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The rth Moment of the Divisor Function: An Elementary Approach

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Abstract

For integer $r \geq 1$ we give an elementary proof for the main term of the asymptotic behavior of the rth moment of the number of divisors of n for positive integers $n \leq x$.

1 Introduction

Let $\tau(n)$ be the number of divisors of n. Ramanujan [2] stated without proof that, given any real number $\varepsilon > 0$, the estimate

$$\sum_{n\le x} \tau(n)^2 = x(A(\log x)^3 + B(\log x)^2 + C\log x + D) + O(x^{3/5+\varepsilon})$$

holds with $A = \pi^{-2}$. An elementary proof of the asymptotic formula

$$\sum_{n\le x} \tau(n)^2 \sim Ax(\log x)^3,$$

as $x \to \infty$, appears in several places (see, for example, [1, Thm. 7.8]). Wilson [3] proved Ramanujan's claim and generalized it by showing that for any integer $r \ge 2$ one has

$$\sum_{n\le x} \tau(n)^r = x(C_{r,1}(\log x)^{2^r-1} + C_{r,2}(\log x)^{2^r-2} + \cdots + C_{r,2^r}) + O(x^{\frac{2^r-1}{2^r+2}+\varepsilon}).$$

Note that when $r = 2$, Wilson's error term is better than the one claimed by Ramanujan. We are not aware even of elementary proofs for the asymptotic formula

$$\sum_{n\le x} \tau(n)^r \sim C_r x(\log x)^{2^r-1}$$

as $x \to \infty$ for any $r \ge 2$. In this note, we give an elementary proof of the following more general result.

Theorem 1. *Let k be a positive integer and $f(n)$ be a multiplicative function which on prime powers p^α satisfies*

$$f(p) = k \quad \text{and} \quad f(p^\alpha) = \alpha^{O(1)} \quad \textit{for all primes } p \textit{ and integers } \alpha \ge 2,$$

where the constant implied by the above O is uniform in p. Then

$$\sum_{n\le x} f(n) = xC_f(\log x)^{k-1} + O(x(\log x)^{k-2})$$

where

$$C_f = \frac{1}{(k-1)!}\left(\prod_{p\ge 2}\left(1-\frac{1}{p}\right)^k\left(\sum_{\alpha\ge 0}\frac{f(p^\alpha)}{p^\alpha}\right)\right).$$

In the case $f(n) = \tau(n)^r$ for integer $r \ge 1$, Theorem 1 applies with $k = 2^r$.

The only facts that we use are Abel's summation formula, the Möbius inversion formula, the elementary estimate

$$\sum_{n\le t} \frac{1}{n} = \log t + \gamma + O(1/t) \tag{1}$$

valid for all real $t \ge 1$, and the fact that the counting function of the *squarefull* numbers $s \le t$ is $O(t^{1/2})$, where s is squarefull if and only if $p^2 \mid s$ for all prime factors p of s, all provable by elementary means.

2 A lemma

Lemma 2. *Assume that r is a positive integer and $f(n)$ is some arithmetic function such that*

$$\sum_{n\le x} f(n) = \sum_{j=0}^{r} c_j(\log x)^j + O(x^{-1/2+o(1)}), \tag{2}$$

for some constants c_j, $j = 0,\ldots,r$. Then

$$\sum_{n\le x} f(n)(\log(x/n))^k = \sum_{\ell=0}^{k+r} C_\ell(\log x)^\ell + O(x^{-1/2+o(1)}), \tag{3}$$

holds for all positive integers k with some constants $C_0,\ldots,C_{k+r}$. Here, if $\ell \in \{k, k+1,\ldots,k+r\}$, then

$$C_\ell := c_{\ell-k}\left(1 + (\ell-k)\sum_{i=1}^{k} \frac{(-1)^i}{\ell-k+i}\binom{k}{i}\right). \tag{4}$$

Furthermore, if $r \ge t \ge 1$ are positive integers and

$$\sum_{n\le x} f(n) = \sum_{j=t}^{r} c_j(\log x)^j + O((\log x)^{t-1}), \tag{5}$$

then

$$\sum_{n\le x} f(n)(\log(x/n))^k = \sum_{j=k+t}^{k+r} C_j(\log x)^j + O((\log x)^{t+k-1}). \tag{6}$$

Proof. We show how to deduce (3) out of (2) with the leading coefficients given by (4). Let

$$A(x) = \sum_{n\le x} f(n).$$

Then

$$A(x) = \sum_{j=0}^{r} c_j(\log x)^j + R(x),$$

where $|R(x)| = x^{-1/2+o(1)}$ as $x \to \infty$. Let $i \ge 1$. Put

$$B_i(x) := \sum_{n\le x} f(n)(\log n)^i.$$

Then, by the Abel summation formula and by interchanging the order between the summation and the integration, we get

$$
\begin{aligned}
B_i(x) &= A(x)(\log x)^i - i\int_1^x A(t)\left(\frac{(\log t)^{i-1}}{t}\right)dt \\
&= \sum_{j=0}^{r}\left(c_j(\log x)^{j+i} - i\int_1^x \left(\frac{c_j(\log t)^{j+i-1}}{t}\right)dt\right) \\
&- i\int_1^x \frac{(\log t)^{i-1}R(t)}{t}dt + R(x)(\log x)^i \\
&= \sum_{j=0}^{r}\left(c_j(\log x)^{j+i} - \frac{c_j i}{j+i}(\log t)^{j+i}\Big|_1^x\right) + \\
&- i\int_1^\infty \frac{(\log t)^{i-1}R(t)}{t}dt + i\int_x^\infty \frac{(\log t)^{i-1}R(t)}{t}dt + R(x)(\log x)^i \\
&= \sum_{j=0}^{r}\frac{c_j j}{j+i}(\log x)^{j+i} + D_i + O(x^{-1/2+o(1)}),
\end{aligned}
$$

where

$$
D_i := -i\int_1^\infty \frac{(\log t)^{i-1}R(t)}{t}dt
$$

In the above, we used the fact that $|R(t)| \le t^{-1/2+o(1)}$ as $t \to \infty$ to deduce that the above integral converges and that its tail from x to infinity as well as the other errors are $O(x^{-1/2+o(1)})$ as $x \to \infty$. Using the binomial formula and the above arguments, we have

$$
\begin{aligned}
C_k(x) &:= \sum_{n\le x} f(n)(\log(x/n))^k \\
&= \sum_{i=0}^{k}(-1)^i\binom{k}{i}(\log x)^{k-i}\sum_{n\le x} f(n)(\log n)^i \\
&= \sum_{n\le x} f(n) + \sum_{i=1}^{k}(-1)^i\binom{k}{i}(\log x)^{k-i}B_i(x) \\
&= \sum_{\ell=0}^{k+r} C_\ell(\log x)^\ell + O(x^{-1/2+o(1)}),
\end{aligned}
$$

where C_ℓ are given by formula (4) for $\ell \ge k$. For $\ell = 1, \ldots, k-1$, the coefficient C_ℓ involves the expression D_ℓ. The deduction of (6) out of (5) is immediate by similar arguments. $\square$

3 The proof of Theorem 1

Let $f_0(n) := f(n)$. Recursively define $f_j(n)$ such that

$$f_{j-1}(n) = \sum_{d|n} f_j(d), \quad j = 1, 2, \ldots.$$

By Möbius inversion,

$$f_j(n) = \sum_{d|n} \mu(d) f_{j-1}(n/d).$$

On primes

$$f_j(p) = f_{j-1}(p) - 1, \quad j = 1, 2, \ldots.$$

Since $f_0(p) = k$, we get that $f_j(p) = k - j$. In particular, $f_k(p) = 0$. Further, for $\alpha \geq 2$, we have that

$$f_j(p^\alpha) = f_{j-1}(p^\alpha) - f_{j-1}(p^{\alpha-1}).$$

Since $f_0(p^\alpha) = \alpha^{O(1)}$ it follows that $f_j(p^\alpha) = \alpha^{O(1)}$ for all $j \geq 2$. The constant in $O(1)$ might depend on j. Further,

$$\sum_{\alpha \geq 0} \frac{f_j(p^\alpha)}{p^\alpha} = \left(1 - \frac{1}{p}\right) \sum_{\alpha \geq 0} \frac{f_{j-1}(p^\alpha)}{p^\alpha}, \quad j = 1, 2, \ldots,$$

therefore

$$\sum_{\alpha \geq 0} \frac{f_j(p^\alpha)}{p^\alpha} = \left(1 - \frac{1}{p}\right)^j \sum_{\alpha \geq 0} \frac{f(p^\alpha)}{p^\alpha}, \quad j = 0, 1, \ldots$$

Put

$$E_j := \prod_{p \geq 2} \left(\sum_{\alpha \geq 0} \frac{f_j(p^\alpha)}{p^\alpha} \right) = \prod_{p \geq 2} \left(\left(1 - \frac{1}{p}\right)^j \sum_{\alpha \geq 0} \frac{f(p^\alpha)}{p^\alpha} \right).$$

Fix $j \geq 1$. Then

$$F_{j-1}(x) := \sum_{n \leq x} \frac{f_{j-1}(n)}{n} = \sum_{n \leq x} \frac{1}{n} \sum_{d|n} f_j(d) = \sum_{d \leq x} f_j(d) \sum_{\substack{n \leq x \\ d|n}} \frac{1}{n}.$$

In the inner sum, we write an $n \leq x$ which is a multiple of d as $n = dm$ for some integer $m \leq x$. We get

$$\begin{aligned} F_{j-1}(x) &= \sum_{d \leq x} \frac{f_j(d)}{d} \sum_{m \leq x/d} \frac{1}{m} = \sum_{d \leq x} \frac{f_j(d)}{d} \left(\log(x/d) + \gamma + O(d/x)\right) \\ &= \sum_{d \leq x} \frac{f_j(d)}{d} \log(x/d) + \gamma \sum_{d \leq x} \frac{f_j(d)}{d} + O\left(\frac{1}{x} \sum_{d \leq x} |f_j(d)| \right) \qquad (7) \end{aligned}$$

for $j = 1, 2, \ldots$. When $j = k$, since $f_k(p) = 0$, it follows that $f_k(d) = 0$ if d is not squarefull. Thus, when $j = k$ in the right–hand side of (7), we have

$$\sum_{d \le x} \frac{f_k(d)}{d} \log(x/d) + \gamma \sum_{d \le x} \frac{f_k(d)}{d} + O\left(\frac{1}{x} \sum_{d \le x} |f_k(d)|\right).$$

Note that

$$\begin{aligned} \sum_{d \le x} \frac{f_k(d)}{d} &= \sum_{d \ge 1} \frac{f_k(d)}{d} + O\left(\sum_{d > x} \frac{|f_k(d)|}{d}\right) = E_k + O\left(\sum_{\substack{d \ge x \\ d \text{ squarefull}}} \frac{1}{d^{1+o(1)}}\right) \\ &= E_k + O(x^{-1/2+o(1)}), \end{aligned} \tag{8}$$

where for the error term we used the fact that $|f_k(d)| = |\tau(d)|^{O(1)} = d^{o(1)}$ as $d \to \infty$ and the Abel summation formula to conclude that

$$\sum_{\substack{d > x \\ d \text{ squarefull}}} \frac{1}{d^{1+o(1)}} \le x^{-1/2+o(1)} \quad \text{as} \quad x \to \infty.$$

Further, we have

$$\begin{aligned} \sum_{d \le x} \frac{f_k(d)}{d}(-\log d + \gamma) &= \sum_{d \ge 1} \frac{f_k(d)(-\log d + \gamma)}{d} + O\left(\sum_{\substack{d > x \\ d \text{ squarefull}}} \frac{|f_k(d)| \log d}{d}\right) \\ &:= F_k + O(x^{-1/2+o(1)}) \end{aligned} \tag{9}$$

as $x \to \infty$, by a similar argument since $|f_k(d)| \log d \le d^{o(1)}$ as $d \to \infty$. Finally

$$\sum_{d \le x} |f_k(d)| \le x^{1/2+o(1)}, \tag{10}$$

again since $f_k(d) = 0$ if d is not squarefull. Collecting (8), (9) and (10) and putting them into (7) with $j = k$, we get

$$F_{k-1}(x) = \sum_{n \le x} \frac{f_{k-1}(n)}{n} = E_k \log x + F_k + O(x^{-1/2+o(1)}).$$

In a similar way,

$$G_{k-1}(x) := \sum_{n \le x} \frac{|f_{k-1}(n)|}{n} = E'_k \log x + F'_k + O(x^{-1/2+o(1)}).$$

for some (maybe different) constants E'_k and F'_k. We now apply Lemma 2 in order to find recursively $F_{k-2}(x), F_{k-3}(x), \ \ldots, \ F_0(x)$. We claim, by induction on j, that

$$F_{k-j}(x) = A_j(\log x)^j + B_j(\log x)^{j-1} + O((\log x)^{j-2}) \tag{11}$$

for $j = 2, \ldots, k$. At $j = 1$, this is so with $A_1 = E_k$, $B_1 = F_k$ and the error term is better, namely $O(x^{-1/2+o(1)})$. In order to realize the induction step from $j = 1$ to $j = 2$, we use the first part of Lemma 1 with $r = 1$, whereas for the induction step from $j \geq 2$ to $j+1$ we use the second part of Lemma 2 with $r = j$ and $t = j-1$. Assuming that (11) holds for $j \geq 1$, we have, by (7),

$$\begin{aligned} F_{k-j-1}(x) &= \sum_{d\leq x} \frac{f_{k-j-1}(d)}{d} = \sum_{d\leq x} \frac{f_{k-j}(d)}{d}\log(x/d) + \gamma \sum_{d\leq x}\frac{f_{k-j}(d)}{d} \\ &+ O\left(\frac{1}{x}\sum_{d\leq x}|f_{k-j}(d)|\right). \end{aligned}$$

By Lemma 2, we get that the right hand side is

$$\begin{aligned} &\frac{A_j}{j+1}(\log x)^{j+1} + \left(\frac{B_j}{j} + \gamma A_j\right)(\log x)^j \\ + \quad & O\left((\log x)^{j-1} + \frac{1}{x}\sum_{d\leq x}|f_{k-j}(d)|\right) \\ &:= A_{j+1}(\log x)^{j+1} + B_{j+1}(\log x)^j + O\left((\log x)^{j-1} + \frac{1}{x}\sum_{d\leq x}|f_{k-j}(d)|\right), \end{aligned}$$

where

$$A_{j+1} = \frac{A_j}{j+1}, \quad \text{and} \quad B_{j+1} = \gamma A_j + \frac{B_j}{j}.$$

Thus, we note that $A_j = E_k/j!$. It remains to deal with the sum in the error term. But the exact same approach applies to $|f_{k-j}(n)|$. That is $g_0(n) = |f_{k-j}(n)|$ satisfies the same conditions as our initial $f_0(n)$ with k replaced by $k-j$. Thus,

$$\sum_{d\leq x}\frac{|f_{k-j}(d)|}{d} = C_j(\log x)^j + D_j(\log x)^{j-1} + O((\log x)^{j-2}),$$

where for $j = 1$, the error term is $O(x^{-1/2+o(1)})$ as $x \to \infty$. By Abel summation, we get that

$$\begin{aligned} \sum_{d\leq x}|f_{k-j}(d)| &= x(C_j(\log x)^j + D_j(\log x)^{j-1} + O((\log x)^{j-2})) \\ &- \int_1^x (C_j(\log t)^j + D_j(\log t)^{j-1} + O((\log t)^{j-2}))dt \\ &= O(x(\log x)^{j-1}), \end{aligned}$$

which is sufficient for us. This completes the induction procedure and shows that at $j = k$ we have

$$\sum_{n \le x} \frac{f(n)}{n} = \frac{1}{k!} E_k (\log x)^k + B_k (\log x)^{k-1} + O((\log x)^{k-2}).$$

Abel summation formula once again gives

$$\begin{aligned} \sum_{n \le x} f(n) &= \left(\frac{E_k}{k!} (\log x)^k + B_k (\log x)^{k-1} + O((\log x)^{k-2}) \right) x \\ &- \int_1^x \left(\frac{E_k}{k!} (\log t)^k + B_k (\log t)^{k-1} + O((\log t)^{k-2}) \right) dt \\ &= \frac{E_k}{(k-1)!} x (\log x)^{k-1} + O(x (\log x)^{k-2}), \end{aligned}$$

which is what we wanted.

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