}\right)\ll\frac{x}{\log x}\exp\left(\frac{\left(\log_2 x\right)^{1-\beta}\left(\mathcal{C}(x)\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-2\beta}}\right). \qquad (36)$$

Combining upper bounds (35) and (36), we obtain

$$\begin{aligned}\sum_{n\in\mathcal{N}_4(x)}\frac{1}{p^{(\beta)}(n)} \ll &\frac{x\log_2 x}{\log x}\sum_{p\in[q_1,\log x]}\frac{1}{p^2}\sum_{k\in J}\left(\frac{e\left(\log_2 x\right)^{1-\beta}\left(\log_2 p+c\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}k}\right)^{k}\\ &+\frac{x}{\log x}\exp\left(\frac{\left(\log_2 x\right)^{1-\beta}\left(\log_3 x-\frac{4}{1-\beta}\log_4 x+O\left(\log_5 x\right)\right)^{\beta}}{\beta^{\beta}\left(1-\beta\right)^{1-2\beta}}\right),\end{aligned} \qquad (37)$$

since the second term in the right-hand side of estimate (35) is smaller than the one of the bound (37). When the prime number p satisfies $q_2 \leq p \leq \log x$, we have that $\log_2 p \leq \log_3 x$ and that $\sum_{p \geq q_2} \frac{1}{p^2} \ll \frac{1}{q_2 \log q_2}$, where $q_2 = \exp\left((\log_2 x)^{1-\beta} \log_3 x\right) + O(1)$. Hence, as $x \to \infty$,

$$\sum_{p \in [q_2, \log x]} \frac{\exp\left(\frac{(\log_2 x)^{1-\beta}(\log_2 p + c)^{\beta}}{\beta^{\beta}(1-\beta)^{1-\beta}}\right)}{p^2} \ll \frac{\exp\left(\frac{(\log_2 x)^{1-\beta}(\log_3 x)^{\beta}}{\beta^{\beta}(1-\beta)^{1-\beta}}(1+o(1))\right)}{\exp\left((\log_2 x)^{1-\beta}\log_3 x\right)} = o(1). \tag{38}$$

It follows from estimates (37) and (38) that

$$\begin{aligned} \sum_{n \in \mathcal{N}_4(x)} \frac{1}{p^{(\beta)}(n)} \ll & \frac{x \log_2 x}{\log x} \sum_{p \in I} \frac{1}{p^2} \sum_{k \in J} \left(\frac{e (\log_2 x)^{1-\beta} (\log_2 p + c)^{\beta}}{\beta^{\beta} (1-\beta)^{1-\beta} k} \right)^k \\ & + \frac{x}{\log x} \exp\left(\frac{(\log_2 x)^{1-\beta} \left(\log_3 x - \frac{4}{1-\beta} \log_4 x + O(\log_5 x)\right)^{\beta}}{\beta^{\beta} (1-\beta)^{1-2\beta}} \right). \end{aligned} \tag{39}$$

Finally, in light of equation (24), since

$$g(t_0) = \exp\left(\frac{(\log_2 x)^{1-\beta} \left(\log_3 x - \frac{2-\beta}{1-\beta} \log_4 x + O(1)\right)^{\beta}}{\beta^{\beta} (1-\beta)^{1-2\beta}} + O\left(\left(\frac{\log_2 x}{\log_3 x}\right)^{1-\beta}\right)\right),$$

it follows from upper bound (39) that

$$\frac{x}{\log x} \exp\left(\frac{(\log_2 x)^{1-\beta} \left(\log_3 x - \frac{4}{1-\beta} \log_4 x - \log_5 x (1 + o(1))\right)^{\beta}}{\beta^{\beta} (1-\beta)^{1-2\beta}} \right) = o\left(\frac{x}{\log x} g(t_0)\right).$$

Hence, we can conclude that $E(x) = o\left(\frac{x}{\log x} g(t_0)\right)$, which completes the proof of Theorem 1.

5 Final remarks

The error term $\mathcal{R}$ in Theorem 1 seems difficult to improve if one wants to obtain an explicit result, the reason being that our result comes directly from the estimation of ρ and $F(\rho, p)$ provided in Erdős and Tenenbaum [6]. In fact, estimate (17) is the same as the one given by Erdős and Tenenbaum [6, Lemma 1], so that obtaining an explicit result better than the one given in Theorem 1 would require improving the Erdős and Tenenbaum estimates. However, it would still be interesting to obtain an estimate for $\#\mathcal{N}_{p,k}(x)$ in a range wider than the one provided by the primes $p \in I$ and integers $k \in J$.

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