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Some Theorems and Applications of the (q, r)-Whitney Numbers

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Abstract

The (q, r)-Whitney numbers were recently defined in terms of the q-Boson operators, and several combinatorial properties which appear to be q-analogues of similar properties were studied. In this paper, we obtain elementary and complete symmetric polynomial forms for the (q, r)-Whitney numbers, and give combinatorial interpretations in the context of A-tableaux. We also obtain convolution-type identities using the combinatorics of A-tableaux. Lastly, we present applications and theorems related to discrete q-distributions.

1 Introduction

In a recent paper, the author and Katriel [21] introduced a new approach to generate q-analogues of Stirling and Whitney-type numbers. In this paper, the (q, r)-Whitney numbers of the first and second kinds were defined as coefficients in

$$m^n(a^\dagger)^n a^n = \sum_{k=0}^{n} w_{m,r,q}(n,k)(ma^\dagger a + r)^k \tag{1}$$

and

$$(ma^\dagger a + r)^n = \sum_{k=0}^{n} m^k W_{m,r,q}(n,k)(a^\dagger)^k a^k, \tag{2}$$

respectively (cf. [21]), by using as framework, the q-Boson operators $a^\dagger$ and a of Arik and Coon [2] which satisfy the commutation relation

$$[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1. \tag{3}$$

By convention, $w_{m,r,q}(0,0) = W_{m,r,q}(0,0) = 1$ and $w_{m,r,q}(n,k) = W_{m,r,q}(n,k) = 0$ for $k > n$ and for $k < 0$. Several combinatorial properties were already established, including the following triangular recurrence relations [21, Theorem 6]:

$$w_{m,r,q}(n+1,k) = q^{-n}\Big(w_{m,r,q}(n,k-1) - (m[n]_q + r)w_{m,r,q}(n,k)\Big), \tag{4}$$

with $[n]_q = \frac{q^n-1}{q-1}$, the q-integer, and

$$W_{m,r,q}(n+1,k) = q^{k-1}W_{m,r,q}(n,k-1) + (m[k]_q + r)W_{m,r,q}(n,k). \tag{5}$$

From here, one readily obtains

$$w_{m,r,q}(n,0) = (-1)^n q^{-\binom{n}{2}} \prod_{i=0}^{n-1}(m[i]_q + r), \tag{6}$$

$$w_{m,r,q}(n,n) = q^{-\binom{n}{2}}, \tag{7}$$

$$W_{m,r,q}(n,0) = r^n, \tag{8}$$

and

$$W_{m,r,q}(n,n) = q^{\binom{n}{2}}. \tag{9}$$

The identities presented in Eqs. (4) and (5) can be used as tools to obtain further combinatorial identities for $w_{m,r,q}(n,k)$ and $W_{m,r,q}(n,k)$. For instance, with the aid of these recurrence relations, the vertical recurrence relations

$$w_{m,r,q}(n+1,k+1) = \sum_{j=k}^{n}(-1)^{n-j}q^{\binom{j}{2}-\binom{n+1}{2}}w_{m,r,q}(j,k)\prod_{i=j+1}^{n}(m[i]_q + r), \tag{10}$$

with $\prod_{i=j+1}^{n}(m[i]_q + r) = 1$ when $j = n$, and

$$W_{m,r,q}(n+1,k+1) = q^k \sum_{j=k}^{n}(m[k+1]_q + r)^{n-j}W_{m,r,q}(j,k), \tag{11}$$

can be proved by induction, as well as the rational generating function of the (q,r)-Whitney numbers of the second kind given by

$$\sum_{n=k}^{\infty} W_{m,r,q}(n,k)t^n = \frac{q^{\binom{k}{2}}t^k}{\prod_{i=0}^{k}\left(1 - (m[i]_q + r)t\right)}. \tag{12}$$

On the other hand, the horizontal recurrence relations

$$w_{m,r,q}(n,k) = q^n \sum_{j=0}^{n-k} (m[n]_q + r)^j w_{m,r,q}(n+1, k+j+1) \tag{13}$$

and

$$W_{m,r,q}(n,k) = \sum_{j=0}^{n-k} (-1)^j q^{\binom{k}{2} - \binom{k+j+1}{2}} \frac{\prod_{i=0}^{k+j}(m[i]_q + r)}{\prod_{i=0}^{k}(m[i]_q + r)} W_{m,r,q}(n+1, k+j+1) \tag{14}$$

can be verified by evaluating the right-hand sides using Eqs. (4) and (5). Before proceeding, we note that Eqs. (10) and (11) follow a behaviour similar to that of the Chu-Shi-Chieh's identity (see [6]) for the classical binomial coefficients given by

$$\binom{n+1}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k},$$

while Eqs. (13) and (14) are analogous with

$$\binom{n}{k} = \binom{n+1}{k+1} - \binom{n+1}{k+2} + \cdots + (-1)^{n-k}\binom{n+1}{n+1},$$

another known identity for the classical binomial coefficients.

The purpose of this paper is to express the (q,r)-Whitney numbers of both kinds in symmetric polynomial forms. This proves to be useful in establishing combinatorial interpretations in terms of A-tableaux. In return, remarkable convolution-type identities are obtained and several other interesting theorems are also presented.

2 Explicit formulas in symmetric polynomial forms

2.1 (q,r)-Whitney numbers of the first kind

Expanding the falling factorial $(x)_n = x(x-1)\cdots(x-n+1)$ in powers of x, we obtain

$$(x)_n = \sum_{k=0}^{n} (-1)^{n-k} x^k \sum_{1 \le i_1 < i_2 < \cdots < i_{n-k} \le n-1} \prod_{j=1}^{n-k} i_j\,,$$

which yield the well-known expression for the Stirling numbers of the first kind in terms of elementary symmetric functions. This relation can be generalized to the q-Stirling numbers as follows:

$$\begin{aligned}
[x]_q[x-1]_q \cdots [x-n+1]_q &= [x]_q([x]_q + q^x)([x]_q + q^x[2]_q)\cdots([x]_q + q^x[n-1]_q) \\
&= \sum_{k=0}^{n} [x]_q^k \cdot q^{x(n-k)} \sum_{1 \le i_1 < i_2 < \cdots < i_{n-k} \le n-1} \prod_{j=1}^{n-k} [i_j]_q\,.
\end{aligned}$$

To further generalize this procedure to the (q,r)-Whitney numbers of the first kind, recall that application of both sides of the defining relation in Eq. (1) on the q-boson number state $|\ell\rangle$ gives

$$m^n[\ell]_q[\ell-1]_q\cdots[\ell-n+1]_q=\sum_{k=0}^{n}w_{m,r,q}(n,k)\Big(m[\ell]_q+r\Big)^k.$$

Since both sides of this relation are finite polynomials in ℓ, and since the relation is valid for all integer ℓ, it remains valid when ℓ is replaced by the real number x, i.e.,

$$m^n[x]_q[x-1]_q\cdots[x-n+1]_q=\sum_{k=0}^{n}w_{m,r,q}(n,k)\Big(m[x]_q+r\Big)^k. \tag{15}$$

Now, defining $y=[x]_q+\alpha$, where $\alpha=\frac{r}{m}$, we note that $[x-i]_q=q^{-i}(y-\alpha-[i]_q)$. Hence,

$$m^n[x]_q[x-1]_q\cdots[x-n+1]_q=\sum_{k=0}^{n}(m[x]_q+r)^k q^{-\binom{n}{2}}(-1)^{n-k}\sum_{0\le i_1<i_2<\cdots<i_{n-k}\le n-1}\prod_{j=1}^{n-k}(m[i_j]_q+r). \tag{16}$$

The identity in the next theorem is obtained by comparing the right-hand-sides of Eqs. (15) and (16).

Theorem 1. *The (q,r)-Whitney numbers of the first kind satisfy the following explicit form*

$$w_{m,r,q}(n,k)=(-1)^{n-k}q^{-\binom{n}{2}}\sum_{0\le i_1<i_2<\cdots<i_{n-k}\le n-1}\prod_{j=1}^{n-k}(r+[i_j]_q m). \tag{17}$$

Remark 2. The sum within this theorem is the symmetric polynomial of degree $n-k$ in the n variables $\{(r+[i]_q m); i=0,1,\ldots,n-1\}$. For $r=0$ all the terms with $i_1=0$ vanish so the summation starts at 1, which is consistent with the expressions presented above for the Stirling and q-Stirling numbers of the first kind.

The above theorem can also be proved by induction as follows:

Alternative proof of Theorem 1. The theorem readily yields $w_{m,r,q}(0,0)=1$. Making the induction hypothesis that the theorem is true up to n, for all $k=0,1,\ldots,n$, we prove it for

$n+1$ and $k=0,1,\ldots,n$, via the recurrence relation (4). Thus,

$$\begin{aligned}
w_{m,r,q}(n+1,k) &= q^{-n}\Big((-1)^{n+1-k}q^{-\binom{n}{2}}\sum_{0\le i_1<i_2<\cdots<i_{n+1-k}\le n-1}\prod_{j=1}^{n+1-k}(r+[i_j]_q m)\\
&\quad -(m[n]_q+r)(-1)^{n-k}q^{-\binom{n}{2}}\sum_{0\le i_1<i_2<\cdots<i_{n-k}\le n-1}\prod_{j=1}^{n-k}(r+[i_j]_q m)\Big)\\
&= q^{-\binom{n+1}{2}}(-1)^{n+1-k}q^{-\binom{n}{2}}\Big(\sum_{0\le i_1<i_2<\cdots<i_{n+1-k}\le n-1}\prod_{j=1}^{n+1-k}(r+[i_j]_q m)\\
&\quad +(m[n]_q+r)\sum_{0\le i_1<i_2<\cdots<i_{n-k}\le n-1}\prod_{j=1}^{n-k}(r+[i_j]_q m)\Big)
\end{aligned}$$

The first term within the large paretheses contains all products of $n+2-k$ distinct factors out of $\{(r+[i]_q m); i=0,1,\ldots,n-1\}$, whereas the second term contains all products of $n+2-k$ distinct factors, one of which is $(r+m[n]_q)$ and the others chosen out of $\{(r+[i]_q m); i=0,1,\ldots,n-1\}$. Together, these sums yield

$$\sum_{0\le i_1<i_2<\cdots<i_{n+1-k}\le n}\prod_{j=0}^{n+1-k}(r+[i_j]_q m),$$

thus establishing the theorem for the range of indices specified above. Finally, the theorem yields $w_{m,r,q}(n+1,n+1)=q^{-\binom{n+1}{2}}$, in agreement with (7). $\square$

As $q\to 1$, the explicit formula (17) reduces to an expression for the r-Whitney numbers of the first kind given by

$$w_{m,r}(n,k)=(-1)^{n-k}\sum_{0\le i_1<i_2<\cdots<i_{n-k}\le n-1}\prod_{j=1}^{n-k}(r+i_j m). \tag{18}$$

An equivalent of this identity was reported by Mangontarum et al. [18, Theorem 6]. For $m=1$ and $r=0$, (17) reduces to an explicit formula for a q-analogue of the Stirling numbers of the first kind, viz,

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=(-1)^{n-k}q^{-\binom{n}{2}}\sum_{0\le i_1<i_2<\cdots<i_{n-k}\le n-1}\prod_{j=1}^{n-k}[i_j]_q, \tag{19}$$

where $\begin{bmatrix} n\\ k\end{bmatrix}_q$ denote the q-Stirling numbers of the first kind defined by

$$[x]_{q,n}=\sum_{k=1}^{n}(-1)^{n-k}\begin{bmatrix} n\\ k\end{bmatrix}_q[x]_q^k, \tag{20}$$

$[x]_{q,n} = [x]_q[x-1]_q[x-2]_q \cdots [x-n+1]_q$ (cf. [4]). For any given set of $n-k$ integers that satisfy $1 < i_2 < \cdots < i_{n-k} < n-1$, let

$$\{\ell_1, \ell_2, \ldots, \ell_k\} \equiv \{1, 2, 3, \ldots, n-1\} - \{i_1, i_2, \ldots, i_{n-k}\}$$

be the complement with respect to $\{1, 2, 3, \ldots, n-1\}$. It follows that

$$\prod_{j=0}^{n-k} [i_j]_q = \frac{[n-1]_q!}{\prod_{j=0}^{k} [\ell_j]_q}. \tag{21}$$

This allows (19) to be written in the form

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{-\binom{n}{2}} [n-1]_q! \sum_{0 \le i_1 < i_2 < \cdots < i_{n-k} \le n-1} \frac{1}{\prod_{j=0}^{k} [\ell_j]_q}. \tag{22}$$

As $q \to 1$, one recovers from (19) Comtet's [8] identity given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} \sum_{0 \le i_1 < i_2 < \cdots < i_{n-k} \le n-1} \prod_{j=1}^{n-k} i_j, \tag{23}$$

while (22) yields Adamchik's [1] identity for the Stirling numbers of the first kind given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)! \sum_{0 \le i_1 < i_2 < \cdots < i_{n-k} \le n-1} \frac{1}{\prod_{j=0}^{k} \ell_j}. \tag{24}$$

2.2 (q, r)-Whitney numbers of the second kind

Theorem 3. *The (q, r)-Whitney numbers of the second kind satisfy the following explicit form:*

$$W_{m,r,q}(n, k) = q^{\binom{k}{2}} \sum_{c_0 + c_1 + \cdots + c_k = n-k} \prod_{j=0}^{k} (m[j]_q + r)^{c_j}, \tag{25}$$

where $c_0, c_1, \ldots, c_k$ are non-negative integers.

Proof. We proceed by induction over n. First, we note that the theorem is satisfied when $n = k = 0$. That is, $W_{m,r,q}(0, 0) = 1$. Making the induction hypothesis that the theorem holds up to n (for all $k = 0, 1, \ldots, n$) we show, using the recurrence relation (5), that it holds for $n+1$ and $k=0,1,\ldots,n$. Thus,

$$\begin{aligned} W_{m,r,q}(n+1,k) &= q^{k-1}q^{\binom{k-1}{2}} \sum_{c_0+c_1+\cdots+c_{k-1}=n+1-k} \prod_{j=0}^{k-1} (m[j]_q+r)^{c_j} \\ &\quad +(m[k]_q+r)q^{\binom{k}{2}} \sum_{c_0+c_1+\cdots+c_k=n-k} \prod_{j=0}^{k} (m[j]_q+r)^{c_j} \\ &= q^{\binom{k}{2}} \Big(\sum_{c_0+c_1+\cdots+c_{k-1}=n+1-k} \prod_{j=0}^{k-1} (m[j]_q+r)^{c_j} \\ &\quad +(m[k]_q+r) \sum_{c_0+c_1+\cdots+c_k=n-k} \prod_{j=0}^{k} (m[j]_q+r)^{c_j} \Big) \end{aligned}$$

Now, the first term within the bigparetheses is a sum of products of $n+1-k$ factors, non of which contains $(m[k]_q+r)$. The second term is again a sum of $n+1-k$ factors, each one of which containing $(m[k]_q+r)$ at least once. Thus,

$$W_{m,r,q}(n+1,k) = q^{\binom{k}{2}} \sum_{c_0+c_1+\cdots+c_k=n+1-k} \prod_{j=0}^{k} (m[j]_q+r)^{c_j}.$$

To complete the proof we need to show that the theorem holds for $n+1$ and $k=n+1$. For this case the theorem yields $W_{m,r,q}(n+1,n+1) = q^{\binom{n+1}{2}}$, which is in agreement with (9). $\square$

Apart from $q^{\binom{k}{2}}$, (25) is a homogeneous complete symmetric polynomial of degree $n-k$ in the variables $\{(r+[j]_q m); j=0,1,2,\ldots,k\}$. As $q\to 1$, we obtain

$$W_{m,r}(n,k) = \sum_{c_0+c_1+\cdots+c_k=n-k} \prod_{j=0}^{k} (r+mj)^{c_j}, \tag{26}$$

and for $r=0$, (25) reduces to an expression for the q-Stirling numbers of the second kind, viz,

$$\left\{ {n \atop k} \right\}_q = q^{\binom{k}{2}} \sum_{c_0+c_1+\cdots+c_k=n-k} [1]_q^{i_1}[2]_q^{i_2}\cdots[k]_q^{i_k}. \tag{27}$$

The q-Stirling numbers of the second kind were originally defined as

$$[x]_q^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}_q [x]_{q,k} \tag{28}$$

(cf. [4]). Moreover, when $q\to 1$, Eq. (27) yields an expression for the classical Stirling numbers of the second kind reported by Comtet [8].

Notice that from the inner product

$$\prod_{j=0}^{k}(m[j]_q + r)^{c_j} = (m[0]_q + r)^{c_0}(m[1]_q + r)^{c_1}(m[2]_q + r)^{c_2}\cdots(m[k]_q + r)^{c_k} \tag{29}$$

in the explicit formula in (25), we observe that there are exactly $n-k$ factors of $(m[j]_q + r)$ which is repeated c_j times for each j. From here, we write

$$(m[0]_q + r)^{c_0} = (m[j_1]_q + r)(m[j_2]_q + r)\cdots(m[j_{c_0}]_q + r),$$

where $j_i = 0$, $i = 1, 2, \ldots, c_0$;

$$(m[1]_q + r)^{c_1} = (m[j_{c_0+1}]_q + r)(m[j_{c_0+2}]_q + r)\cdots(m[j_{c_0+c_1}]_q + r),$$

where $j_{c_0+i} = 1$, $i = 1, 2, \ldots, c_1$;

$$(m[2]_q + r)^{c_2} = (m[j_{c_0+c_1+1}]_q + r)(m[j_{c_0++c_1+2}]_q + r)\cdots(m[j_{c_0+c_1+c_2}]_q + r),$$

where $j_{c_0+c_1+i} = 2$, $i = 1, 2, \ldots, c_2$ and so on until

$$(m[k]_q + r)^{c_k} = (m[j_{c_0+c_1+\cdots+c_{k-1}+1}]_q + r)(m[j_{c_0+c_1+\cdots+c_{k-1}+2}]_q + r)\cdots(m[j_{c_0+c_1+\cdots+c_k}]_q + r),$$

where $j_{c_0+c_1+\cdots+c_{k-1}+i} = k$, $i = 1, 2, \ldots, c_k$ and $c_0 + c_1 + c_2 + \cdots + c_{k-1} + c_k = n-k$. Thus, $0 \leq j_1 \leq j_2 \leq \cdots \leq j_{n-k} \leq k$ and we have

$$W_{m,r,q}(n,k) = q^{\binom{k}{2}} \sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{n-k}\leq k} \prod_{i=1}^{n-k}(m[j_i]_q + r). \tag{30}$$

We formally state this result in the next theorem.

Theorem 4. *The (q,r)-Whitney numbers of the second kind satisfy the following explicit form:*

$$W_{m,r,q}(n,k) = q^{\binom{k}{2}} \sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{n-k}\leq k} \prod_{i=1}^{n-k}(m[j_i]_q + r). \tag{31}$$

Notice that when $q \to 1$, we obtain an identity similar to the result obtained by Mangontarum et al. [18, Theorem 11].

3 On the context of A-tableaux

De Medicis and Leroux [23] defined a 0-1 tableau to be a pair $\varphi = (\lambda, f)$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition of an integer m and $f = (f_{ij})_{1\leq j\leq\lambda_i}$ is a "filling" of the cells of the corresponding Ferrers diagram of shape λ with 0's and 1's such that exactly one 1 in

0	0	0	1	0	1	0	1
0	1	0	0	1	0	1	
0	0	1	0	0			
1	0	0	0				
0							

Figure 1: A 0-1 tableau φ

each column. For instance, the figure below represents the 0-1 tableau $\varphi = (\lambda, f)$, where $\lambda = (8, 7, 5, 4, 1)$ with

$$f_{14} = f_{16} = f_{18} = f_{22} = f_{25} = f_{27} = f_{33} = f_{41} = 1$$

and $f_{ij} = 0$ elsewhere for $1 \leq j \leq \lambda_i$. In the same paper, an A-tableau is defined to be a list Φ of columns c of a Ferrers diagram of a partition λ (by decreasing order of length) such that the length $|c|$ is part of the sequence $A = (a_i)_{i\geq 0}$, a strictly increasing sequences of non-negative integers. Combinatorial interpretations of Stirling-type numbers in terms of A-tableaux are already done in the past. Similar works can be seen in [9, 12, 14, 17, 23] and some of the references therein. In particular, Corcino and Montero [14] defined a q-analogue of the Rucinski-Voigt numbers (an equivalent of the r-Whitney numbers of the second kind) and then presented a combinatorial interpretation using the theory of A-tableaux. The same type of interpretation was obtained by Mangontarum et al. [17] for the case of the translated Whitney numbers (see [20]) and their q-analogues. It is important to note that the q-analogues of these authors follow motivations which differ from that of the (q, r)-Whitney numbers. Furthermore, the numbers considered in the paper of Ramírez and Shattuck [26] belong to p, q-analogues, a natural extension of q-analogues.

Now, we let ω be a function from the set of non-negative integers N to a ring K, and suppose that Φ is an A-tableau with r columns of length $|c|$. Also, it is known that Φ might contain a finite number of columns whose lengths are zero since $0 \in A$ and if $\omega(0) \neq 0$ (cf. [23]). Before proceeding, we denote by $T^A(x, y)$ the set of A-tableaux with $A = \{0, 1, 2, \ldots, x\}$ and exactly y columns (with some columns possibly of zero length), and by $T^A_d(x, y)$ the subset of $T^A(x, y)$ which contains all A-tableaux with columns of distinct lengths. The next theorem relates the (q, r)-Whitney numbers of both kinds to certain sets of A-tableaux.

Theorem 5. *Let $\Omega : N \longrightarrow K$ and $\omega : N \longrightarrow K$ be functions from the set of non-negative integers N to a ring K (column weights according to length) defined by*

$$\Omega(|c|) = m[|c|]_q + r$$

and

$$\omega(|c|) = m[|\bar{c}|]_q + r,$$

where m and r are complex numbers, $|c|$ is the length of column c of an A-tableau in $T_d^A(n-1,n-k)$, and $|\bar{c}|$ is the length of column c of an A-tableau in $T^A(k,n-k)$. Then

$$(-1)^{n-k}q^{\binom{n}{2}}w_{m,r,q}(n,k)=\sum_{\Phi\in T_d^A(n-1,n-k)}\prod_{c\in\Phi}\Omega(|c|) \tag{32}$$

and

$$q^{-\binom{k}{2}}W_{m,r,q}(n,k)=\sum_{\phi\in T^A(k,n-k)}\prod_{\bar{c}\in\phi}\omega(|\bar{c}|). \tag{33}$$

Proof. Let $\Phi\in T_d^A(n-1,n-k)$. This means that Φ has exactly $n-k$ columns, say $c_1,c_2,\ldots,c_{n-k}$ whose lengths are $j_1,j_2,\ldots,j_{n-k}$, respectively. Now, for each column $c_i\in\Phi$, $i=1,2,3,\ldots,n-k$, we have $|c_i|=j_i$ and

$$\Omega(|c_i|)=m[|j_i|]_q+r.$$

Thus,

$$\begin{aligned}\prod_{c\in\Phi}\Omega(|c|) &= \prod_{i=1}^{n-k}\Omega(|c_i|)\\ &= \prod_{i=1}^{n-k}(m[j_i]_q+r).\end{aligned}$$

Since $\Phi\in T_d^A(n-1,n-k)$, then

$$\begin{aligned}\sum_{\Phi\in T_d^A(n-1,n-k)}\prod_{c\in\Phi}\Omega(|c|) &= \sum_{0\le j_1<j_2<\cdots<j_{n-k}\le n-1}\prod_{c\in\Phi}\Omega(|c|)\\ &= \sum_{0\le j_1<j_2<\cdots<j_{n-k}\le n-1}\prod_{i=1}^{n-k}(m[j_i]_q+r)\\ &= (-1)^{n-k}q^{\binom{n}{2}}w_{m,r,q}(n,k).\end{aligned}$$

The second result is obtained similarly. □

3.1 Combinatorics of A-tableaux

In the following theorem, we will demonstrate the simple combinatorics of A-tableaux. To start, note that Eqs. (32) and (33) are equivalent to

$$(-1)^{n-k}q^{\binom{n}{2}}w_{m,r,q}(n,k)=\sum_{\Phi\in T_d^A(n-1,n-k)}\Omega_A(\Phi) \tag{34}$$

and

$$q^{-\binom{k}{2}} W_{m,r,q}(n,k) = \sum_{\phi \in T^A(k,n-k)} \omega_A(\phi), \tag{35}$$

respectively, where

$$\Omega_A(\Phi) = \prod_{c\in\Phi} \Omega(|c|) = \prod_{c\in\Phi} (m[|c|]_q + r),\ |c| \in \{0, 1, 2, \ldots, n-1\} \tag{36}$$

and

$$\omega_A(\phi) = \prod_{\bar{c}\in\phi} \omega(|\bar{c}|) = \prod_{\bar{c}\in\phi} (m[|c|]_q + r),\ |\bar{c}| \in \{0, 1, 2, \ldots, k\}. \tag{37}$$

Theorem 6. *For nonnegative integers n and k, and complex numbers m and r, the following identities hold:*

$$w_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{j}{k} (-r_2)^{j-k} w_{m,r_1,q}(n,j) \tag{38}$$

$$W_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{n}{j} r_2^{n-j} W_{m,r_1,q}(j,k), \tag{39}$$

where $r_1 + r_2 = r$.

Proof. Let $\Phi \in T_d^A(n-1)$. Substituting $j_i = |c|$ in Eq. (36) gives

$$\Omega_A(\Phi) = \prod_{i=1}^{n-k} (m[j_i]_q + r),$$

where $j_i \in \{0, 1, 2, \ldots, n-1\}$. Suppose that for some numbers r_1 and r_2, $r = r_1 + r_2$. Then, with $\Omega^*(j_i) = m[j_i]_q + r_1$, we may write

$$\begin{aligned} \Omega_A(\Phi) &= \prod_{i=1}^{n-k} [(m[j_i]_q + r_1) + r_2] \\ &= \prod_{i=1}^{n-k} (\Omega^*(j_i) + r_2) \\ &= \sum_{\ell=0}^{n-k} r_2^{n-k-\ell} \sum_{j_1 \le q_1 < q_2 < \cdots < q_\ell \le j_{n-k}} \prod_{i=1}^{\ell} \Omega^*(q_i). \end{aligned}$$

Let B_Φ be the set of all A-tableaux corresponding to Φ such that for each $\psi \in B_\Phi$, one of the following is true:

ψ has no column whose weight is r_2;

ψ has one column whose weight is r_2;

ψ has two columns whose weight are r_2;

$\vdots$

ψ has $n-k$ columns whose weight are r_2.

Then,

$$\Omega_A(\Phi) = \sum_{\psi \in B_\Phi} \Omega_A(\psi).$$

Now, if ℓ columns in ψ have weights other than r_2, then

$$\begin{aligned} \Omega_A(\psi) &= \prod_{c\in\psi} \Omega^*(|c|) \\ &= r_2^{n-k\ell} \prod_{i=1}^{\ell} \Omega^*(q_i), \end{aligned}$$

where $q_1, q_2, q_3, \dots, q_\ell \in \{j_1, j_2, j_3, \dots, j_{n-k}\}$. Hence, Eq. (34) may be written as

$$\begin{aligned} (-1)^{n-k} q^{\binom{n}{2}} w_{m,r,q}(n,k) &= \sum_{\Phi \in T_d^A(n-1,n-k)} \Omega_A(\Phi) \\ &= \sum_{\Phi \in T_d^A(n-1,n-k)} \sum_{\psi \in B_\Phi} \Omega_A(\psi). \end{aligned}$$

For each ℓ, it is known that there correspond $\binom{n-k}{\ell}$ tableaux with ℓ distinct columns with weights $\Omega^*(q_i)$, $q_i \in \{j_1, j_2, \dots, j_{n-k}\}$. Since $T_d^A(n-1, n-k)$ contains $\binom{n}{k}$ tableaux, then for each $\Phi \in T_d^A(n-1, n-k)$, the total number of A-tableaux ψ corresponding to Φ is

$$\binom{n}{k}\binom{n-k}{\ell}.$$

However, only $\binom{n}{\ell}$ tableaux in B_Φ with ℓ distinct columns of weights other than r_2 are distinct. It then follows that every distinct tableau ψ appears

$$\frac{\binom{n}{k}\binom{n-k}{\ell}}{\binom{n}{\ell}} = \binom{n-\ell}{k}$$

times in the collection (cf. [12]). Thus, we consequently obtain

$$(-1)^{n-k} q^{\binom{n}{2}} w_{m,r,q}(n,k) = \sum_{\ell=0}^{n-k} \binom{n-\ell}{k} r_2^{n-k-\ell} \sum_{\psi \in B_\ell} \prod_{c\in\psi} \Omega^*(|c|),$$

where B_ℓ denotes the set of all tableaux ψ having ℓ distinct columns whose lengths are in the set $\{0, 1, 2, \ldots, n-1\}$. Reindexing the double sum yields

$$(-1)^{n-k} q^{\binom{n}{2}} w_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{j}{k} r_2^{j-k} \sum_{\psi \in B_{n-j}} \prod_{c \in \psi} \Omega^*(|c|). \tag{40}$$

Since $B_{n-j} = T_d^A(n-1, n-j)$, then

$$\sum_{\psi \in B_{n-j}} \prod_{c \in \psi} \Omega^*(|c|) = (-1)^{n-j} q^{\binom{n}{2}} w_{m,r_1,q}(n,j). \tag{41}$$

Combining Eqs. (40) and (41) gives

$$(-1)^{n-k} q^{\binom{n}{2}} w_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{j}{k} r_2^{j-k} (-1)^{n-j} q^{\binom{n}{2}} w_{m,r_1,q}(n,j) \tag{42}$$

which is equivalent to the desired result in Eq. (38). Similarly, if $\phi \in T^A(n-1)$, then substituting $j_i = |\bar{c}|$ in Eq. (37) gives

$$\omega_A(\phi) = \prod_{i=1}^{n-k} (m[j_i]_q + r),$$

where $j_i \in \{0, 1, 2, \ldots, k\}$. If for some numbers r_1 and r_2, $r = r_1 + r_2$, then

$$\begin{aligned}
\omega_A(\phi) &= \prod_{i=1}^{n-k} [(m[j_i]_q + r_1) + r_2] \\
&= \prod_{i=1}^{n-k} (\omega^*(j_i) + r_2),\ \omega^*(j_i) = m[j_i]_q + r_1 \\
&= \sum_{\ell=0}^{n-k} r_2^{n-k-\ell} \sum_{j_1 \le q_1 \le q_2 \le \cdots \le q_\ell \le j_{n-k}} \prod_{i=1}^{\ell} \omega^*(q_i).
\end{aligned}$$

Suppose $\bar{B}_\phi$ is the set of all A-tableaux corresponding to ϕ such that for each $\zeta \in \bar{B}_\phi$, one of the following is true:

ζ has no column whose weight is r_2;

ζ has one column whose weight is r_2;

ζ has two columns whose weight are r_2;

$\vdots$

ζ has $n-k$ columns whose weight are r_2.

Then, we may write

$$\omega_A(\phi) = \sum_{\zeta \in \bar{B}_\phi} \omega_A(\zeta).$$

If there are ℓ columns in ζ with weights other than r_2, then we have

$$\begin{aligned} \omega_A(\zeta) &= \prod_{\bar{c} \in \zeta} \omega^*(|\bar{c}|) \\ &= r_2^{n-k\ell} \prod_{i=1}^{\ell} \omega^*(q_i), \end{aligned}$$

where $q_1, q_2, q_3, \ldots, q_\ell \in \{j_1, j_2, j_3, \ldots, j_{n-k}\}$. It then follows that Eq. (35) may be expressed as

$$\begin{aligned} q^{-\binom{k}{2}} W_{m,r,q}(n,k) &= \sum_{\phi \in T^A(k,n-k)} \omega_A(\phi) \\ &= \sum_{\phi \in T^A(k,n-k)} \sum_{\zeta \in \bar{B}_\phi} \omega_A(\zeta). \end{aligned}$$

Like in the previous, for each ℓ, there correspond $\binom{n-k}{\ell}$ tableaux with ℓ columns having weights $\omega^*(q_i)$, $q_i \in \{j_1, j_2, \ldots, j_{n-k}\}$. Since the set $T^A(k, n-k)$ contains $\binom{n}{k}$ tableaux, then for each $\phi \in T^A(k, n-k)$, there are

$$\binom{n}{k}\binom{n-k}{\ell}$$

A-tableaux corresponding to ϕ. But only $\binom{\ell+k}{\ell}$ of these tableaux are distinct. Hence, every distinct tableau ζ with ℓ columns of weights other than r_2 appears

$$\frac{\binom{n}{k}\binom{n-k}{\ell}}{\binom{\ell+k}{\ell}} = \binom{n}{\ell+k}$$

times in the collection (cf. [9]). It implies that

$$q^{-\binom{k}{2}} W_{m,r,q}(n,k) = \sum_{\ell=0}^{n-k} \binom{n}{\ell+k} r_2^{n-k-\ell} \sum_{\zeta \in \bar{B}_\ell} \prod_{\bar{c} \in \zeta} \omega^*(|\bar{c}|),$$

where $\bar{B}_\ell$ is the set of all tableaux ζ having ℓ columns of weights $\omega^*(j_i)$. Reindexing the sums yield

$$q^{-\binom{k}{2}} W_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{n}{j} r_2^{n-j} \sum_{\zeta \in \bar{B}_{j-k}} \prod_{\bar{c} \in \zeta} \omega^*(|\bar{c}|). \tag{43}$$

Since $\bar{B}_{n-j} = T^A(k, n-j)$, then

$$\sum_{\zeta \in \bar{B}_{j-k}} \prod_{\bar{c} \in \zeta} \omega^*(|\bar{c}|) = q^{-\binom{k}{2}} W_{m,r_1,q}(j,k). \tag{44}$$

Moreover, by Eqs. (43) and (44), we obtain

$$q^{-\binom{k}{2}} W_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{n}{j} r_2^{n-j} q^{-\binom{k}{2}} W_{m,r_1,q}(j,k) \tag{45}$$

which is equivalent to the second desired result. □

Let $r_1 = r-1$ and $r_2 = 1$ in Eqs. (38) and (39). Then.

$$w_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{j}{k} (-1)^{j-k} w_{m,r-1,q}(n,j) \tag{46}$$

and

$$W_{m,r,q}(n,k) = \sum_{j=k}^{n} \binom{n}{j} W_{m,r-1,q}(j,k). \tag{47}$$

These identities were first seen in [21, Theorem 9]. Now, using Eq. (39), the (q,r)-Dowling numbers $D_{m,r,q}(n)$ [21] may be expressed as

$$\begin{aligned} D_{m,r,q}(n) &= \sum_{k=0}^{n} W_{m,r,q}(n,k) \\ &= \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} W_{m,r-1,q}(j,k) \\ &= \sum_{j=0}^{n} \binom{n}{j} \sum_{k=0}^{j} W_{m,r-1,q}(j,k) \\ &= \sum_{j=0}^{n} \binom{n}{j} D_{m,r-1,q}(j). \end{aligned}$$

Moreover, by applying the binomial inversion formula [8]

$$f_n = \sum_{j=0}^{n} \binom{n}{j} g_j \iff g_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f_j$$

to this identity gives

$$D_{m,r-1,q}(n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} D_{m,r,q}(j).$$

These results are formally stated in the following corollary:

Corollary 7. *The (q, r)-Dowling numbers satisfy the recurrence relations with respect to r given by*

$$D_{m,r+1,q}(n) = \sum_{j=0}^{n} \binom{n}{j} D_{m,r,q}(j) \tag{48}$$

and

$$D_{m,r,q}(n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} D_{m,r+1,q}(j). \tag{49}$$

Remark 8. When $q \to 1$ and $m = \beta$, we obtain the following identities by Corcino and Corcino [11]:

$$G_{n,\beta,r+1} = \sum_{j=0}^{n} \binom{n}{j} G_{j,\beta,r} \tag{50}$$

$$G_{n,\beta,r} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} G_{j,\beta,r+1}, \tag{51}$$

where $G_{n,\beta,r} := D_{\beta,r,1}(n)$ is the generalized Bell numbers in [10, 11]. These identities were used to identify the Hankel transform of $G_{n,\beta,r}$.

Looking at the previous corollary, we see that the sequence $\big(D_{m,r+1,q}(n)\big)$ is the binomial transform of the sequence $\big(D_{m,r,q}(n)\big)$, for $r = 0, 1, 2, \ldots$. Using "Layman's Theorem" [16], $\big(D_{m,0,q}(n)\big)$, $\big(D_{m,1,q}(n)\big)$, $\big(D_{m,2,q}(n)\big), \ldots, \big(D_{m,r,q}(n)\big), \ldots$ have the same Hankel transform. This directs our attention to the following open problem:

Problem 9. Is it possible to identify the Hankel transform of $D_{m,r,q}(n)$ using a method parallel to what is being done in [11] for $G_{n,\beta,r}$?

3.2 Convolution-type identities

Recall that for any two sequences a_n and b_n, we call the sequence c_n as convolution sequence if

$$c_n = \sum_{k=1}^{n} a_n b_{n-k}, \;\; n = 0, 1, 2, \ldots. \tag{52}$$

One of the most famous convolution-type identity is the Vandermonde's formula [6, 8] given by

$$\binom{a+b}{n} = \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k}. \tag{53}$$

The following theorem contains convolution-type identities for the (q, r)-Whitney numbers of the first kind which will be proved using the combinatorics of A-tableaux:

Theorem 10. *The (q,r)-Whitney numbers of the first kind have convolution-type identities given by*

$$w_{m,r,q}(p+j,n) = q^{-pj}\sum_{k=0}^{n} w_{m,r,q}(p,k)w_{\bar{m},\bar{r},q}(j,n-k) \tag{54}$$

and

$$w_{m,r,q}(n+1,j+p+1) = \sum_{k=0}^{n} q^{k^2-nk-n} w_{m,r,q}(k,p)w_{\bar{m},\bar{r},q}(n-k,j), \tag{55}$$

where $\bar{m} = mq^p$ and $\bar{r} = m[p]_q + r$.

Proof. For $A_1 = \{0,1,2,\ldots,p-1\}$ and $A_2 = \{p,p+1,p+2,\ldots,p+j-1\}$, let $\Phi_1 \in T_d^{A_1}(p-1,p-k)$ and $\Phi_2 \in T_d^{A_2}(j-1,j-n+k)$. Note that by joining the columns of the tableaux Φ_1 and Φ_2, we may generate an A-tableau Φ with $p+j-n$ distinct columns whose lengths are in the set $A = \{0,1,2,\ldots,p+j-1\}$. That is, $\Phi \in T_d^A(p+j-1,p+j-n)$. Hence,

$$\sum_{\Phi\in T_d^A(p+j-1,p+j-n)} \Omega_A(\Phi) = \sum_{k=0}^{n}\left\{\sum_{\Phi_1\in T_d^{A_1}(p-1,p-k)}\Omega_{A_1}(\Phi_1)\right\}\left\{\sum_{\Phi_2\in T_d^{A_2}(j-1,j-n+k)}\Omega_{A_2}(\Phi_2)\right\}.$$

Note that in the right-hand side, we get

$$\begin{aligned}
\sum_{\Phi_2\in T_d^{A_2}(j-1,j-n+k)}\Omega_{A_2}(\Phi_2) &= \sum_{p\le g_1<g_2<\cdots<g_{j-n+k}\le p+j-1}\prod_{i=1}^{j-n+k}(m[g_i]_q+r)\\
&= \sum_{0\le g_1<g_2<\cdots<g_{j-n+k}\le j-1}\prod_{i=1}^{j-n+k}(m[p+g_i]_q+r)\\
&= \sum_{0\le g_1<g_2<\cdots<g_{j-n+k}\le j-1}\prod_{i=1}^{j-n+k}(mq^p[g_i]_q+([p]_q+r))\\
&= (-1)^{j-n+k}q^{\binom{j}{2}}w_{\bar{m},\bar{r},q}(j,n-k),
\end{aligned}$$

where $\bar{m} = mq^p$ and $\bar{r} = m[p]_q + r$. Also, using Eq. (34),

$$\sum_{\Phi_1\in T_d^{A_1}(p-1,p-k)}\Omega_{A_1}(\Phi_1) = (-1)^{p-k}q^{\binom{p}{2}}w_{m,r,q}(p,k)$$

and

$$\sum_{\Phi\in T_d^A(p+j-1,p+j-n)}\Omega_A(\Phi) = (-1)^{p+j-n}q^{\binom{p+j}{2}}w_{m,r,q}(p+j,n).$$

Hence, by simplification, we obtain the convolution identity (54). Similarly, we let Φ_1 be a tableau with $k-p$ columns whose lengths are in $B_1 = \{0, 1, 2, \ldots, k-1\}$ and Φ_2 be a tableau with $n-k-j$ columns whose lengths are in $B_2 = \{k+1, k+2, \ldots, n\}$ so that $\Phi \in T_d^{B_1}(k-1, k-p)$ and $\Phi \in T_d^{B_2}(n-k-1, n-k-j)$. Note that we may generate an A-tableau Φ by joining the columns of Φ_1 and Φ_2 whose lengths are in $A = \{0, 1, 2, \ldots, n\}$. Hence, we have

$$\sum_{\Phi \in T_d^A(n,n-j-p)} \Omega_A(\Phi) = \sum_{k=0}^{n} \left\{ \sum_{\Phi_1 \in T_d^{B_1}(k-1,k-p)} \Omega_{B_1}(\Phi_1) \right\} \left\{ \sum_{\Phi_2 \in T_d^{B_2}(n-k-11,n-k-j)} \Omega_{B_2}(\Phi_2) \right\}.$$

Applying Eq. (34) gives

$$\sum_{\Phi \in T_d^A(n,n-j-p)} \Omega_A(\Phi) = (-1)^{n-j-p} q^{\binom{n+1}{2}} w_{m,r,q}(n+1, j+p+1)$$

and

$$\sum_{\Phi_1 \in T_d^{B_1}(k-1,k-p)} \Omega_{B_1}(\Phi_1) = (-1)^{k-p} q^{\binom{k}{2}} w_{m,r,q}(k, p).$$

Also, in the right-hand side, we get

$$\begin{aligned}
\sum_{\Phi_2 \in T_d^{B_2}(n-k-11,n-k-j)} \Omega_{B_2}(\Phi_2) &= \sum_{p \le g_1 < g_2 < \cdots < g_{n-k-j} \le p+n-k-1} \prod_{i=1}^{n-k-j} (m[g_i]_q + r) \\
&= \sum_{0 \le g_1 < g_2 < \cdots < g_{n-k-j} \le n-k-1} \prod_{i=1}^{n-k-j} (m[p+g_i]_q + r) \\
&= \sum_{0 \le g_1 < g_2 < \cdots < g_{n-k-j} \le n-k-1} \prod_{i=1}^{n-k-j} (mq^p[g_i]_q + ([p]_q + r)) \\
&= (-1)^{n-k-j} q^{\binom{n-k}{2}} w_{\bar{m},\bar{r},q}(n-k, j),
\end{aligned}$$

where $\bar{m} = mq^p$ and $\bar{r} = m[p]_q + r$. This completes the proof. □

The next theorem can be proved similarly.

Theorem 11. *The (q, r)-Whitney numbers of the second kind have convolution-type identities given by*

$$W_{m,r,q}(n+1, j+p+1) = \sum_{k=0}^{n} q^{p+pj+j} W_{m,r,q}(k, p) W_{\hat{m},\hat{r},q}(n-k, j) \tag{56}$$

and

$$W_{m,r,q}(p+j,n)=\sum_{k=0}^{n}q^{nk-k^2}W_{m,r,q}(p,k)W_{\hat{m},\hat{r},q}(j,n-k), \tag{57}$$

where $\hat{m}=mq^{p+1}$ *and* $\hat{r}=m[p+1]_q+r$*.*

As $q\to 1$, we recover from Theorems 10 and 11 the results recently obtained by Xu and Zhou [27, Theorems 2.1 and 2.4].

4 On Heine and Euler distributions

Consider the Poisson distribution

$$f_X(x)=e^{-\lambda}\frac{\lambda^x}{x!}, \tag{58}$$

for $x=0,1,2,\ldots$. The factorial moment of a Poisson random variable is readily evaluated, i.e.,

$$E\big[(X)_n\big]=\lambda^n \tag{59}$$

the mean, $E\big[X\big]=\lambda$, being the special case $n=1$. Expanding x^n in terms of falling factorials (using the Stirling numbers of the second kind), we obtain the n-th moment of X given by

$$E\big[X^n\big]=B_n(\lambda), \tag{60}$$

where $B_n(\lambda)$ are the Bell polynomials. The q-analogues of the Poisson distribution introduced by Kemp [15], and Benkherouf and Bather in [3] are given by

$$f_Y(y)=e_q(-\lambda)q^{\binom{y}{2}}\frac{\lambda^y}{[y]_q!}, y=0,1,2,\ldots \tag{61}$$

and

$$f_Z(z)=\widehat{e}_q(-\lambda)\frac{\lambda^z}{[z]_q!}, z=0,1,2,\ldots. \tag{62}$$

These are called Heine and Euler distributions, respectively, where

$$e_q(t)=\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!} \tag{63}$$

and

$$\widehat{e}_q(t)=\sum_{k=0}^{\infty}q^{\binom{k}{2}}\frac{t^k}{[k]_q!}. \tag{64}$$

In line with this, Charalambides and Papadatos [5] obtained the following important results:

$$E\big[[Y]_{r,q}\big]=\frac{q^{\binom{r}{2}}\lambda^r}{\prod_{i=1}^{r}(1+\lambda(1-q)q^{i-1})}, \tag{65}$$

$$E\big[[Z]_{r,q}\big] = \lambda^r, \tag{66}$$

where $[x]_{r,q} = [x]_q[x-1]_q[x-2]_q\cdots[x-r+1]_q$ is the q-falling factorial of x of order r. Considering these, we now state the following theorem:

Theorem 12. *If Y and Z are random variables with Heine and Euler distributions, respectively, and if the mean of Y is $\phi = \frac{\lambda}{1+\lambda(1-q)}$ and the mean of Z is λ, then*

$$E_\phi\big[(m[Y]_q + r)^n\big] = \sum_{\ell=0}^{n}\sum_{i=0}^{n}(-\lambda)^i q^{-\binom{\ell}{2}-\ell i}\frac{\lambda^\ell}{[\ell]_q![i]_q!}\frac{(m[\ell]_q + r)^n}{\prod_{j=1}^{\ell+i}(1+\lambda(1-q)q^{j-1})}, \tag{67}$$

$$E_\lambda\big[(m[Z]_q + r)^n\big] = \widehat{e}_q(-\lambda)\sum_{\ell=0}^{n}\frac{\lambda^\ell}{[\ell]_q!}(m[\ell]_q + r)^n. \tag{68}$$

Proof. From the defining relation in (1) and the result in (65),

$$E_\lambda\big[(m[Y]_q + r)^n\big] = \sum_{k=0}^{n} m^k W_{m,r,q}(n,k)\frac{q^{\binom{k}{2}}\lambda^k}{\prod_{j=1}^{k}(1+\lambda(1-q)q^{j-1})}.$$

Using the explicit formula for the (q,r)-Whitney numbers of the second kind [21, Theorem 16] given by

$$W_{m,r,q}(n,k) = \frac{1}{m^k[k]_q!}\sum_{\ell=0}^{k}(-1)^{k-\ell}q^{\binom{k-\ell}{2}}\binom{k}{\ell}_q (m[\ell]_q + r)^n, \tag{69}$$

we obtain

$$\begin{aligned}
E_\lambda\big[(m[Z]_q + r)^n\big] &= \sum_{k=0}^{n}\left\{\frac{1}{[k]_q!}\sum_{\ell=0}^{k}(-1)^{k-\ell}q^{\binom{k-\ell}{2}}\binom{k}{\ell}_q (m[\ell]_q + r)^n\right\} \\
&\quad \times\frac{q^{\binom{k}{2}}\lambda^k}{\prod_{j=1}^{k}(1+\lambda(1-q)q^{j-1})} \\
&= \sum_{\ell=0}^{n}\sum_{k=\ell}^{n}(-1)^{k-\ell}q^{\binom{k-\ell}{2}-\binom{k}{2}}\frac{\lambda^k}{[\ell]_q![k-\ell]_q!}\frac{(m[\ell]_q + r)^n}{\prod_{j=1}^{k}(1+\lambda(1-q)q^{j-1})}.
\end{aligned}$$

Reindexing the second sum yields (67). Eq. (68) may be shown similarly. □

Remark 13. When $m = 1$ and $r = 0$ in the previous theorem, we have

$$E_\phi\big[[Y]_q^n\big] = \sum_{\ell=0}^{n}\sum_{i=0}^{n}(-\lambda)^i q^{-\binom{\ell}{2}-\ell i}\frac{\lambda^\ell}{[\ell]_q![i]_q!}\frac{[\ell]_q^n}{\prod_{j=1}^{\ell+i}(1+\lambda(1-q)q^{j-1})}, \tag{70}$$

and

$$E_\lambda\big[[Z]_q^n\big] = \widehat{e}_q(-\lambda)\sum_{\ell=0}^{n}\frac{\lambda^\ell}{[\ell]_q!}[\ell]_q^n \equiv B_{n,q}(\lambda), \tag{71}$$

where $B_{n,q}(\lambda)$ is the q-Bell polynomials. On the other hand, if the mean is $\lambda = \frac{x}{m}$,

$$E_{x/m}\left\{(m[Z]_q + r)^n\right\} = \widehat{e}_q\left(-\frac{x}{m}\right)\sum_{\ell=0}^{n}\frac{x^\ell}{m^\ell}\frac{(m[\ell]_q + r)^n}{[\ell]_q!}.$$

This explicit formula is due to Mangontarum and Katriel [21]. Thus

$$E_{x/m}\big[(m[Z]_q + r)^n\big] = D_{m,r,q}(n,x),$$

where

$$D_{m,r,q}(n,x) = \sum_{k=0}^{n} W_{m,r,q}(n,k)x^k \tag{72}$$

is the (q,r)-Dowling polynomials.

It is worth mentioning that Mangontarum and Corcino [19] obtained the following pair of n-th order generalized factorial moments

$$E_\lambda\big[(\beta X + \gamma|\alpha)_n\big] = e^{-\lambda}\sum_{i=0}^{\infty}\frac{(i\beta + \gamma|\alpha)_n}{i!}\lambda^i \tag{73}$$

$$E_\lambda\big[(\alpha X - \gamma|\beta)_n\big] = e^{-\lambda}\sum_{i=0}^{\infty}\frac{(i\alpha - \gamma|\beta)_n}{i!}\lambda^i, \tag{74}$$

where X is a Poisson random variable with mean λ and α, β and γ may be real or complex numbers. Here,

$$(t|\alpha)_n = t(t-\alpha)(t-2\alpha)\cdots(t-n\alpha+\alpha), \tag{75}$$

with initial conditions $(t|\ \alpha)_n = 0$ when $n \leq 0$ and $(t|\ \alpha)_0 = 1$. Notice that (73) unifies the factorial moment in (59) and the n-th moment in (60). More precisely,

- when $\beta = 1$, $\gamma = 0$ and $\alpha = 0$,

$$E_\lambda\big[(\beta X + \gamma|\alpha)_n\big] = E_\lambda\big[X^n\big];$$

- when $\beta = 1$, $\gamma = 0$ and $\alpha = 1$,

$$E_\lambda\big[(\beta X + \gamma|\alpha)_n\big] = E_\lambda\big[(X)_n\big].$$

Other known "Bell-type" and "Dowling-type" polynomials (see [7, 10, 18, 22, 24, 25]) can be shown to be particular cases of Eqs. (73) and (74). Furthermore, Corcino and Mangontarum [13] obtained the generalized q-factorial moments

$$E_\phi\big[\,[[\beta Y]_q+[\gamma]_q|[\alpha]_q]_{n,q}\,\big]=\sum_{j=0}^{\infty}\hat{e}_{q^\beta,j}(-\lambda)\frac{(q^\beta\lambda)^j\,[[\beta j]_q+[\gamma]_q|[\alpha]_q]_{n,q}}{[j]_{q^\beta}!\prod_{i=1}^{j}(1+\lambda(1-q^\beta)q^{\beta(i-1)})}\tag{76}$$

and

$$E_\lambda\big[\,[[\beta Z]_q+[\gamma]_q|[\alpha]_q]_{n,q}\,\big]=\hat{e}_b(-\lambda)\sum_{j=0}^{\infty}[[\beta j]_q+[\gamma]_q|[\alpha]_q]_{n,q}\,\frac{\lambda^j}{[j]_b!},\tag{77}$$

where Y is a random variable with Heine distribution and mean $\phi=\frac{\lambda}{1+\lambda(1-q^\beta)}$, and Z is a random variable with an Euler distribution and mean λ. The notations

$$[[\beta Z]_q+[\gamma]_q|[\alpha]_q]_{n,q}=\prod_{j=0}^{n-1}([\beta t]_q+[\gamma]_q-[\alpha j]_q)\tag{78}$$

and

$$\hat{e}_{q^\beta,j}(-\lambda)=\sum_{l=0}^{\infty}\left[\frac{q^{\beta\binom{j}{2}}(-\lambda)^l}{[l]_{q^\beta}!\prod_{i=1}^{l}(q^{\beta(i-1)}+\lambda(1-q^\beta)q^{\beta j})}\right]\tag{79}$$

are used. (76) and (77) are found to be q-analogues of (73). By thoroughly investigating (68), it is obvious that this result is not generalized by (76) and (77).

Privault [25] defined an extension of the classical Bell numbers as

$$e^{ty-\lambda(e^t-t-1)}=\sum_{k=0}^{\infty}B_n(y,\lambda)\frac{t^k}{k!}.$$

Moreover, he obtained the following n-th moment of a Poisson random variable

$$E_\lambda\big[(X+y-\lambda)^n\big]=B_n(y,-\lambda),\tag{80}$$

where

$$B_n(y,-\lambda)=\sum_{k=0}^{n}\binom{n}{k}(y-\lambda)^{n-k}\sum_{j=0}^{k}\left\{{k \atop j}\right\}\lambda^j,\tag{81}$$

Corcino and Corcino [10] showed that the (r,β)-Bell polynomials satisfy

$$G_{n,\beta,r}(x)=\sum_{k=0}^{n}\binom{n}{k}r^{n-k}\sum_{j=0}^{k}\beta^{k-j}\left\{{k \atop j}\right\}x^j.\tag{82}$$

It then follows that

$$G_{n,1,y-\lambda}(\lambda)=B_n(y,-\lambda).$$

The next theorem is analogous to these identities.

Theorem 14. *The (q,r)-Dowling polynomials satisfy the identity*

$$D_{m,r,q}(n,x) = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \sum_{j=0}^{k} m^{k-j} \left\{ {k \atop j} \right\}_q x^j. \tag{83}$$

Proof. Using the binomial theorem, we have

$$\begin{aligned} E_{x/m}\big[(m[Z]_q + r)^n\big] &= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} m^k E_{x/m}\big[[Z]_q^k\big] \\ &= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} m^k B_{n,q}\left(\frac{x}{m}\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} r^{n-k} m^k \sum_{j=0}^{k} \left\{ {k \atop j} \right\}_q \left(\frac{x}{m}\right)^j. \end{aligned}$$

The desired result follows from the fact that $E_{x/m}\big[(m[Z]_q + r)^n\big] = D_{m,r,q}(n,x)$. ☐

Remark 15. As $q \to 1$, we obtain the (r,β)-Bell polynomial identity in Eq. (82). If the mean is replaced with λ, the for an Euler random variable Z,

$$E_\lambda\big[(m[Z]_q + r)^n\big] = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \sum_{j=0}^{k} m^k \left\{ {k \atop j} \right\}_q \lambda^j.$$

As $q \to 1$, we get [19, Eq. 34]

$$E_\lambda\big[(mX + r)^n\big] = \sum_{k=0}^{n} \binom{n}{k} r^{n-k} \sum_{j=0}^{k} m^k \left\{ {k \atop j} \right\} \lambda^j.$$

When $m = 1$ and $r = y - \lambda$,

$$D_{1,y-\lambda,q}(n,x) = \sum_{k=0}^{n} \binom{n}{k} (y-\lambda)^{n-k} \sum_{j=0}^{k} \left\{ {k \atop j} \right\}_q x^j. \tag{84}$$

This is a q-analogue of Privault's identity since (84) $\to$ (81) as $q \to 1$.

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