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Width-k Generalizations of Classical Permutation Statistics

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Abstract

We introduce new natural generalizations of the classical descent and inversion statistics for permutations, called *width-k descents* and *width-k inversions*. These variations induce generalizations of the excedance and major statistics, providing a framework in which well-known equidistributivity results for classical statistics are paralleled. We explore additional relationships among the statistics providing specific formulas in certain special cases. Moreover, we explore the behavior of these width-k statistics in the context of pattern avoidance.

1 Introduction

Let $\mathfrak{S}_n$ denote the set of permutations $\sigma = a_1 \cdots a_n$ of $[n] = \{1, \ldots, n\}$, and let $\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \cdots$. A function $\mathrm{st} : \mathfrak{S}_n \to \mathbb{N}$ is called a *statistic*, and the systematic study of permutation statistics is generally accepted to have begun with MacMahon [6]. In particular, four of the most well-known statistics are the *descent*, *inversion*, *major*, and *excedance* statistics, defined respectively by

$$\begin{aligned}
\operatorname{des}\sigma &= |\{i \in [n-1] \mid a_i > a_{i+1}\}| \\
\operatorname{inv}\sigma &= |\{(i,j) \in [n]^2 \mid i < j \text{ and } a_i > a_j\}| \\
\operatorname{maj}\sigma &= \sum_{i \in \operatorname{Des}\sigma} i \\
\operatorname{exc}\sigma &= |\{i \in [n] \mid a_i > i\}|,
\end{aligned}$$

where $\operatorname{Des}\sigma = \{i \in [n-1] \mid a_i > a_{i+1}\}$.

Given any statistic st, one may form the generating function

$$F_n^{\mathrm{st}}(q) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{st}\,\sigma}.$$

A famous result due to MacMahon [6] states that $F_n^{\mathrm{des}}(q) = F_n^{\mathrm{exc}}(q)$, and that both are equal to the *Eulerian polynomial* $A_n(q)$. The Eulerian polynomials themselves may be defined via the identity

$$\sum_{j \geq 0} (1+j)^n q^j = \frac{A_n(q)}{(1-q)^{n+1}}.$$

Another well-known result, due to MacMahon [5] and Rodrigues [7], states that $F_n^{\mathrm{inv}}(q) = F_n^{\mathrm{maj}}(q) = [n]_q!$. Foata famously provided combinatorial proofs of the equidistributivity of each; Lothaire [4] gives a treatment of these. Thus any statistic st for which $F_n^{\mathrm{st}}(q) = A_n(q)$ is called *Eulerian*, and if $F_n^{\mathrm{st}}(q) = [n]_q!$ then st is called *Mahonian*. These four statistics have many generalizations; in this article, we discuss new variations, induced from a simple generalization of des.

For each of the following definitions, we assume $n \in \mathbb{Z}_{>0}$, $k \in [n-1]$, $\emptyset \neq K \subseteq [n-1]$, and $\sigma = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$. We define a *width-$k$ descent* of σ to be an index $i \subseteq [n-k]$ for which $a_i > a_{i+k}$. Thus the width-1 descents are the usual descents of a permutation. Let

$$\operatorname{Des}_k(\sigma) = \{i \in [n-k] \mid a_i > a_{i+k}\}$$

denote the set of all width-k descents of σ, and set

$$\operatorname{des}_k(\sigma) = |\operatorname{Des}_k(\sigma)|.$$

If one is interested in descents of σ of various widths, first let $K \subseteq [n-1]$ denote the set of widths under consideration. Then, define $\operatorname{Des}_K(\sigma)$ to be the multiset $\bigcup_{k \in K} \operatorname{Des}_k(\sigma)$, and $\operatorname{des}_K(\sigma) = |\operatorname{Des}_K(\sigma)|$.

Now, define a *width-k inversion* of σ to be a pair (i,j) for which $a_i > a_j$ and $j - i = mk$ for some positive integer m. Let

$$\operatorname{Inv}_k(\sigma) = \{(i,j) \in [n]^2 \mid a_i > a_j \text{ and } j - k = mk, k \in \mathbb{Z}\}$$

denote the set of width-k inversions of σ, and set

$$\operatorname{inv}_k(\sigma) = |\operatorname{Inv}_k(\sigma)|.$$

Again, one may be interested in width-k inversions for multiple values of k, so for $K \subseteq [n-1]$ define $\operatorname{Inv}_K(\sigma) = \bigcup_{k \in K} \operatorname{Inv}_k(\sigma)$. Additionally, let $\operatorname{inv}_K(\sigma) = |\operatorname{Inv}_K(\sigma)|$.

Example 1. If $\sigma = 4136572$, then

$$\operatorname{Des}_{\{2,3\}}(\sigma) = \{1,4,5\} \text{ and } \operatorname{Inv}_{\{2,3\}}(\sigma) = \{(1,3),(1,7),(3,7),(4,7),(5,7)\}.$$

Thus, $\operatorname{des}_{\{2,3\}}(\sigma) = 3$, $\operatorname{inv}_{\{2,3\}}(\sigma) = 5$.

As we will see in the next section, the above definitions motivate us to generalize exc and maj in such a way that well-known known relationships among $\mathrm{des}, \mathrm{inv}, \mathrm{maj}, \mathrm{exc}$ are paralleled. However, it is not very convenient to work with exc_K and maj_K directly. So, this paper will focus much more on des_K and inv_K.

The main focus of this paper is to explore these new statistics and their relationships among each other. While they are well-behaved for special cases of $K \subseteq [n-1]$, formulas for more general cases have been more elusive. Section 3 continues this exploration by considering the same statistics for classes of permutations avoiding a variety of patterns.

2 Main results

We begin with simple expressions for $\mathrm{des}_K(\sigma)$ and $\mathrm{inv}_K(\sigma)$ for a fixed $\sigma \in \mathfrak{S}_n$. First, we point out that if $K \subseteq [n-1]$, $j \in K$ and $ij \in K$ for some positive integer i, then for any $\sigma \in \mathfrak{S}_n$, $\mathrm{inv}_K(\sigma) = \mathrm{inv}_{K\setminus\{ij\}}(\sigma)$. This occurs because for any $k \in [n-1]$, $\mathrm{inv}_k(\sigma)$ counts all descents whose widths are multiples of k. Thus ij is already accounted for in $\mathrm{inv}_{K\setminus\{ij\}}(\sigma)$. This follows quickly from the definitions of inv_k and des_k.

Proposition 2. *For any nonempty $K \subseteq [n-1]$,*

$$\mathrm{inv}_K(\sigma) = \sum_{\emptyset \subsetneq K' \subseteq K} (-1)^{|K'|+1} \mathrm{inv}_{\mathrm{lcm}(K')}(\sigma),$$

where we set $\mathrm{inv}_{\mathrm{lcm}(K')}(\sigma) = 0$ if $\mathrm{lcm}(K') \geq n$.

Proof. We first consider the case where $K = \{k\}$. The elements of $\mathrm{Inv}_k(\sigma)$ are pairs of the form $(i, i + jk)$ for some positive integer j. Such an element exists if and only if there is a width-(jk) descent of σ at i. Thus, $\mathrm{inv}_k(\sigma)$ simply counts the number of width-jk descents of σ for all possible j. This leads to the equality

$$\mathrm{inv}_k(\sigma) = \sum_{j\geq 1} \mathrm{des}_{jk}(\sigma).$$

The formula for general K then follows from inclusion-exclusion: by adding the number of all width-k inversions for all $k \in K$, we are twice counting any instances of a width-$\mathrm{lcm}(k_1, k_2)$ inversion since such an inversion is also of widths k_1 and k_2. If K contains three distinct elements k_1, k_2, k_3, then $\mathrm{lcm}(k_1, k_2, k_3)$ would have been added three times (once for each $\mathrm{inv}_{k_i}(\sigma)$) and subtracted three times (once for each $\mathrm{inv}_{\{k_i,k_j\}}(\sigma)$, $i < j$), so it must be added again for the sum. Extending this argument to range over larger subsets of K results in the claimed formula. ∎

Example 3. Let us return to $\sigma = 4136572$. We saw from Example 1 that $\mathrm{inv}_{\{2,3\}}(\sigma) = 5$, where four inversions have width 2 and two have width 3. But since the inversion $(1, 7)$ has width both 2 and 3, it must also have width $\mathrm{lcm}(2, 3) = 6$. So, it contributes two summands of 1 and one summand of -1.

We come now to a function which helps demonstrate the interactions among the width-k statistics. Let n and k be positive integers for which $n = dk + r$ for some $d, r \in \mathbb{Z}$ with $0 \leq r < k$. To each $\sigma = a_1 \cdots a_n \in \mathfrak{S}_n$ we may then associate the set of disjoint substrings $\beta_{n,k}(\sigma) = \{\beta^1_{n,k}(\sigma), \ldots, \beta^k_{n,k}(\sigma)\}$ where

$$\beta^i_{n,k}(\sigma) = \begin{cases} a_i a_{i+k} a_{i+2k} \cdots a_{i+dk}, & \text{if } i \leq r; \\ a_i a_{i+k} a_{i+2k} \cdots a_{i+(d-1)k}, & \text{if } r < i < k. \end{cases}$$

Now, define

$$\phi : \mathfrak{S}_n \to \mathfrak{S}^r_{d+1} \times \mathfrak{S}^{k-r}_d$$

by setting $\phi(\sigma) = (\operatorname{std} \beta^1_{n,k}(\sigma), \ldots, \operatorname{std} \beta^k_{n,k}(\sigma))$, where std is the *standardization* map, that is, the permutation obtained by replacing the smallest element of σ with 1, the second-smallest element with 2, etc. Note in particular that each $\operatorname{std} \beta^i_{n,k}(\sigma)$ is a permutation of $[d+1]$ or $[d]$.

Example 4. Suppose that $\sigma = 829317645$, suppose $k = 4$. We then have

$$\begin{aligned} \beta_{9,4}(\sigma) &= (\operatorname{std} \beta^1_{9,4}(\sigma), \operatorname{std} \beta^2_{9,4}(\sigma), \operatorname{std} \beta^3_{9,4}(\sigma), \operatorname{std} \beta^4_{9,4}(\sigma)) \\ &= (\operatorname{std} 815, \operatorname{std} 27, \operatorname{std} 96, \operatorname{std} 34) \\ &= (312, 12, 21, 12). \end{aligned}$$

The first of the identities in the following proposition was originally established by Sack and Úlfarsson [8], though with slightly different notation. We provide an alternate proof and extend their result to width-k inversions.

Theorem 5. *Let n and k be positive integers such that $n = dk + r$, where $0 \leq r < k$, and let $A_i(q)$ denote the i^{th} Eulerian polynomial. Also let $M_{n,k}$ denote the multinomial coefficient*

$$M_{n,k} = \binom{n}{(d+1)^{*r}, d^{*(k-r)}}$$

where i^{*j} indicates i repeated j times. We then have the identities

$$\begin{aligned} F^{\operatorname{des}_k}_n(q) &= M_{n,k} A^r_{d+1}(q) A^{k-r}_d(q) \\ F^{\operatorname{inv}_k}_n(q) &= M_{n,k} [d+1]^r_q! [d]^{k-r}_q! \end{aligned}$$

Proof. Let $k \in [n-1]$ and consider ϕ defined above. Note that ϕ is an $M_{n,k}$-to-one function since, given $(\sigma_1, \ldots, \sigma_k) \in \mathfrak{S}^r_{d+1} \times \mathfrak{S}^{k-r}_d$, there are $M_{n,k}$ ways to partition $[n]$ into the subsequences $\beta^i_{n,k}(\sigma)$ such that $\operatorname{std} \beta^i_{n,k}(\sigma) = \sigma_i$ for all i. Also note that

$$\operatorname{des}_k(\sigma) = \sum_{i=1}^{k} \operatorname{des}(\operatorname{std} \beta^i_{n,k}(\sigma)).$$

Thus,

$$\begin{aligned} F_n^{\mathrm{des}_k}(q) &= \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{des}_k \sigma} \\ &= M_{n,k}\left(\sum_{(\sigma_1,\ldots,\sigma_k)\in\mathfrak{S}_{d+1}^r\times\mathfrak{S}_d^{k-r}} q^{\mathrm{des}\,\sigma_1}\cdots q^{\mathrm{des}\,\sigma_k}\right) \\ &= M_{n,k}A_{d+1}^r(q)A_d^{k-r}(q). \end{aligned}$$

This proves the first identity.

The second identity follows completely analogously, with the main modification being that an element of $\mathrm{Inv}_k(\sigma)$ corresponds to a usual inversion in some unique $\mathrm{std}\,\beta_{n,k}^j(\sigma)$. □

We note that the sequences of coefficients of $F_n^{\mathrm{des}_2}(q)$ appear as OEIS sequence A180887. No other nontrivial choice of k appears to have been studied before, for either des_k for inv_k.

Given des_K and inv_K, one must wonder what the corresponding generalizations of exc and maj are whose relationships with des_K and inv_K parallel that of the classical statistics. To do this, we define the multiset

$$\mathrm{Exc}_K(\sigma) = \bigcup_{k\in K}\biguplus_{i=1}^{k}\{\{\lceil j/k\rceil \in [n-1] \mid \lceil j/k\rceil \in \mathrm{Exc}(\mathrm{std}\,\beta_{n,k}^i(\sigma))\}\}$$

and set $\mathrm{exc}_K(\sigma) = |\,\mathrm{Exc}_K(\sigma)|$, and also set

$$\mathrm{maj}_K(\sigma) = \sum_{k\in K}\sum_{i\in\mathrm{Des}_k(\sigma)}\left\lceil\frac{i}{k}\right\rceil.$$

We could equivalently state that

$$\mathrm{maj}_K(\sigma) = \sum_{k\in K}\sum_{i=1}^{k}\mathrm{maj}(\mathrm{std}\,\beta_{n,k}^i(\sigma)) \text{ and } \mathrm{exc}_K(\sigma) = \sum_{k\in K}\sum_{i=1}^{k}\mathrm{exc}(\mathrm{std}\,\beta_{n,k}^i(\sigma)).$$

These are exactly the definitions needed in order to obtain identities that parallel $F_n^{\mathrm{des}}(q) = F_n^{\mathrm{exc}}(q)$ and $F_n^{\mathrm{inv}}(q) = F_n^{\mathrm{maj}}(q)$, as we will soon see.

One important distinction to make between exc and exc_K is the following. If $\sigma = a_1a_2\cdots a_n$ and $\tau = b_1b_2\cdots b_n$, then even if $a_i = b_i$ for some i, one cannot say $i \in \mathrm{Exc}_K(\sigma)$ if and only if $i\in\mathrm{Exc}_K(\tau)$. For example, if $\sigma = 4136572$, then $1\in\mathrm{Exc}_2(\sigma)$, but if $\tau = 4153627$ then $1\notin\mathrm{Exc}_2(\tau)$.

Example 6. Again let $\sigma = 4136572$. We then have

$$\mathrm{exc}_{\{2,3\}}(\sigma) = |\{\{1,3,1,2\}\}| = 4$$

and

$$\mathrm{maj}_{\{2,3\}}(\sigma) = \left\lceil\frac{1}{2}\right\rceil + \left\lceil\frac{5}{2}\right\rceil + \left\lceil\frac{4}{3}\right\rceil = 6.$$

k	$G_{6,k}(q)$	$G_{8,k}(q)$	$G_{9,k}(q)$
1	$6A_5(q)$	$8A_7(q)$	$9A_9(q)$
2	$180A_2(q)^2$	$1120A_3(q)^2$	$9q^{-1}A_9(q)$
3	$6!$	$8q^{-2}A_7(q)$	$45360A_2(q)^3$
4	$180q^{-2}A_2(q)^2$	$8!$	$9q^{-3}A_9(q)$
5	$6q^{-4}A_5(q)$	$8q^{-4}A_7(q)$	$9q^{-4}A_9(q)$
6		$1120q^{-4}A_3(q)^2$	$45360q^{-3}A_2(q)^3$
7		$8q^{-6}A_7(q)$	$9q^{-6}A_9(q)$
8			$9q^{-7}A_9(q)$

Table 1: The polynomials $G_{n,k}(q)$ for $n = 6, 8, 9$ and $1 \leq k \leq n$.

By constructing a nearly identical argument as in the proof Theorem 5, and using the facts that $F_n^{\mathrm{des}}(q) = F_n^{\mathrm{exc}}(q)$ and $F_n^{\mathrm{inv}}(q) = F_n^{\mathrm{maj}}(q)$, we have the following corollary.

Corollary 7. *The identities* $F_n^{\mathrm{des}_k}(q) = F_n^{\mathrm{exc}_k}(q)$ *and* $F_n^{\mathrm{inv}_k}(q) = F_n^{\mathrm{maj}_k}(q)$ *hold.*

Now that we have established the analogous parallels between the four classical statistics and their width-k counterparts, we wish to explore what other structure is present. A simple proposition relates des_k and inv_k when k is large.

Corollary 8. *For all* $k \geq n/2$, $F_n^{\mathrm{des}_k}(q) = F_n^{\mathrm{inv}_k}(q)$.

Proof. Since $k \geq n/2$, the sets $\beta^i_{n,k}(\sigma)$ contain at most two elements. So, width-k descents and width-k inversions of σ are identical. □

We now show that interesting behavior occurs when considering the function

$$G_{n,k}(q) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{des}_k(\sigma) - \mathrm{des}_{n-k}(\sigma)}.$$

According to computational data, the following conjecture holds for all $n \leq 9$ and $1 \leq k < n$ for which $\gcd(k, n) = 1$.

Conjecture 9. If $\gcd(k, n) = 1$, then $G_{n,k}(q) = nq^{1-k}A_{n-1}(q)$.

Several illustrative polynomials are given in Table 2. This does not hold more generally, and it would be interesting to determine if a general formula exists.

Question 10. For which values of n, k does there exist a closed formula for $G_{n,k}(q)$, and what is the formula?

3 Pattern avoidance

We say that a permutation $\sigma \in \mathfrak{S}_n$ *contains the pattern* $\pi \in \mathfrak{S}_m$ if there exists a subsequence σ' of σ such that $\mathrm{std}(\sigma') = \pi$. If no such subsequence exists, then we say that σ *avoids the pattern* π. If $\Pi \subseteq \mathfrak{S}$, then we say σ *avoids* Π if σ avoids every element of Π. Let $\mathrm{Av}_n(\Pi)$ denote the set of all permutations of $\mathfrak{S}_n$ avoiding Π. In a mild abuse of notation, if $\Pi = \{\pi\}$, we will write $\mathrm{Av}_n(\pi)$.

In this section, we consider the functions

$$F_n^{\mathrm{st}}(\Pi; q) = \sum_{\sigma \in \mathrm{Av}_n(\Pi)} q^{\mathrm{st}\,\sigma},$$

which specializes to $F_n^{\mathrm{st}}(q)$ if $\Pi = \emptyset$. In most instances, $F_n^{\mathrm{des}_k}$ will be the main focus, but $F_n^{\mathrm{des}_K}$ and $F_n^{\mathrm{inv}_k}$ will also make appearances.

An important concept within pattern avoidance is that of Wilf equivalence. Two sets $\Pi, \Pi' \subset \mathfrak{S}$ are said to be *Wilf equivalent* if $|\mathrm{Av}_n(\Pi)| = |\mathrm{Av}_n(\Pi')|$ for all n. In this case, we write $\Pi \equiv \Pi'$ to denote this Wilf equivalence, which is indeed an equivalence relation. For example, independent work of MacMahon and Knuth [3, 6] show that that whenever $\pi, \pi' \in \mathfrak{S}_3$, then $|\mathrm{Av}_n(\pi)| = |\mathrm{Av}_n(\pi')| = C_n$, where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the n^{th} Catalan number.

Proving whether $\Pi \equiv \Pi'$ is often quite difficult, and their Wilf equivalence does not imply that $F_n^{\mathrm{st}}(\Pi; q) = F_n^{\mathrm{st}}(\Pi'; q)$. However, in some instances, the problems of establishing these identities have straightforward solutions by applying basic transformations on the elements of the avoidance classes. Given $\pi = a_1 \cdots a_n \in \mathfrak{S}_m$, let π^r denote its *reversal* and π^c denote its *complement*, respectively defined by

$$\pi^r = a_m a_{m-1} \cdots a_1 \text{ and } \pi^c = (m+1-a_1)(m+1-a_2)\cdots(m+1-a_m).$$

Similarly, given $\Pi \subseteq \mathfrak{S}$, we let

$$\Pi^r = \{\pi^c \mid \pi \in \Pi\} \text{ and } \Pi^c = \{\pi^r \mid \pi \in \Pi\}$$

be the *reversal* and *complement* of Π, respectively.

Our results of this section begin with a multivariate generalization of $F_n^{\mathrm{st}}(\Pi; q)$, and with showing how its specializations describe relationships among Π, Π^r, and Π^c.

Definition 11. Fix $\Pi \subseteq \mathfrak{S}$. Define

$$T_n(\Pi; t_1, \ldots, t_{n-1}) = \sum_{\sigma \in \mathrm{Av}_n(\Pi)} \prod_{k=1}^{n-1} t_k^{\mathrm{des}_k(\sigma)}.$$

The function T_n specializes to $F_n^{\mathrm{des}_K}(\Pi;q)$ by setting $t_i = q$ for $i \in K$ and $t_i = 1$ for all $i \notin K$. We can also recover $F_n^{\mathrm{inv}_K}(\Pi;q)$ from T_n by setting $t_i = q$ whenever $i \in [n-1] \cap k\mathbb{Z}$ for some $k \in K$, and setting $t_i = 1$ otherwise. Additionally, when $\Pi = \{\pi\}$, a nice duality appears, providing a mild generalization of [1, Lemma 2.1].

Proposition 12. *For any $\Pi \subseteq \mathfrak{S}$, let Π' denote either Π^r or Π^c. We then have*

$$T_n(\Pi'; t_1, \ldots, t_{n-1}) = t_1^{n-1} t_2^{n-2} \cdots t_{n-1} T_n(\Pi; t_1^{-1}, \ldots, t_{n-1}^{-1}).$$

Consequently,

$$T_n((\Pi^r)^c; t_1, \ldots, t_{n-1}) = T_n(\Pi; t_1, \ldots, t_{n-1}).$$

Proof. It is enough to prove the claim when the set of patterns being avoided is $\{\pi\}$ for some $\pi \in \mathfrak{S}_m$, since the full result follows by applying the argument to all elements of Π simultaneously.

First consider when $\pi' = \pi^c$ and $\sigma \in \mathrm{Av}_n(\pi)$. Because $\sigma \in \mathrm{Av}_n(\pi)$ if and only if $\sigma^c \in \mathrm{Av}_n(\pi^c)$, we have that for each k, $i \in \mathrm{Des}_k(\sigma)$ if and only if $i \notin \mathrm{Des}_k(\sigma^c)$. This implies $\mathrm{Des}_k(\sigma^c) = [n-k] \setminus \mathrm{Des}_k(\sigma)$, hence $\mathrm{des}_k(\sigma^c) = n - k - \mathrm{des}_k(\sigma)$. So,

$$\begin{aligned}
T_n(\pi^c; t_1, \ldots, t_{n-1}) &= \sum_{\sigma \in \mathrm{Av}_n(\pi^c)} \prod_{k=1}^{n-1} t_k^{\mathrm{des}_k(\sigma)} \\
&= \sum_{\sigma \in \mathrm{Av}_n(\pi)} \prod_{k=1}^{n-1} t_k^{\mathrm{des}_k(\sigma^c)} \\
&= \sum_{\sigma \in \mathrm{Av}_n(\pi)} \prod_{k=1}^{n-1} t_k^{n-k-\mathrm{des}_k(\sigma)} \\
&= t_1^{n-1} t_2^{n-2} \cdots t_{n-1} T_n(\pi; t_1^{-1}, \ldots, t_{n-1}^{-1}).
\end{aligned}$$

Proving that the result holds for $\pi' = \pi^r$ follows similarly.

The second identity in the proposition statement holds by applying the first identity twice: first for Π^r and then for Π^c. $\square$

The above identities significantly reduce the amount of work needed to study $F_n^{\mathrm{des}_K}(\Pi;q)$ for all $\Pi \subseteq \mathfrak{S}_n$. For the remainder of this paper, we systematically approach Π for $|\Pi| \leq 2$.

3.1 Avoiding singletons

By Proposition 12, we immediately get

$$F_n^{\mathrm{des}_k}(123;q) = q^{n-k} F_n^{\mathrm{des}_k}(321;q^{-1})$$

and

$$F_n^{\mathrm{des}_k}(132;q) = F_n^{\mathrm{des}_k}(213;q) = q^{n-k} F_n^{\mathrm{des}_k}(231;q^{-1}) = q^{n-k} F_n^{\mathrm{des}_k}(312;q^{-1}).$$

So, studying $F_n^{\mathrm{des}_k}(\pi;q)$ for $\pi \in \mathfrak{S}_3$ reduces to studying the function for a choice of one pattern from $\{123, 321\}$ and one from the remaining patterns. For some choices of Π, the permutations in $\mathrm{Av}_n(\Pi)$ are especially highly structured, which leads to similar arguments throughout the rest of this paper.

We begin with $\pi = 312$. Notice that if $a_1 \cdots a_n \in \mathrm{Av}_n(312)$ and $a_i = 1$, then we know $\mathrm{std}(a_1 \cdots a_{i-1}) \in \mathrm{Av}_{i-1}(312)$, $\mathrm{std}(a_{i+1} \cdots a_n) \in \mathrm{Av}_{n-i}(312)$, and

$$\max\{a_1, \ldots, a_{i-1}\} < \min\{a_{i+1}, \ldots, a_n\}.$$

Proposition 13. *For all n,*

$$F_n^{\mathrm{des}_k}(312;q) = \sum_{i=1}^{k} C_{i-1} F_{n-i}^{\mathrm{des}_k}(312;q) + \sum_{i=k+1}^{n} q F_{i-1}^{\mathrm{des}_k}(312;q) F_{n-i}^{\mathrm{des}_k}(312;q)$$

where C_i is the i^{th} Catalan number.

Proof. First consider when $\sigma = a_1 \cdots a_n \in \mathrm{Av}_n(312)$ and $a_i = 1$ for some $i \leq k$. By the discussion preceding this proposition, $j \notin \mathrm{Des}_k(\sigma)$ for any $j \leq i$. So, none of the C_{i-1} possible permutations that make up $\mathrm{std}(a_1 \cdots a_{i-1})$ contribute to $\mathrm{des}_k(\sigma)$. The only contributions to $\mathrm{des}_k(\sigma)$ come from $\mathrm{std}(a_{i+1} \cdots a_n) \in \mathrm{Av}_{n-i}(312;q)$. The overall contribution to $F_n^{\mathrm{des}_k}(312;q)$ is the second summand of the identity.

Now suppose $a_i = 1$ for some $i > k$. Each choice of $a_1 \cdots a_i$, contributes to $\mathrm{des}_k(\sigma)$ as usual, but there will be an additional width-k descent produced at $i - k$. The elements $a_{i+1} \cdots a_n$ contribute to $\mathrm{des}_k(\sigma)$ as before. The overall contribution to $F_n^{\mathrm{des}_k}(312;q)$ is the first summand of the identity. Since we have considered all possible indices i for which we could have $a_i = 1$, we add the two cases together and are done. □

Note that when we set $q = 1$, the above recursion specializes to the well-known recursion for Catalan numbers $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. For more about the Catalan numbers see A000108. Also, when $k = 1$, the coefficients of $F_n^{\mathrm{des}_k}(312;q)$ are the Narayana numbers, appearing in the OEIS as sequence A001263. The sequence of coefficients of the polynomials $F_n^{\mathrm{des}_{n-1}}(312;q)$ appear as A026008, where the coefficients are listed with the degree-1 summand first and then the constant term second.

We can use the previous proposition in conjunction with Proposition 12 to produce formulae for $F_n^{\mathrm{des}_k}(\Pi;q)$ and $F_n^{\mathrm{inv}_k}(\Pi;q)$ whenever Π is a single element of $\mathfrak{S}_3$ other than 123 or 321. The only nontrivial work required, then, is to compute the degrees of the two polynomials.

Corollary 14. *For all n,*

$$\deg F_n^{\mathrm{des}_k}(312;q) = n - k \text{ and } \deg F_n^{\mathrm{inv}_k}(312;q) = \sum_{i=1}^{n-k} \left\lfloor \frac{n-i}{k} \right\rfloor.$$

Proof. The degrees of the above polynomials are given by identifying a permutation in $\mathrm{Av}_n(312)$ with the most possible descents. This is satisfied by $n(n-1)\cdots 21 \in \mathrm{Av}_n(312)$ which has $n-k$ descents of width k. Determining the number of width-k inversions in this permutation is done similarly. $\square$

Although 321 is Wilf equivalent to 312, it is not so obvious how to construct a recurrence relation for $F_n^{\mathrm{des}_k}(321;q)$ in general. In the special case of $k=1$, the coefficients are given by sequence A166073, but the coefficients for other k are not easily identifiable. To help describe the elements of $\mathrm{Av}_n(321)$, first let $\sigma = a_1 \cdots a_n \in \mathfrak{S}$ and call a_i a *left-right maximum* if $a_i > a_j$ for all $j < i$. Thus $\sigma \in \mathrm{Av}_n(321)$ if and only if its set of non-left-right maxima form an increasing subsequence of σ. Indeed, if the non-left-right maxima did contain a descent, then there would be some left-right maximum preceding both elements, which violates the condition that σ avoid 321. Despite such a description, using it to reveal $F_n^{\mathrm{des}_k}(321;q)$ has thus far been unsuccessful. So, we leave the following as an open question.

Question 15. Is there a closed formula or simple recursion for $F_n^{\mathrm{des}_k}(321;q)$ (equivalently, for $F_n^{\mathrm{des}_k}(123;q)$)?

3.2 Avoiding doubletons

At this point, we begin studying the functions $F_n^{\mathrm{des}_k}(\Pi;q)$ when avoiding doubletons from $\mathfrak{S}_3$. Recall that, by the Erdős-Szekeres theorem [2], there are no permutations in $\mathfrak{S}_n$ for $n \geq 5$ that avoid both 123 and 321. Thus, we will not consider this pair. Additionally, the functions $F_n^{\mathrm{inv}_k}(\Pi;q)$ are quite unwieldy, so we will not consider these either.

For our first nontrivial example, we begin with $\{123, 132\}$. Permutations $a_1 \cdots a_n \in \mathrm{Av}_n(123,132)$ have the following structure: for any $i = 1, \ldots, n$, if $a_i = n$, then the substring $a_1 a_2 \cdots a_{i-1}$ is decreasing and consists of the elements from the interval $[n-i+1, n-1]$. Additionally, the substring $a_{i+1} \cdots a_n$ is an element of $\mathrm{Av}_{n-i}(123,132)$. This structure makes it easy to show that there are 2^{n-1} elements of $\mathrm{Av}_n(123,132)$ [9].

Proposition 16. *For all n and $1 \leq k \leq n-1$,*

$$\begin{aligned} F_n^{\mathrm{des}_k}(123,132;q) &= \sum_{i=1}^{k} q^{\min(i,n-k)} F_{n-i}^{\mathrm{des}_k}(123,132;q) \\ &\quad + \sum_{i=k+1}^{n-k} q^{\min(i-1,n-k-1)} F_{n-i}^{\mathrm{des}_k}(123,132;q) \\ &\quad + 2^{n-\max(k+1,n-k+1)} q^{n-k-1}. \end{aligned}$$

Proof. Let $\sigma = a_1 \cdots a_n \in \mathrm{Av}_n(123,132;q)$. If $a_i = n$ for $i \leq k$, then it is clear from the preceding description of the elements in $\mathrm{Av}_n(123,132)$ that all of $1, \ldots, \min(i, n-k)$ are elements of $\mathrm{Des}_k(\sigma)$. If $j > n-k$ for some j, then $j \notin \mathrm{Des}_k(\sigma)$ since a_{j+k} does not exist.

The remaining elements of $\mathrm{Des}_k(\sigma)$ arise as a width-k descent of $a_{i+1}\cdots a_n$, which accounts for the factor of $F_{n-i}^{\mathrm{des}_k}(123,132;q)$ in the first summand.

Now, if $k+1 \le n-k$ and $a_i = n$ for $i = k+1,\ldots,n-k$, then every element of $1,\ldots,i$ except $i-k$ is an element of $\mathrm{Des}_k(\sigma)$. If $k+1 > n-k$, then the second summand is empty and nothing is lost by continuing to the case of $a_i = n$ for $i \ge \max(k+1, n-k+1)$. Again, the remaining elements of $\mathrm{Des}_k(\sigma)$ are the width-k descents of $a_{i+1}\cdots a_n$, hence the additional factor of $F_{n-i}^{\mathrm{des}_k}(123,132;q)$. This accounts for the second summand.

Finally, let $m = \max(k+1, n-k+1)$. If $a_i = n$ for $i \ge m$, then all of $1,\ldots,n-k$ are descents except $i-k$, and these are the only possible width-k descents. In particular, none of $i+1, i+2, \ldots, n$ can be the index for a width-k descent. This leads to the sum

$$\sum_{i=m}^{n} q^{n-k-1}|\,\mathrm{Av}_{n-i}(123,132)| = \left(1 + \sum_{i=m}^{n-1} 2^{n-i-1}\right) q^{n-k-1} = 2^{n-m}q^{n-k-1},$$

which accounts for the third summand. $\square$

Again, the sequence of coefficients for $k=1$ has appeared already, this time as sequence A109446, but the sequences for nontrivial $k>1$ appear to be entirely unstudied.

Next, we consider $\{123, 312\}$. We proceed similarly as before but with some minor differences, reflecting the new structure we encounter. If $\sigma = a_1\cdots a_n \in \mathrm{Av}_n(123,312)$ and $a_i = 1$ for some $i < n$, then σ is of the form

$$\sigma = i(i-1)\cdots 21n(n-1)\cdots(i+2)(i+1),$$

since neither subsequence $a_1\cdots a_{i-1}$ and $a_{i+1}\cdots a_n$ may contain an ascent. If $a_n = 1$, though, then $\mathrm{std}(a_1\cdots a_{n-1}) \in \mathrm{Av}_{n-1}(123,312)$.

Proposition 17. *For all n and $1 \le k < n$,*

$$\begin{aligned} F_n^{\mathrm{des}_k}(123,312;q) &= \sum_{i=1}^{k} q^{\max(0,n-k-i)} + \sum_{i=k+1}^{n-1} q^{\max(n-2k,i-k)} \\ &\quad + qF_{n-1}^{\mathrm{des}_k}(123,312;q) \end{aligned}$$

Proof. Suppose $a_i = 1$ for some $i \le k$. Using the description of elements in $\mathrm{Av}_n(123,312)$, we know there are $n-k-i$ width-k descents. Since there is only one such permutation for each i, we simply add all of the q^{n-k-i}, so long as $n-k-i \ge 0$. If this inequality does not hold, then there are no width-k descents in this range. This accounts for the first summand.

The second summand is computed very similarly to that of the second summand in Proposition 16. The final summand is a direct result of noting that when $a_n = 1$, then $(a_1-1)\cdots(a_{n-1}-1)$ may be any element of $\mathrm{Av}_{n-1}(123,312;q)$, which accounts for the factor $F_{n-1}^{\mathrm{des}_k}(123,312;q)$. For each of these choices, we know $n-k \in \mathrm{Des}_k(\sigma)$, hence the factor of q. Adding the sums results in the identity claimed. $\square$

Now we consider $\{132, 231\}$. Note that if $a_1 \cdots a_n \in \operatorname{Av}_n(132, 231)$, then $a_i \neq n$ for any $1 < i < n$. If $a_1 = n$, then $\operatorname{std}(a_2 \cdots a_n) \in \operatorname{Av}_{n-1}(132, 231)$, and similarly if $a_n = n$. Once again, this allows us to quickly compute that $|\operatorname{Av}_n(132, 231)| = 2^{n-1}$.

Proposition 18. *Let $K = \{k_1, \ldots, k_l\}$ be a nonempty subset of $[n-1]$ whose elements are listed in increasing order. We then have*

$$F_n^{\operatorname{des}_K}(132, 231; q) = \prod_{i=1}^{l+1} (1 + q^{i-1})^{k_i - k_{i-1}}$$

where $k_0 = 1$ and $k_{l+1} = n$.

Proof. It follows from the description of elements in $\operatorname{Av}_n(132, 231)$ that there are two summands in a recurrence for $F_n^{\operatorname{des}_K}(132, 231; q)$: one corresponding to $a_1 = n$ and one corresponding to $a_n = n$. When $a_1 = n$, then there are $|K| = l$ copies of $1 \in \operatorname{Des}_K(\sigma)$; if $a_n = n$, then a_n makes no contribution to $\operatorname{Des}_K(\sigma)$. Thus, by deleting n from σ, we get the recurrence

$$F_n^{\operatorname{des}_K}(132, 231; q) = (1 + q^l) F_{n-1}^{\operatorname{des}_K}(132, 231; q).$$

Repeating this, a factor of $1 + q^l$ appears until we get to

$$F_n^{\operatorname{des}_K}(132, 231; q) = (1 + q^l)^{n-k_l} F_{k_l}^{\operatorname{des}_K}(132, 231; q).$$

At this point, note that

$$F_{k_l}^{\operatorname{des}_K}(132, 231; q) = F_{k_l}^{\operatorname{des}_{K \setminus \{k_l\}}}(132, 231; q).$$

Repeating the previous argument ends up with the identity claimed. $\square$

Next, consider $\{132, 213\}$. If $\sigma = a_1 \cdots a_n \in \operatorname{Av}_n(132, 213)$, then if $a_i = n$ for any i, then $a_1 \cdots a_{i-1}$ must be increasing in order to avoid 213. Moreover,

$$\max(a_{i+1}, \cdots, a_n) < a_1$$

in order to avoid 132, and $\operatorname{std}(a_{i+1} \cdots a_n) \in \operatorname{Av}_{n-i}(132, 213)$.

Proposition 19. *For all n and $1 \leq k < n$,*

$$\begin{aligned} F_n^{\operatorname{des}_k}(132, 213; q) &= \sum_{i=1}^{k} q^{\min(i, n-k)} F_{n-i}^{\operatorname{des}_k}(132, 213; q) \\ &\quad + \sum_{i=k+1}^{n-k} q^{\min(k, n-i)} F_{n-i}^{\operatorname{des}_k}(132, 213; q) \\ &\quad + \sum_{i=\max(k+1, n-k+1)}^{n} q^{n-i} F_{n-i}^{\operatorname{des}_k}(132, 213; q). \end{aligned}$$

Proof. The proof of this is entirely analogous to the proof of Proposition 16. ☐

In this case, when $k = 2$, the sequence of coefficients as n grows is the sequence A208343. No other nontrivial choice of k appears to have been previously studied. Additionally, for all remaining choices of Π we consider, the instances of previously-studied sequences appear to be only when $k = 1$.

Next, we consider $\{132, 312\}$. For each $i = 1, \ldots, n-1$, either $a_{i+1} = \max\{a_1, \ldots, a_i\} + 1$ or $a_{i+1} = \min\{a_1, \ldots, a_i\} - 1$. This doubleton often results in especially pleasant formulas, and our results are no exception.

Proposition 20. *Let $K = \{k_1, \ldots, k_l\}$ be a nonempty subset of $[n-1]$ whose elements are listed in increasing order. We then have*

$$F_n^{\mathrm{des}_K}(132, 312; q) = \prod_{i=1}^{l+1} (1 + q^{i-1})^{k_i - k_{i-1}}$$

where $k_0 = 1$ and $k_{l+1} = n$.

Proof. From the description of elements $\sigma = a_1 \cdots a_n \in \mathrm{Av}_n(132, 312)$, we know that either $a_n = 1$ or $a_n = n$. In the former case, $n - k \in \mathrm{Des}_K(\sigma)$ for each $k \in K$, and in the latter case, $n - k \notin \mathrm{Des}_K(\sigma)$ for each $k \in K$. This leads to the recurrence

$$F_n^{\mathrm{des}_K}(132, 312; q) = (1 + q^l) F_{n-1}^{\mathrm{des}_K}(132, 312; q).$$

Following an analogous argument as in the proof of Proposition 18 obtains the result. ☐

From the general formula given above, we are able to quickly determine $F_n^{\mathrm{inv}_k}(132, 312; q)$.

Corollary 21. *For fixed n, k, write $n = dk + r$ for unique nonnegative integers d, r such that $0 \le r < k$. We then have*

$$F_n^{\mathrm{inv}_k}(132, 312; q) = F_n^{\mathrm{inv}_k}(132, 231; q) = 2^{k-1}(1 + q^d)^r \prod_{i=1}^{d-1} (1 + q^i)^k.$$

Proof. This is a direct consequence of the general formulas from Propositions 18 and 20, and recalling that

$$F_n^{\mathrm{inv}_k}(\Pi; q) = F_n^{\mathrm{des}_{[n-1] \cap k\mathbb{Z}}}(\Pi; q).$$

☐

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