Journal of Integer Sequences, Vol. 19 (2016),
Article 16.1.4

Sequences of Spiral Knot Determinants

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Abstract

Spiral knots are a generalization of the well-known class of torus knots indexed by strand number and base word repetition. By fixing the strand number and varying the repetition index, we obtain integer sequences of spiral knot determinants. In this paper, we examine such sequences for spiral knots of up to four strands using a new periodic crossing matrix method. Surprisingly, the resulting sequences vary widely in character and, even more surprisingly, nearly every one of them is a known integer sequence in the *Online Encyclopedia of Integer Sequences*. We also develop a general form for these sequences in terms of recurrence relations that exhibits a pattern which is potentially generalizable to all spiral knots.

1 Introduction

Torus knots are a well-understood infinite class of periodic knots that admit braid projections whose base words involve the first strand passing over all other strands in order. Torus knots constitute one of the most basic classes of knots, and much is known about them; in particular, they have known determinant sequences. Spiral knots are a generalization of torus knots that were developed recently in Brothers et al. [4], and differ from torus knots in that the traveling strand may weave over and under in various patterns as it crosses each of the other strands in order instead of always crossing over.

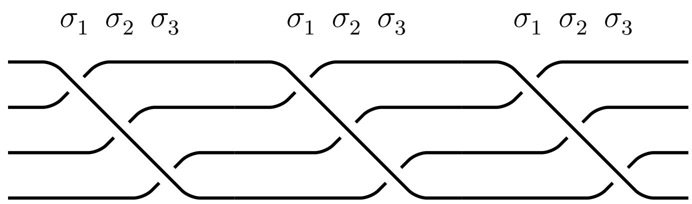

Figure 1: The standard braid projection of the torus knot $T(4,3) = (\sigma_1\sigma_2\sigma_3)^3$.

Determinants of spiral knots have not yet been characterized, and this paper provides an initial attempt to do so by providing a partial characterization. The *determinant* of a knot is defined to be the determinant of any minor of its crossing matrix, and is known to be a knot invariant, as described Livingston [9, Section 3.4]. This means that two knots with different determinants must necessarily be different knots. Thus, the study of spiral knot determinants provides a way to distinguish and characterize spiral knots. This paper will focus on the determinant sequences themselves and not the classification of spiral knots based on those sequences; however, the results in this paper do prove that infinitely many spiral knots are nontrivial and distinct from each other.

Knot determinants are also related to the knot invariant of p-colorability, which we will leverage in Section 2 to obtain a more streamlined method of determinant calculation for the class of spiral knots. A knot is *p-colorable* if its strands can be labeled with numbers $0, 1, 2, \ldots, p-1$ such that at least two distinct numbers are used in the labeling and, at each crossing with understrands labeled a and b and overstrand labeled c, we have $a + b \equiv 2c \pmod{p}$. It follows directly from the definition of crossing matrices that a knot is p-colorable if and only if its knot determinant is divisible by p. For more information, see Livingston [9, Sections 3.3, 3.4].

The standard braid projection of the torus knot $T(n,k)$ with n strands and k repetitions has braid word $(\sigma_1\sigma_2\cdots\sigma_{n-1})^k$ and is shown in Figure 1. Torus knots have exceptionally nice and well-understood properties. For example, the torus knot $T(n,k)$ is equivalent to the torus knot $T(k,n)$, whose standard braid projection has k strands, n repetitions, and braid word $(\sigma_1\sigma_2\cdots\sigma_{k-1})^n$. The crossing number of the torus knot $T(n,k)$ is always realized by one of the standard torus braid projections; this crossing number is $c(T(n,k)) = \min(k(n-1), n(k-1))$. For more background on torus knots, see Adams [1, Section 5.1].

Determinants and p-colorability classes of torus knots are also known. Breiland et al. [3] and Isidro et al. [7] independently show that, for n and k relatively prime, we have

$$\det(T(n,k)) = \begin{cases} 1, & \text{if } n \text{ and } k \text{ are both odd;} \\ n, & \text{if } n \text{ is odd and } k \text{ is even;} \\ k, & \text{if } k \text{ is odd and } n \text{ is even.} \end{cases}$$

When n and k fail to be relatively prime, the torus link $T(n,k)$ has more than one component;

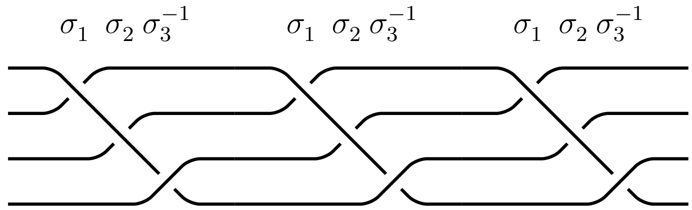

Figure 2: A braid projection of the spiral knot $S(4, 3, (1, 1, -1)) = (\sigma_1\sigma_2\sigma_3^{-1})^3$.

however, it is still possible to calculate its determinant. By considering this additional data, we can examine sequences of determinants of torus knots $T(n, k)$ as n is fixed and k varies.

Example 1. Consider the 4-strand torus knots of the form $T(4, k)$. If k is relatively prime to 4, then k must be odd and, thus, the determinant of a torus knot of the form $T(4, k)$ is simply k [3]. In this paper, we will show that when k is not relatively prime to 4, that is, when $k = 2j$ is even, the determinant of $T(4, k)$ is $2k$ when j is odd and 0 when j is even. Taking these results together, we see that the determinants of 4-strand torus knots form the following sequence, starting from $k = 1$:

$$1, 4, 3, 0, 5, 12, 7, 0, 9, 20, 11, 0, 13, 28, 15, 0, \ldots.$$

Surprisingly, although torus knots are very well understood, this sequence had not yet been recorded in the Online Encyclopedia of Integer Sequences (OEIS) until recently, when the authors of this paper added it as A251610.

Spiral knots are a generalization of torus knots in which the base word is permitted to have both overcrossings and undercrossings. For example, if we change the base word $\sigma_1\sigma_2\sigma_3$ of the torus knot $T(4, 3)$ to $\sigma_1\sigma_2\sigma_3^{-1}$ so that the last crossing in each segment of the standard braid projection is an undercrossing, then we get the spiral knot $S(4, 3, (1, 1, -1))$ shown in Figure 2.

In general, a spiral knot $S(n, k, \epsilon)$ is a knot that admits a braid projection of the periodic form $(\sigma_1^{\epsilon_1}\sigma_2^{\epsilon_2}\cdots\sigma_{n-1}^{\epsilon_{n-1}})^k$, where the vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})$ records the overcrossings and undercrossings of each period, and each ϵ_i is either 1 or -1. Torus knots are a subclass of spiral knots with $T(n, k) = S(n, k, (1, 1, \ldots, 1))$.

Determinants of the general class of spiral knots are not yet well understood. Dowdall et al. [6], Lopes and Matias [10], and Oesper [11] made discoveries about Turk's Head knots and weaving knots, both of which are subclasses of spiral knots, but no general formula for the determinant of a spiral knot $S(n, k, \epsilon)$ is known. In specific examples, we can fix the strand count n and pattern vector ϵ and calculate the resulting sequence of spiral knot determinants as the repetition index k varies. For example, by fixing $n = 4$ and $\epsilon = (1, 1, -1)$, we could calculate the sequence of determinants for the 4-strand "almost-torus" knots. As we will see in Theorem 2, this particular sequence has a strikingly simple closed form.

In Section 2, we give an overview of our streamlined method of finding determinants of spiral knots. Determinants of spiral knots with $n \leq 4$ strands and $1 \leq k \leq 16$ repetitions are given in Section 3. In Section 4, we derive formulae for determinant sequences of spiral knots through $n = 4$ strands and all k using an extension of the methods used in DeLong et al. [5] and Kauffman and Lopes [8]. Perhaps even more surprising than the fact that the sequence of torus knot determinants from Example 1 was *not* a previously-listed sequence in the OEIS is that every other sequence of spiral knot determinants for up to $n = 4$ strands *was*, although not, until recent updates by the authors, in relation to knot theory. The information in the existing OEIS listings provided clues as to how the varied patterns of spiral knot determinant sequences might be related. In Section 5, we focus on those relationships and prove that, for $n \leq 4$, the sequences of spiral knot determinants can each be written in terms of one particular type of linear homogeneous recurrence relation. This, in turn, hints at a possible generalization to determinants of all spiral knots.

2 Finding determinants of periodic braids

Calculating the determinant of a spiral knot $S(n, k, \epsilon)$ from its standard spiral braid projection typically involves finding the determinant of a square matrix of size $k(n-1) - 1$. This calculation becomes cumbersome as k grows large. Moreover, it is difficult to identify patterns by examining matrices of varying sizes. We will use the fact that spiral knots are in periodic braid forms to streamline the process of calculating their determinants.

The relationship between determinants and colorability provides the key to our streamlined determinant method for spiral knots.

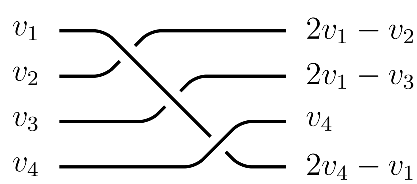

Figure 3: Coloring-compatible labeling for $S(4, 1, (1, 1, -1))$.

For example, each period of the braid projection of $S(4, 3, (1, 1, -1))$, shown in Figure 2, has the form shown in Figure 3. If the leftmost strands are colored top to bottom with numbers v_1, v_2, v_3, and v_4, and if we want to preserve colorability rules as we pass from left to right, then after passing through the crossings on the interior of the diagram the rightmost strands must be colored top to bottom with $2v_1 - v_2$, $2v_1 - v_3$, v_4, and $2v_4 - v_1$. The *base word matrix* that will take the leftmost vector of colors to the rightmost in this example is:

$$M = \left[\begin{array}{rrrr} 2 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \end{array} \right].$$

If the pattern in Figure 3 repeats k times, it forms the braid projection of $S(4, k, (1, 1, -1))$ shown in Figure 4. Now the exponentiated matrix M^k takes the initial, leftmost coloring vector to the final, rightmost coloring vector of the diagram. It is possible to close this braid with a coherent p-coloring if and only if $M^k(v) \equiv v \pmod p$, where $v = (v_1, v_2, v_3, v_4)$ is the leftmost coloring vector.

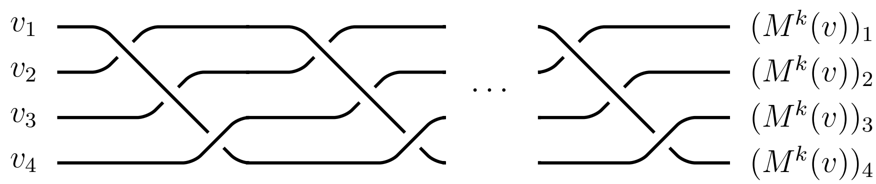

Figure 4: Coloring-compatible labeling for $S(4, k, (1, 1, -1))$.

In general, for any spiral knot $S(n, k, \epsilon)$, we can use the one-period pattern of $S(n, 1, \epsilon)$ to find a base word matrix M corresponding to the coloring rule $a + b \equiv 2c \pmod p$ at each crossing within the pattern, and then solve $(M^k - I_n)v \equiv 0 \pmod p$ to determine if $S(n, k, \epsilon)$ is p-colorable. The argument in Livingston [9, Section 3.4] then implies that the determinant of the original spiral knot is equal to the determinant of any i, j minor of the $n \times n$ matrix $M^k - I_n$:

$$\det(S(n, k, \epsilon)) = \det((M^k - I_n)_{ij}).$$

This method is equivalent to the process used by Delong et al. [5] for twisted torus knots, as well as the “black-box” approach used in Kauffman and Lopes [8] for plat closures of braids. Note that the usual method of computing the determinant of a spiral knot would involve a matrix whose size depended on the repetition index k, but our new method involves only matrices whose size depends on the number of strands n, regardless of the value of k. The computational complexity will instead come from exponentiating the base word matrix M; in Section 4 we will use Jordan canonical forms to mitigate that complexity.

3 Spiral knot determinant sequences

With the new base word matrix method from Section 2, we used Mathematica [14] to generate initial values of determinant sequences for spiral knots $S(n, k, \epsilon)$ as n and ϵ are fixed and k varies. This data is the jumping-off point for the determinant formulas we will prove in Sections 4 and 5. Table 1 shows spiral knot determinants for all possible values of ϵ corresponding to $2 \leq n \leq 4$ strands, with repetition index through $k = 16$. Note that reversing (reading backwards) or inverting (swapping the roles of $+1$ and -1) an ϵ-vector preserves spiral knot type; see the inaugural paper on spiral knots from Brothers et al. [4]. Therefore, when $n = 2$, there is only one family of spiral knots up to equivalence: the torus knots $T(2,k) = S(2,k,(1))$. When $n = 3$, the ϵ-vectors $(1,1)$ and $(-1,-1)$ produce equivalent knots, as do the ϵ-vectors $(1,-1)$ and $(-1,1)$. Therefore, there are only two $n = 3$ classes of spiral knots: the torus knots $T(3,k) = S(3,k,(1,1))$ and the almost-torus (or, equivalently in this case, "weaving") knots $S(3,k,(1,-1))$. For $n = 4$, we have three classes: the torus knots $T(4,k) = S(4,k,(1,1,1))$, the almost-torus knots $S(4,k,(1,1,-1))$, and the weaving knots $S(4,k,(1,-1,1))$.

k	(1)	$(1,1)$	$(1,-1)$	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$
1	1	1	1^2	$1 \cdot 1$	1^3	$1 \cdot 1^2$
2	2	3	$5 \cdot 1^2$	$2 \cdot 2$	2^3	$2 \cdot 6 \cdot 1^2$
3	3	4	4^2	$3 \cdot 1$	3^3	$3 \cdot 5^2$
4	4	3	$5 \cdot 3^2$	$4 \cdot 0$	4^3	$4 \cdot 6 \cdot 4^2$
5	5	1	11^2	$5 \cdot 1$	5^3	$5 \cdot 19^2$
6	6	0	$5 \cdot 8^2$	$6 \cdot 2$	6^3	$6 \cdot 6 \cdot 15^2$
7	7	1	29^2	$7 \cdot 1$	7^3	$7 \cdot 71^2$
8	8	3	$5 \cdot 21^2$	$8 \cdot 0$	8^3	$8 \cdot 6 \cdot 56^2$
9	9	4	76^2	$9 \cdot 1$	9^3	$9 \cdot 265^2$
10	10	3	$5 \cdot 55^2$	$10 \cdot 2$	10^3	$10 \cdot 6 \cdot 209^2$
11	11	1	199^2	$11 \cdot 1$	11^3	$11 \cdot 989^2$
12	12	0	$5 \cdot 144^2$	$12 \cdot 0$	12^3	$12 \cdot 6 \cdot 780^2$
13	13	1	521^2	$13 \cdot 1$	13^3	$13 \cdot 3691^2$
14	14	3	$5 \cdot 377^2$	$14 \cdot 2$	14^3	$14 \cdot 6 \cdot 2911^2$
15	15	4	1364^2	$15 \cdot 1$	15^3	$15 \cdot 13775^2$
16	16	3	$5 \cdot 987^2$	$16 \cdot 0$	16^3	$16 \cdot 6 \cdot 10864^2$

Table 1: Initial values of determinant sequences for spiral knots $S(n,k,\epsilon)$ with $n \leq 4$ strands and $1 \leq k \leq 16$ repetitions, listed in columns according to ϵ-vectors.

The numbers in Table 1 are partially factored in order to make certain patterns evident. For example, in the rightmost column, it is clear that each entry is divisible by k, alternate entries are divisible by 6, and the remaining factor of each entry is a perfect square. In order to better understand the initial spiral knot determinant data in this table, we searched for each sequence—as well as what we will call *embedded sequences*, such as the pattern of the numbers being squared in the rightmost column—in the OEIS. Based on our experience using the OEIS for other projects, we did not expect to find many known sequences associated with these determinant patterns.

To our amazement, we found that, with only one exception (see Example 1), every column in Table 1, as well as every embedded sequence, was part of an existing sequence in the OEIS. Each of these existing OEIS sequences at the time had no recorded connection to knot determinants, but provided clues that suggested formulas not only for individual sequences, but possibly for spiral knot determinants in general. Some of the sequences associated with the data in Table 1 are described below, with descriptions taken directly from the OEIS [13]:

- A000027: $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 \ldots$
  The sequence of positive integers clearly matches the initial terms of the determinant sequence for the torus knots $T(2, k) = S(2, k, (1))$.

- A131027: $1, 3, 4, 3, 1, 0, 1, 3, 4, 3, 1, 0, 1, 3, 4, 3, \ldots$
  This period 6 sequence is the third column of the triangular array in A131022, and an index to sequences with linear recurrences with constant coefficients, signature $(2, -2, 1)$. It matches the initial terms of the determinant sequence for the torus knots $T(3, k) = S(3, k, (1, 1))$.

- A004146: $1, 5, 16, 45, 121, 320, 841, 2205, 5776, 15125, 39601, 103680, 271441, \ldots$
  The sequence of alternate Lucas numbers minus 2 matches the initial terms of the determinant sequence for the almost-torus (and weaving) knots $S(3, k, (1, -1))$. This connection is related to a result on weaving knots in Oesper [11]; see the proof at the end of this section. This sequence also represents the number of spanning trees of the wheel W_k on $k + 1$ vertices, as featured in Rebman [12].

- A098149: $-1, -1, 4, -11, 29, -76, 199, -521, 1364, -3571, 9349, -24476, 64079, \ldots$
  This sequence relates bisections of Lucas and Fibonacci numbers, and is defined by the recursive formula $a_k = -3a_{k-1} - a_{k-2}$, with $a_0 = -1$ and $a_1 = -1$. After the first term, the absolute values of the terms in this sequence match the initial odd terms of the embedded sequence of numbers to be squared in the determinant sequence for $S(3, k, (1, -1))$. The absolute values are also the sequence A002878, which is the L_{2k+1} bisection of the Lucas sequence.

- A001906: $0, 1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, 17711, 46368, 121393, 317811, \ldots$
  This sequence is the F_{2k} bisection of the Fibonacci sequence, given by the recursive formula $a_k = 3a_{k-1} - a_{k-2}$, with $a_0 = 0$ and $a_1 = 1$. After the first term, it matches the initial even terms of the embedded sequence of numbers to be squared in the determinant sequence for $S(3, k, (1, -1))$.

- A005013: $1, 1, 4, 3, 11, 8, 29, 21, 76, 55, 199, 144, 521, 377, 1364, 987, 3571, 2584, \ldots$
  Putting the previous two sequences together results, unsurprisingly, in a sequence

that alternates Lucas and Fibonacci numbers. It is given by the recursive formula $a_k = 3a_{k-2} - a_{k-4}$, with $a_0 = 0, a_1 = 1, a_2 = 1$, and $a_3 = 4$, and matches the initial terms of the embedded sequence of numbers to be squared in the determinant sequence for $S(3, k, (1, -1))$.

- A251610: $1, 4, 3, 0, 5, 12, 7, 0, 9, 20, 11, 0, 13, 28, 15, 0, \ldots$
This is the only sequence in this list that had not appeared in the OEIS before the authors' contribution, although it matches the initial terms of one of the most well-known knot types in the table: the torus knots $T(4, k) = S(4, k, (1, 1, 1))$.

- A007877: $1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \ldots$
This period 4 repeating sequence is also an index to sequences with linear recurrences with constant coefficients, signature $(1, -1, 1)$. It matches the initial terms of the embedded sequence of numbers that are multiplied by k in the determinant sequence for $S(4, k, (1, 1, 1))$.

- A000578: $1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, 2197, 2744, 3375, \ldots$
The sequence of perfect cubes clearly matches the initial terms of the determinants of the 4-strand almost-torus spiral knots $S(4, k, (1, 1, -1))$.

- A006235: $1, 12, 75, 384, 1805, 8100, 35287, 150528, 632025, 2620860, 10759331, \ldots$
This sequence for the complexity of a doubled cycle matches the initial terms of the determinant sequence for the 4-strand weaving spiral knots $S(4, k, (1, -1, 1))$. This sequence is described in the OEIS with the formula $a_k = \frac{k}{2}(-2 + (2 - \sqrt{3})^k + (2 + \sqrt{3})^k)$, which we will see in Section 4. It is also the number of spanning trees of the k-prism graph; note that this is the second sequence in this list to involve spanning trees. Although not pursued in this paper, spanning trees of checkerboard graphs may provide a key to ultimately finding a unifying formula for all spiral knot determinants; see Dowdall et al. [6] and Kauffman and Lopes [8].

- A001834: $1, 5, 19, 71, 265, 989, 3691, 13775, 51409, 191861, 716035, 2672279, \ldots$
This sequence is given by the recurrence relation $a_k = 4a_{k-1} - a_{k-2}$, with $a_0 = 1$ and $a_1 = 5$, and matches the initial odd terms of the embedded sequence of squared numbers in the determinant sequence for $S(4, k, (1, -1, 1))$. This sequence is also described in the OEIS as $a_k = \frac{((1+\sqrt{3})^{2k+1} + (1-\sqrt{3})^{2k+1})}{2^{k+1}}$, and as the index entries for sequences related to linear recurrences with constant coefficients, signature $(4, -1)$.

- A001353: $1, 4, 15, 56, 209, 780, 2911, 10864, 40545, 151316, 564719, 2107560, \ldots$
This sequence is given by the recurrence relation $a_k = 4a_{k-1} - a_{k-2}$ with $a_0 = 0$ and $a_1 = 1$, the same formula as in the previous sequence with different initial conditions. It matches the initial even terms of the embedded sequence of squared numbers in the determinant sequence for $S(4, k, (1, -1, 1))$.

- A108412: $1, 1, 5, 4, 19, 15, 71, 56, 265, 209, 989, 780, 3691, 2911, 13775, 10864, \ldots$
  Combining the previous two sequences forms the sequence $a_k = 4a_{k-2} - a_{k-4}$, with $a_0 = 1$, $a_1 = 1$, $a_2 = 5$, and $a_3 = 4$. It matches the initial terms of the embedded sequence of squared numbers in the determinant sequence for $S(4, k, (1, -1, 1))$.

Although, of course, matching initial sequence terms does not prove anything about the long-term behavior of these determinant sequences, the connections to so many OEIS sequences encouraged us to seek formulas for all of them. As we will see in Section 4, this is indeed possible. Moreover, the connections to sequences in the OEIS that involve linear recurrences with constant coefficients suggested that, perhaps, we could recast all of these formulas recursively. We will do this in Section 5, and we will see that every one of the spiral knot determinant formulas for $n \leq 4$ can be expressed in terms of the same linear recurrence relation.

In the third item listed above, we mentioned that the initial terms of the sequence of determinants for $S(3, k, (1, -1))$ matched sequence A004146, which can be expressed in terms of alternate Lucas numbers as $L_{2k} - 2$. The fact that this particular formula holds for all values of k follows from a result on weaving knots in Oesper [11], in which the weaving knot $W(k, 3)$ is equal to $S(3, k, (1, -1))$.

Proof. Oesper [11] shows that we have $\det(W(k, 3)) = -(C_{2k-2} + 1)C_{2k} + C_{2k-1}^2$, where $C_j = \sum_{i=1}^{j}(-1)^{i+1} f_i$, and f_i is the i^{th} Fibonacci number (with $f_1 = 1$ and $f_2 = 1$). Using Fibonacci identities and the definition of the Lucas numbers, we can make the following reduction of Oesper's formula:

$$\begin{aligned}
\det(W(k,3)) &= -(C_{2k-2} + 1)C_{2k} + C_{2k-1}^2 \\
&= -(-f_{2k-3} + 2)(-f_{2k-1} + 1) + (f_{2k-2} + 1)(f_{2k-2} + 1) \\
&= -f_{2k-3}f_{2k-1} + f_{2k-3} + 2f_{2k-1} + f_{2k-2}^2 + 2f_{2k-2} - 1 \\
&= -f_{2k-3}f_{2k-1} + f_{2k-1} + f_{2k+1} - 1 + f_{2k-2}^2 \\
&= (-f_{2k-2}^2 - 1) + L_{2k} + (f_{2k-2}^2 - 1) \\
&= L_{2k} - 2.
\end{aligned}$$

In particular, the second, fourth and fifth equalities follow from identities 21, 8, and 32 in Benjamin and Quinn [2, Appendix], respectively.

□

4 Formulas for spiral knot determinants

We now state and prove formulas for all of the determinant sequences in Table 1 of Section 3. According to the method described in Section 2, for each type of spiral knot $S(n, k, \epsilon)$, we must compute a base word matrix M depending on n and ϵ, and then find a formula for the determinant of any minor of the $n \times n$ matrix $M^k - I_n$. Our strategy will be to use the Jordan decomposition of M to simplify the exponentiation.

Theorem 2. *The determinants of the spiral knots $S(n, k, \epsilon)$ with $n \leq 4$ are given by the following formulas:*

(i) $\det(S(2, k, (1))) = k$,

(ii) $\det(S(3, k, (1, 1))) = 2 - \frac{(1-i\sqrt{3})^k + (1+i\sqrt{3})^k}{2^k}$,

(iii) $\det(S(3, k, (1, -1))) = \frac{(3-\sqrt{5})^k + (3+\sqrt{5})^k}{2^k} - 2$,

(iv) $\det(S(4, k, (1, 1, 1))) = k\left(1 - \frac{i^k + i^{-k}}{2}\right)$,

(v) $\det(S(4, k, (1, 1, -1))) = k^3$,

(vi) $\det(S(4, k, (1, -1, 1))) = \frac{k((2-\sqrt{3})^k + (2+\sqrt{3})^k - 2)}{2}$.

Proof. We will sketch proofs of parts *(iii)* and *(v)* of Theorem 2; the remaining proofs are entirely similar.

Part *(iii)* gives yet another formula for the determinant sequence of the 3-strand weaving knots, which we proved at the end of Section 3 to be equal to $L_{2k} - 2$. We will use this simple example to illustrate our general method of proof. By the method outlined in Section 2, the base word matrix for $S(3, k, (1, -1))$ is

$$M = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{bmatrix}.$$

We are interested first in determining M^k, which we will do by finding the Jordan decomposition QJQ^{-1} and then calculating QJ^kQ^{-1}. The Jordan decomposition of M is

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3-\sqrt{5}}{2} & 0 \\ 0 & 0 & \frac{3+\sqrt{5}}{2} \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ 1 & 1 & 1 \end{bmatrix}.$$

In this case, J is a diagonal matrix and, thus, J^k is easy to find:

$$J^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\frac{3-\sqrt{5}}{2})^k & 0 \\ 0 & 0 & (\frac{3+\sqrt{5}}{2})^k \end{bmatrix}.$$

The matrix $M^k - I_3 = QJ^kQ^{-1} - I_3$ is unwieldy; however, it is a simple but tedious exercise to calculate and simplify the determinant of any of its minors to obtain

$$\det(S(3, k, (1, -1))) = \frac{(3-\sqrt{5})^k + (3+\sqrt{5})^k}{2^k} - 2,$$

as given in Theorem 2. Of course, if desired, we could also use algebra and the fact that

$$L_k = \frac{(1+\sqrt{5})^k + (1-\sqrt{5})^k}{2^k}$$

to show that this expression for $S(3, k, (1, -1))$ is also equal to $L_{2k} - 2$.

The proof of part *(v)* of Theorem 2 is similar. In this case, the base word matrix M for the 4-strand almost-torus knots $S(4, k, (1, 1, -1))$ is

$$M = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \end{bmatrix}.$$

Again, we want to find the determinant of the minors of $M^k - I_4$. The Jordan decomposition of M is $M = QJQ^{-1}$, where

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

It is a simple exercise to show by induction that

$$J^k = \begin{bmatrix} 1 & k & \frac{k(k-1)}{2} & \frac{k(k-1)(k-2)}{6} \\ 0 & 1 & k & \frac{k(k-1)}{2} \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A straightforward but tedious calculation shows that the matrix $M^k - I_4 = QJ^kQ^{-1} - I_4$ is equal to

$$\frac{k}{6} \begin{bmatrix} -k^2+7 & k^2-3k-4 & -k^2+6k-5 & k^2-3k+2 \\ -k^2+3k+10 & k^2-6k-1 & -k^2+9k-14 & k^2-6k+5 \\ -k^2+1 & k^2-3k+2 & -k^2+6k-11 & k^2-3k+8 \\ -k^2-3k-2 & k^2-1 & -k^2+3k-2 & k^2+5 \end{bmatrix}.$$

Remarkably, taking the determinant of any minor of this complicated matrix results in k^3, and we therefore conclude that

$$\det(S(4, k, (1, 1, -1))) = k^3. \qquad \square$$

5 Recursive formulas for spiral knot determinants

One of the notable aspects of the formulas in Theorem 2 is that they differ widely in both form and complexity. Although these formulas are interesting in their differences, what would be more interesting is if we had some kind of overarching formula or consistent pattern that depended on n, k, or ϵ in some obvious way. That is precisely what we will derive in the remainder of this paper, rewriting the results of Section 4 into a common form. Theorem 3 shows that all of the determinant sequences from Table 1 and Theorem 2 are related to a single recursive formula.

Theorem 3. *The determinants of the spiral knots $S(n, k, \epsilon)$ with $n \leq 4$ can be expressed in terms of sequences defined by the single linear homogeneous recurrence relation*

$$c_k = c_2\, c_{k-1} - c_{k-2}$$

with common initial condition $c_1 = 1$ and initial condition c_2 given in each case as follows:

(i) $\det(S(2, k, (1))) = k$, *with* $c_2 = \sqrt{4}$,

(ii) $\det(S(3, k, (1, 1))) = c_k^2$, *with* $c_2 = \sqrt{3}$,

(iii) $\det(S(3, k, (1, -1))) = c_k^2$, *with* $c_2 = \sqrt{5}$,

(iv) $\det(S(4, k, (1, 1, 1))) = kc_k^2$, *with* $c_2 = \sqrt{2}$,

(v) $\det(S(4, k, (1, 1, -1))) = kc_k^2$, *with* $c_2 = \sqrt{4}$,

(vi) $\det(S(4, k, (1, -1, 1))) = kc_k^2$, *with* $c_2 = \sqrt{6}$.

Proof. As in Theorem 2, we will prove parts *(iii)* and *(v)*. Part *(i)* is trivial, and parts *(ii)*, *(iv)*, and *(vi)* can be proven in a similar manner to part *(iii)*.

To prove part *(iii)* of Theorem 3, we start with part *(iii)* of Theorem 2:

$$\det(S(3, k, (1, -1))) = \frac{(3 - \sqrt{5})^k + (3 + \sqrt{5})^k}{2^k} - 2.$$

Since we already know that $\det(S(3, 1, (1, -1))) = 1$, and $\det(S(3, 2, (1, -1))) = 5$, it suffices to show that, for $c_1 = 1$ and $c_2 = \sqrt{5}$, the recurrence relation $c_k = \sqrt{5}c_{k-1} - c_{k-2}$ satisfies $c_k^2 = \frac{(3-\sqrt{5})^k+(3+\sqrt{5})^k}{2^k} - 2$.

First, suppose $c_k = r^k$ for $k \geq 1$. Then $r^k = \sqrt{5}r^{k-1} - r^{k-2}$, so $r^{k-2}(r^2 - \sqrt{5}r + 1) = 0$, which implies $r = \frac{\sqrt{5}\pm 1}{2}$. These two solutions for r both satisfy the recurrence relation and, thus, any linear combination of them does as well. Therefore, we have

$$c_k = a\left(\frac{\sqrt{5}+1}{2}\right)^k + b\left(\frac{\sqrt{5}-1}{2}\right)^k.$$

Using the fact that $c_1 = 1$ and $c_2 = \sqrt{5}$, we can easily find that $a = 1$ and $b = -1$. Therefore, $c_k = (\frac{\sqrt{5}+1}{2})^k - (\frac{\sqrt{5}-1}{2})^k$. Squaring both sides of the equation and simplifying yields the desired result.

The remaining part *(v)* is an easier case. We already know from Theorem 2 that $\det(S(4, k, (1, 1, -1))) = k^3$ and, clearly, we have $k^3 = k(k^2)$, so it suffices to show that $c_k = k$ can be written as a recurrence relation that abides by the form of Theorem 3. But if $c_k = k$, we have

$$c_k = k = 2(k-1) - (k-2) = 2c_{k-1} - c_{k-2} = \sqrt{4}\, c_{k-1} - c_{k-2},$$

with $c_1 = 1$ and $c_2 = 2 = \sqrt{4}$, as desired. $\square$

Looking forward to a more general formula, notice that, for a given n, the second initial condition c_2 in Theorem 3 seems to grow according to the number of changes in the ϵ-vector from negative to positive or vice-versa. For example, the 3-strand formula in part *(ii)* of Theorem 3 has $\epsilon = (1, 1)$ with no sign changes and $c_2 = \sqrt{3}$, while the 3-strand formula in part *(iii)* has $\epsilon = (1, -1)$ with one sign change and $c_2 = \sqrt{5}$. Similarly, the 4-strand formulas in the last three parts of Theorem 3 have zero, one, and two sign changes, respectively, and their c_2 values increase accordingly.

When examining spiral knots with $n = 5$ strands, we start to encounter more calculational complexity, as well as base word matrices whose characteristic polynomials do not split over the real numbers; however, we have some preliminary evidence that, for $n = 5$, we may be able to write spiral knot determinants as a product $c_k^2\, d_k^2$ of two squared terms of sequences defined by recurrence relations. In future work, we hope to build on these results to extend the patterns in Theorem 3 to obtain a general formula for the determinant of any spiral knot.

6 Acknowledgments

The authors would like to thank James Madison University for support of this project, as well as the reviewers, whose insightful comments greatly enhanced the presentation of this paper.

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2010 *Mathematics Subject Classification:* Primary 57M25; Secondary 11B37, 11B39, 15A21, 15B36.
Keywords: knot determinant, spiral knot, integer sequence, Lucas sequence, Jordan decomposition, recurrence relation.

(Concerned with sequences A000027, A000578, A001353, A001834, A001906, A002878, A004146, A005013, A006235, A007877, A098149, A108412, A121022, A131022, A131027, and A251610.)

Received March 10 2015; revised versions received October 18 2015; November 17 2015. Published in *Journal of Integer Sequences*, December 17 2015.

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