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Links Between Sums Over Paths in Bernoulli's Triangles and the Fibonacci Numbers

Denis Neiter and Amsha Proag
École Polytechnique
Route de Saclay
91128 Palaiseau
France
denis.neiter@polytechnique.org
amsha.proag@polytechnique.org

Abstract

We investigate paths in Bernoulli's triangles, and derive several relations linking the partial sums of binomial coefficients to the Fibonacci numbers.

1 Introduction

Binomial coefficients appear in many identities, some of which are closely connected to the Fibonacci sequence [1, 2]. Pascal's triangle has been explored for links to the Fibonacci sequence as well as to generalized sequences [3]. The partial sums of the binomial coefficients are less well known, although a number of identities have been found regarding sums of their powers [4, 5] and polynomials [6]. To add to the existing corpus, we review Bernoulli's second and third-order triangles for relations pertaining to sums of the binomial coefficients. We contribute several relations that link the Fibonacci numbers to binomial partial sums.

2 Notation, definitions and preliminary lemma

We let $\mathbb{N}^*$ denote the positive natural numbers, i.e., $\{1, 2, \ldots\}$ and we let $\mathbb{N}_0$ refer to $\mathbb{N}^* \cup \{0\}$. For $p, q \in \mathbb{Z}$, $p \leqslant q$, we let $[\![p, q]\!]$ denote the integers between p and q, i.e., $[\![p, q]\!] = \{k \in \mathbb{Z} \mid$

$p \leqslant k \leqslant q\}$. For $p \in \mathbb{Z}$ we also let p^+ denote $\frac{1}{2}(p+|p|)$.

We let $(F_n)_{n\in\mathbb{N}_0}$ denote the *Fibonacci numbers* (A000045 [7]).

Definition 1. Let $B_{n,k}$ denote the sum of the first k binomial coefficients, i.e.,

$$\forall n, k \in \mathbb{N}_0,\ B_{n,k} = \sum_{q=0}^{k} \binom{n}{q}.$$

Definition 1 implies that $\forall n \in \mathbb{N}_0, \forall k \geqslant 1,\ B_{n,k} = B_{n,k-1} + \binom{n}{k}$ and $\forall k \geqslant n,\ B_{n,k} = 2^n$. We can therefore derive the following recurrence relation:

$$\begin{aligned}
\forall n, k \geqslant 1,\ B_{n,k} &= \sum_{q=0}^{k} \binom{n}{q} \\
&= \sum_{q=0}^{k} \binom{n-1}{q} + \sum_{q=0}^{k} \binom{n-1}{q-1} \\
&= \sum_{q=0}^{k} \binom{n-1}{q} + \sum_{q=0}^{k-1} \binom{n-1}{q}, \\
B_{n,k} &= B_{n-1,k} + B_{n-1,k-1}. \qquad (1)
\end{aligned}$$

We recognize the structure of Pascal's rule for the binomial coefficients with the boundary values $B_{n,0} = 1$ and $B_{n,n} = \sum_{q=0}^{n} \binom{n}{q} = 2^n$.

Definition 2. Let *Bernoulli's triangle* denote the triangle formed by $(B_{n,k})_{n\in\mathbb{N}_0, k\in[\![0,n]\!]}$. Bernoulli's triangle is illustrated by Figure 1.

Definition 3. For $n, k \in \mathbb{N}_0$, let $B^{[1]}_{n,k} = \binom{n}{k}$. Then for every $m \geqslant 1$ we define

$$\forall n, k \in \mathbb{N}_0,\ B^{[m+1]}_{n,k} = \sum_{q=0}^{k} B^{[m]}_{n,q}.$$

For $m \geqslant 1$, let *Bernoulli's mth-order triangle* $\mathcal{B}^{[m]}$ denote the triangle formed by the family $\left(B^{[m]}_{n,k}\right)_{n\in\mathbb{N}_0, k\in[\![0,n]\!]}$.

The first and second-order triangles $\mathcal{B}^{[1]}$ and $\mathcal{B}^{[2]}$ correspond respectively to Pascal's triangle and Bernoulli's triangle. For $m \geqslant 1$, the elements of $\mathcal{B}^{[m]}$ verify the following recurrence relation:

$$\forall n \geqslant 2, \forall k \in [\![1, n-1]\!],\ B^{[m]}_{n,k} = B^{[m]}_{n-1,k} + B^{[m]}_{n-1,k-1}. \qquad (2)$$

The proof of Relation (2) is the same as that of Relation (1). The corresponding boundary values are $\forall n \in \mathbb{N}_0, \forall m \geqslant 1,\ B_{n,0}^{[m]} = 1$ and $\forall m \geqslant 2,\ B_{n,n}^{[m]} = \sum_{q=0}^{n} B_{n,q}^{[m-1]}$.

Definition 4. In Bernoulli's mth-order triangle $\mathcal{B}^{[m]}$, for $c, l \in \mathbb{Z},\ n_0 \in \mathbb{N}_0,\ k_0 \in [\![0, n_0]\!]$, let *the path following direction* (c, l) *from* (n_0, k_0) denote the sequence $\left(B_{n_0+kl,k_0-kc}^{[m]}\right)_{k\in\mathbb{N}_0}$ (where $l > 0$ corresponds to increasing row numbers from n_0, and $c < 0$ to increasing column numbers from k_0). For a given pair (n_0, k_0), $\left(B_{n_0+kl,k_0-kc}^{[m]}\right)_{k\in\mathbb{N}_0}$ contains a finite number of elements when either $l \leqslant 0$, or $l > 0$ and $c \notin [\![-l, 0]\!]$.

In this work, we focus our attention on sums of elements along the two types of paths defined below.

Definition 5. Let $m \in \mathbb{N}^*$, $n \in \mathbb{N}_0$.

(i) Let $S_n^{[m]}(c, l)$ denote the sum over the path following direction (c, l) $(c > 0,\ l < 0)$ from (n, n), i.e.,

$$S_n^{[m]}(c, l) = \sum_{k=0}^{\lfloor n/c \rfloor} B_{n+kl,n-kc}^{[m]}.$$

We shall also use

$$\bar{S}_n^{[m]}(c, l) = 2B_{n,n}^{[m]} - S_n^{[m]}(c, l) = B_{n,n}^{[m]} - \sum_{k=1}^{\lfloor n/c \rfloor} B_{n+kl,n-kc}^{[m]}.$$

n \ k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	2								
2	1	3	4							
3	1	4	7	8						
4	1	5	11	15	16					
5	1	6	16	26	31	32				
6	1	7	22	42	57	63	64			
7	1	8	29	64	99	120	127	128		
8	1	9	37	93	163	219	247	255	256	
9	1	10	46	130	256	382	466	502	511	512

Figure 1: Bernoulli's triangle (A008949 [7]): the element in row n and column k corresponds to $B_{n,k}$.

(ii) Let $T_n^{[m]}(c,l)$ denote the sum over the path following direction (c,l) $(c<0,\, l<0)$ from $(n,0)$, i.e.,

$$T_n^{[m]}(c,l) = \sum_{k=0}^{\lfloor -n/(c+l) \rfloor} B_{n+kl,-kc}^{[m]}.$$

Lemma 6. *Let* $u \in \mathbb{Z}^{\mathbb{N}_0}$ *and* $\forall n \in \mathbb{N}^*,\ v_n = u_n - 2u_{n-1}$.

Then $\forall n \in \mathbb{N}_0,\ u_n = 2^n u_0 + \sum_{k=1}^{n} 2^{n-k} v_k$.

Proof. From the definition of $(v_n)_{n\in\mathbb{N}^*}$, we have

$$\begin{aligned}
\sum_{k=1}^{n} 2^{n-k} v_k &= \sum_{k=1}^{n} 2^{n-k}(u_k - 2u_{k-1}) \\
&= \sum_{k=1}^{n} 2^{n-k} u_k - \sum_{k=1}^{n} 2^{n-k+1} u_{k-1} \\
&= \sum_{k=1}^{n} 2^{n-k} u_k - \sum_{k=0}^{n-1} 2^{n-k} u_k \\
&= u_n - 2^n u_0.
\end{aligned}$$

□

Applying Lemma 6 to $u_n = F_{n+2}$ (i.e., $v_n = -F_{n-1}$) or $u_n = F_{2n+4}$ (i.e., $v_n = F_{2n+1}$) yields the following relations for $n \in \mathbb{N}_0$:

$$F_{n+2} = 2^n - \sum_{k=1}^{n} 2^{n-k} F_{k-1}, \tag{3}$$

$$F_{2n+4} = 3 \cdot 2^n + \sum_{k=1}^{n} 2^{n-k} F_{2k+1}. \tag{4}$$

Note that Benjamin and Quinn have reported Relation (3) [8, Id. 10].

3 Bernoulli's second-order triangle

3.1 The path following direction $(2,-1)$ from (n,n)

We first consider the path following direction $(2,-1)$ from (n,n) in Bernoulli's triangle $\mathcal{B}^{[2]}$. As illustrated by Figure 2, the sequence $\left(\bar{S}_n^{[2]}(2,-1)\right)_{n\in\mathbb{N}_0}$ can be related to the Fibonacci sequence by $\bar{S}_n^{[2]}(2,-1) = F_{n+2}$. Given that $\bar{S}_n^{[2]}(2,-1) = 2B_{n,n} - S_n^{[2]}(2,-1)$, we find that $S_n^{[2]}(2,-1) = 2^{n+1} - F_{n+2}$, which we formally express as Theorem 7 below.

$$
\begin{array}{cccccccccc}
1 \\
1 & 2 \\
\boxed{1} & \boxed{3} & 4 \\
\boxed{1} & \boxed{4} & \boxed{7} & \boxed{8} \\
1 & 5 & \boxed{11} & \boxed{15} & \boxed{16} \\
1 & 6 & 16 & 26 & \boxed{31} & \boxed{32} \\
1 & 7 & 22 & 42 & 57 & 63 & \boxed{64} \\
1 & 8 & 29 & 64 & 99 & 120 & 127 & 128 \\
1 & 9 & 37 & 93 & 163 & 219 & 247 & 255 & 256 \\
1 & 10 & 46 & 130 & 256 & 382 & 466 & 502 & 511 & 512
\end{array}
\qquad
\begin{array}{l}
1 = 1 \\
2 = 2 \\
3 = 4 - 1 \\
\boxed{5 = 8 - 3} \\
\boxed{8 = 16 - 7 - 1} \\
\boxed{13 = 32 - 15 - 4} \\
\boxed{21 = 64 - 31 - 11 - 1} \\
34 = 128 - 63 - 26 - 5 \\
55 = 256 - 127 - 57 - 16 - 1 \\
89 = 512 - 255 - 120 - 42 - 6
\end{array}
$$

Figure 2: Four instances of the path following direction $(2,-1)$ from (n,n) in Bernoulli's triangle are highlighted in distinct colors. Each term of $\bar{S}_n^{[2]}(2,-1)$ corresponds to a Fibonacci number.

Theorem 7.

$$\forall n \in \mathbb{N}_0,\ \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{q=0}^{n-2k} \binom{n-k}{q} = 2^{n+1} - F_{n+2}. \tag{5}$$

Proof. We seek to prove that

$$\forall n \in \mathbb{N}_0,\ S_n^{[2]}(2,-1) = 2^{n+1} - F_{n+2}.$$

We have $S_0^{[2]}(2,-1) = 1 = 2^{0+1} - F_{0+2}$, so the formula is valid for $n = 0$.

Let $n \in \mathbb{N}^*$. Then $S_n^{[2]}(2,-1) - 2S_{n-1}^{[2]}(2,-1) = \sum_{k=0}^{\lfloor n/2 \rfloor} B_{n-k,n-2k} - 2 \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} B_{n-1-k,n-1-2k}$. Assume an even n, e.g., $n = 2s$. Then

$$
\begin{aligned}
S_n^{[2]}(2,-1) - 2S_{n-1}^{[2]}(2,-1) &= \sum_{k=0}^{s} B_{2s-k,2s-2k} - 2\sum_{k=0}^{s-1} B_{2s-1-k,2s-1-2k} \\
&= B_{s,0} + \sum_{k=0}^{s-1} (B_{2s-k,2s-2k} - 2B_{2s-1-k,2s-1-2k}).
\end{aligned}
$$

From Relation (1) and Definition 1, we have

$$\forall p, q \geqslant 1,\ B_{p,q} - 2B_{p-1,q-1} = B_{p-1,q} - B_{p-1,q-1} = \binom{p-1}{q}. \tag{6}$$

Therefore

$$\begin{aligned} S_n^{[2]}(2,-1) - 2S_{n-1}^{[2]}(2,-1) &= 1 + \sum_{k=0}^{s-1} \binom{2s-k-1}{2s-2k} \\ &= 1 + \sum_{k=0}^{s-1} \binom{2s-k-1}{k-1} \\ &= \sum_{k=0}^{s-1} \binom{2s-k-2}{k}. \end{aligned}$$

Given that $\forall n \in \mathbb{N}_0,\ \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F_{n+1}$ [8, Id. 4], we have

$$S_n^{[2]}(2,-1) - 2S_{n-1}^{[2]}(2,-1) = F_{(2s-2)+1} = F_{n-1}.$$

The above relation, illustrated by Figure 3, can be proven for odd values of n using the same method. Applying Lemma 6 to $u_n = S_n^{[2]}(2,-1)$ and using Relation (3), we obtain

$$\forall n \in \mathbb{N}_0,\ S_n^{[2]}(2,-1) = 2^n + \sum_{k=1}^{n} 2^{n-k} F_{k-1} = 2^{n+1} - F_{n+2}.$$

$\square$

1								$1 = 1$	
1	2							$2 = 2$	$= 2 \cdot 1 + F_0$
1	3	4						$5 = 4 + 1$	$= 2 \cdot 2 + F_1$
1	4	7	8					$11 = 8 + 3$	$= 2 \cdot 5 + F_2$
1	5	11	15	16				$24 = 16 + 7 + 1$	$= 2 \cdot 11 + F_3$
1	6	16	26	31	32			$51 = 32 + 15 + 4$	$= 2 \cdot 24 + F_4$
1	7	22	42	57	63	64		$107 = 64 + 31 + 11 + 1$	$= 2 \cdot 51 + F_5$
1	8	29	64	99	120	127	128	$222 = 128 + 63 + 26 + 5$	$= 2 \cdot 107 + F_6$

Figure 3: Fibonacci numbers resulting from $S_n^{[2]}(2,-1) - 2S_{n-1}^{[2]}(2,-1)$, forming A027934 [7].

3.2 The path following direction $(c, 1-c)$ from (n, n)

Definition 8. For $c \geqslant 2,\ n \geqslant 1$, let $\lambda_n(c) = S_n^{[2]}(c, 1-c) - 2S_{n-1}^{[2]}(c, 1-c)$.

The sequence $(\lambda_n(c))_{n\in\mathbb{N}^*}$ corresponds to A000930 for $c = 3$ (see Figure 4), A003269 for $c = 4$ and A003520 for $c = 5$ [7]. We observe that $(\lambda_n(c))_{n\in\mathbb{N}^*}$ satisfies the following linear recurrence relation.

Theorem 9.

$$\forall n \in [\![1, c-1]\!],\ \lambda_n(c) = 0;\ \lambda_c(c) = 1 \textit{ and } \forall n > c,\ \lambda_n(c) = \lambda_{n-1}(c) + \lambda_{n-c}(c).$$

The sequence $(\lambda_n(c))_{n\in\mathbb{N}^*}$ is a generalization the Fibonacci sequence, which corresponds to the $c = 2$ case (as shown above, $\forall n \in \mathbb{N}^*, \lambda_n(2) = F_{n-1}$).

Proof. Let $c \geqslant 2$, $n \geqslant 1$. From Definition 5, we have

$$\lambda_n(c) = \sum_{k=0}^{\lfloor n/c \rfloor} B_{n+k(1-c),n-kc} - 2 \sum_{k=0}^{\lfloor (n-1)/c \rfloor} B_{n-1+k(1-c),n-1-kc}.$$

We first ascertain the initial values of $(\lambda_n(c))_{n\in\mathbb{N}^*}$. For $n < c$, $\lfloor n/c \rfloor = \lfloor (n-1)/c \rfloor = 0$ and $\lambda_n(c) = B_{n,n} - 2B_{n-1,n-1} = 2^n - 2 \cdot 2^{n-1} = 0$. For $n = c$, $\lambda_c(c) = B_{c,c} + B_{1,0} - 2B_{c-1,c-1} = 2^c + 1 - 2 \cdot 2^{c-1} = 1$.

We then prove the recurrence relation for $n > c$. We have

$$\lambda_{n-1}(c) = \sum_{k=0}^{\lfloor (n-1)/c \rfloor} B_{n-1+k(1-c),n-1-kc} - 2 \sum_{k=0}^{\lfloor (n-2)/c \rfloor} B_{n-2+k(1-c),n-2-kc}$$

1								$1 = 1$	
$\boxed{1}$	2							$2 = 2$	$= 2 \cdot 1 + 0$
$\boxed{1}$	$\boxed{3}$	4						$4 = 4$	$= 2 \cdot 2 + 0$
1	4	$\boxed{7}$	$\boxed{8}$					$\boxed{9 = 8 + 1}$	$= 2 \cdot 4 + 1$
1	5	11	$\boxed{15}$	$\boxed{16}$				$\boxed{19 = 16 + 3}$	$= 2 \cdot 9 + 1$
1	6	16	26	31	$\boxed{32}$			$\boxed{39 = 32 + 7}$	$= 2 \cdot 19 + 1$
1	7	22	42	57	63	$\boxed{64}$		$\boxed{80 = 64 + 15 + 1}$	$= 2 \cdot 39 + 2$
1	8	29	64	99	120	127	128	$163 = 128 + 31 + 4$	$= 2 \cdot 80 + 3$

Figure 4: Instances of $S_n^{[2]}(3, -2)$ in $\mathcal{B}^{[2]}$ form A099568 [7] and the difference $S_n^{[2]}(3, -2) - 2S_{n-1}^{[2]}(3, -2)$ yields the terms of $(\lambda_n(3))_{n\in\mathbb{N}^*}$, corresponding to A000930 [7].

and

$$\lambda_{n-c}(c) = \sum_{k=0}^{\lfloor (n-c)/c \rfloor} B_{n-c+k(1-c),n-c-kc} - 2 \sum_{k=0}^{\lfloor (n-c-1)/c \rfloor} B_{n-c-1+k(1-c),n-c-1-kc}.$$

The summation upper bounds of $\lambda_n(c)$, $\lambda_{n-1}(c)$ and $\lambda_{n-c}(c)$ depend on the remainder of the Euclidean division of n by c. Let $n = cq + r > c$ with $q = \lfloor n/c \rfloor \in \mathbb{N}^*, r \in [\![0, c-1]\!]$. There are three cases to consider, which are shown in Table 1.

	$\lfloor n/c \rfloor$	$\lfloor (n-1)/c \rfloor$	$\lfloor (n-2)/c \rfloor$	$\lfloor (n-c)/c \rfloor$	$\lfloor (n-c-1)/c \rfloor$
$r=0$	q	$q-1$	$q-1$	$q-1$	$q-2$
$r=1$	q	q	$q-1$	$q-1$	$q-1$
$1<r<c$	q	q	q	$q-1$	$q-1$

Table 1: Summation upper bounds for the various values of $r = n - c\lfloor n/c \rfloor$.

For $r = 0$, we have

$$\begin{aligned} \lambda_n(c) &= \sum_{k=0}^{q} B_{cq+k(1-c),cq-kc} - 2\sum_{k=0}^{q-1} B_{cq-1+k(1-c),cq-1-kc} \\ &= B_{cq+q(1-c),cq-qc} + \sum_{k=0}^{q-1} \left(B_{cq+k(1-c),cq-kc} - 2B_{cq-1+k(1-c),cq-1-kc} \right), \end{aligned}$$

and Relation (6) yields

$$\begin{aligned} \lambda_n(c) &= B_{q,0} + \sum_{k=0}^{q-1} \binom{cq-1+k(1-c)}{cq-kc} \\ &= 1 + \sum_{k=0}^{q-1} \binom{cq-1+k(1-c)}{k-1} \\ &= 1 + \sum_{k=1}^{q-1} \binom{cq-1+k(1-c)}{k-1}. \end{aligned}$$

Similarly,

$$\begin{aligned} \lambda_{n-1}(c) &= \sum_{k=0}^{q-1} B_{cq-1+k(1-c),cq-1-kc} - 2\sum_{k=0}^{q-1} B_{cq-2+k(1-c),cq-2-kc} \\ &= \sum_{k=0}^{q-1} \binom{cq-2+k(1-c)}{k-1}, \end{aligned}$$

and

$$\begin{aligned}
\lambda_{n-c}(c) &= \sum_{k=0}^{q-1} B_{cq-c+k(1-c),cq-c-kc} - 2\sum_{k=0}^{q-2} B_{cq-c-1+k(1-c),cq-c-1-kc} \\
&= \underbrace{B_{cq-c+(q-1)(1-c),cq-c-(q-1)c}}_{B_{q-1,0}} + \sum_{k=0}^{q-2} \underbrace{(B_{cq-c+k(1-c),cq-c-kc} - 2B_{cq-c-1+k(1-c),cq-c-1-kc})}_{\binom{cq-c-1+k(1-c)}{cq-c-kc}} \\
&= 1 + \sum_{k=0}^{q-2} \binom{cq-c-1+k(1-c)}{k-1} \\
&= 1 + \sum_{k=1}^{q-1} \binom{cq-2+k(1-c)}{k-2}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\lambda_{n-1}(c) + \lambda_{n-c}(c) &= 0 + \sum_{k=1}^{q-1} \binom{cq-2+k(1-c)}{k-1} + \sum_{k=1}^{q-1} \binom{cq-2+k(1-c)}{k-2} + 1 \\
&= \sum_{k=1}^{q-1} \binom{cq-1+k(1-c)}{k-1} + 1 \\
&= \lambda_n(c).
\end{aligned}$$

This proves Theorem 9 for the $r = 0$ case. We can use the same method for $r = 1$ or $r > 1$, since only the summation upper bounds are modified in those cases. □

We can write the explicit expression of $\lambda_n(c)$ as $\lambda_n(c) = \sum_{i=0}^{\lfloor (n-c)/(c-1) \rfloor} \binom{n-c+i(1-c)}{i}$.

We obtain, using Lemma 6, $S_n^{[2]}(c, 1-c) = 2^n + \sum_{k=1}^{n} 2^{n-k}\lambda_k(c)$, which leads to the following relation.

Corollary 10.

$$\sum_{k=0}^{\lfloor n/c \rfloor} \sum_{q=0}^{n-kc} \binom{n-(c-1)k}{q} = 2^n + \sum_{k=1}^{n} 2^{n-k} \sum_{i=0}^{\lfloor (k-c)/(c-1) \rfloor} \binom{k-c-(c-1)i}{i}.$$

3.3 The path following direction $(-1,-1)$ from $(n, 0)$

The path following direction $(-1,-1)$ from $(n, 0)$ has a connection to the Fibonacci sequence that appears in the difference between successive terms of $T_n^{[2]}(-1,-1)$, as illustrated by Figure 5. Not only do we notice that

$$\forall p \geqslant 1,\ T_{2p}^{[2]}(-1,-1) = T_{2p-1}^{[2]}(-1,-1) + F_{2p+1},$$

but also that

$$\forall p \geqslant 1,\ T^{[2]}_{2p+1}(-1,-1) = T^{[2]}_{2p}(-1,-1) + T^{[2]}_{2p-1}(-1,-1).$$

Proof. To prove these two recurrence relations, we rely on Relation (1). For odd indices, we have

$$\begin{aligned}
\forall p \geqslant 1,\ T^{[2]}_{2p+1}(-1,-1) &= \sum_{k=0}^{p} B_{2p+1-k,k} \\
&= B_{2p+1,0} + \sum_{k=1}^{p} (B_{2p-k,k-1} + B_{2p-k,k}) \\
&= B_{2p,0} + \sum_{k=1}^{p} B_{2p-k,k} + \sum_{k=0}^{p-1} B_{2p-1-k,k} \\
&= \sum_{k=0}^{p} B_{2p-k,k} + \sum_{k=0}^{p-1} B_{2p-1-k,k} \\
&= T^{[2]}_{2p}(-1,-1) + T^{[2]}_{2p-1}(-1,-1).
\end{aligned}$$

For even indices,

$$\begin{aligned}
\forall p \geqslant 1,\ T^{[2]}_{2p}(-1,-1) &= \sum_{k=0}^{p} B_{2p-k,k} \\
&= \sum_{k=0}^{p} \sum_{q=0}^{k} \binom{2p-k}{q} \\
&= \sum_{k=0}^{p} \left(\sum_{q=0}^{k-1} \binom{2p-k}{q} + \binom{2p-k}{k} \right) \\
&= \sum_{k=0}^{p-1} \sum_{q=0}^{k} \binom{2p-1-k}{q} + \sum_{k=0}^{p} \binom{2p-k}{k}.
\end{aligned}$$

Since $\forall n \in \mathbb{N}_0,\ \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F_{n+1}$ [8, Id. 4], we have $\sum_{k=0}^{p} \binom{2p-k}{k} = F_{2p+1}$. Hence,

$$\forall p \geqslant 1,\ T^{[2]}_{2p}(-1,-1) = T^{[2]}_{2p-1}(-1,-1) + F_{2p+1}.$$

□

These two relations make it possible to derive a recurrence relation that pertains only to the odd subsequence:

$$\forall p \geqslant 1,\ T^{[2]}_{2p+1}(-1,-1) - 2T^{[2]}_{2p-1}(-1,-1) = F_{2p+1}.$$

1									$1=1$	
1	2								$1=1$	
1	$\boxed{3}$	$\boxed{4}$							$3=1+2$	$=1+F_3$
$\boxed{1}$	$\boxed{4}$	$\boxed{7}$	$\boxed{8}$						$\boxed{4=1+3}$	$=3+1$
$\boxed{1}$	$\boxed{5}$	$\boxed{11}$	15	16					$\boxed{9=1+4+4}$	$=4+F_5$
$\boxed{1}$	$\boxed{6}$	16	26	31	32				$\boxed{13=1+5+7}$	$=9+4$
$\boxed{1}$	7	22	42	57	63	64			$\boxed{26=1+6+11+8}$	$=13+F_7$
1	8	29	64	99	120	127	128		$39=1+7+16+15$	$=26+13$
1	9	37	93	163	219	247	255	256	$73=1+8+22+26+16$	$=39+F_9$

Figure 5: Instances of $T_n^{[2]}(-1,-1)$ in Bernoulli's triangle and differences from the preceding terms.

Therefore, using Lemma 6 and Relation (4), we have

$$\begin{aligned}\forall p \geqslant 1,\ T_{2p+1}^{[2]}(-1,-1) &= 2^p \underbrace{T_1^{[2]}(-1,-1)}_{1} + \sum_{k=1}^{p} 2^{p-k} F_{2k+1} \\ &= 2^p + F_{2p+4} - 3 \cdot 2^p \\ &= F_{2p+4} - 2^{p+1}.\end{aligned}$$

We feed back the odd subsequence into the expression for the even subsequence as follows:

$$\begin{aligned}\forall p \geqslant 1,\ T_{2p}^{[2]}(-1,-1) &= T_{2p-1}^{[2]}(-1,-1) + F_{2p+1} \\ &= F_{2p+2} - 2^p + F_{2p+1} \\ &= F_{2p+3} - 2^p.\end{aligned}$$

We have thus proven that $\forall n \in \mathbb{N}_0,\ T_n^{[2]}(-1,-1) = F_{n+3} - 2^{\lfloor (n+1)/2 \rfloor}$, which we can rewrite as the following theorem.

Theorem 11.

$$\forall n \in \mathbb{N}_0,\ \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{q=0}^{k} \binom{n-k}{q} = F_{n+3} - 2^{\lfloor (n+1)/2 \rfloor}.$$

Remark 12. Among the paths of Bernoulli's triangle following other directions, one can find other sequences that follow linear recurrence relations. For example, $\left(\bar{S}_n^{[2]}(3,-1)\right)_{n\in\mathbb{N}_0}$ corresponds to A005251 [7] and $\left(\bar{S}_n^{[2]}(4,-1)\right)_{n\in\mathbb{N}_0}$ to A138653 [7]. We believe that further sequences related to partial sums of binomial coefficients are yet to be uncovered in $\mathcal{B}^{[2]}$.

4 Bernoulli's third-order triangle

We now consider Bernoulli's third-order triangle $\mathcal{B}^{[3]}$ (see Figure 6). From Definition 3,

$$\forall n \in \mathbb{N}_0, k \in [\![0, n]\!],\ B_{n,k}^{[3]} = \sum_{q=0}^{k} B_{n,q} = \sum_{q=0}^{k}\sum_{r=0}^{q}\binom{n}{r}.$$

Recall that $B_{n,0}^{[3]} = 1$ and $\forall n \geqslant 2,\ \forall k \in [\![1, n-1]\!],\ B_{n,k}^{[3]} = B_{n-1,k}^{[3]} + B_{n-1,k-1}^{[3]}$. Furthermore, $B_{n,n}^{[3]} = (n+2)2^{n-1}$.

n \ k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	3								
2	1	4	8							
3	1	5	12	20						
4	1	6	17	32	48					
5	1	7	23	49	80	112				
6	1	8	30	72	129	192	256			
7	1	9	38	102	201	321	448	576		
8	1	10	47	140	303	522	769	1024	1280	
9	1	11	57	187	443	825	1291	1793	2304	2816

Figure 6: Bernoulli's third-order triangle (A193605 [7]): the element in row n and column k corresponds to $B_{n,k}^{[3]}$.

4.1 The path following direction $(2, -1)$ from (n, n)

Definition 5 states that $\bar{S}_n^{[3]}(2,-1) = B_{n,n}^{[3]} - \sum_{k=1}^{\lfloor n/2 \rfloor} B_{n-k,n-2k}^{[3]}$. Figure 7 suggests the following recurrence relation:

$$\bar{S}_n^{[3]}(2,-1) - 2\bar{S}_{n-1}^{[3]}(2,-1) = F_n. \tag{7}$$

Proof. Let $n \in \mathbb{N}^*$. We have

$$\bar{S}_n^{[3]}(2,-1) - 2\bar{S}_{n-1}^{[3]}(2,-1) = B_{n,n}^{[3]} - \sum_{k=1}^{\lfloor n/2 \rfloor} B_{n-k,n-2k}^{[3]} - 2B_{n-1,n-1}^{[3]} + 2\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} B_{n-1-k,n-1-2k}^{[3]}.$$

$$
\begin{array}{cccccccc}
1 & & & & & & & \\
1 & 3 & & & & & & \\
\boxed{1} & \boxed{4} & 8 & & & & & \\
\boxed{1} & \boxed{5} & \boxed{12} & \boxed{20} & & & & \\
1 & 6 & \boxed{17} & \boxed{32} & \boxed{48} & & & \\
1 & 7 & 23 & 49 & \boxed{80} & \boxed{112} & & \\
1 & 8 & 30 & 72 & 129 & 192 & \boxed{256} & \\
1 & 9 & 38 & 102 & 201 & 321 & 448 & 576
\end{array}
\qquad
\begin{array}{ll}
1 = 1 & \\
3 = 3 & = 2 \cdot 1 + F_1 \\
7 = 8 - 1 & = 2 \cdot 3 + F_2 \\
\boxed{16 = 20 - 4} & = 2 \cdot 7 + F_3 \\
\boxed{35 = 48 - 12 - 1} & = 2 \cdot 16 + F_4 \\
\boxed{75 = 112 - 32 - 5} & = 2 \cdot 35 + F_5 \\
\boxed{158 = 256 - 80 - 17 - 1} & = 2 \cdot 75 + F_6 \\
329 = 576 - 192 - 49 - 6 & = 2 \cdot 158 + F_7
\end{array}
$$

Figure 7: Instances of $\bar{S}_n^{[3]}(2,-1)$ in $\mathcal{B}^{[3]}$ that illustrate Relation (7).

Firstly, $B_{n,n}^{[3]} - 2B_{n-1,n-1}^{[3]} = (n+2)2^{n-1} - 2(n+1)2^{n-2} = 2^{n-1}$.

Let us assume that n is even, e.g., $n = 2s$ (the method is the same for odd values of n). We have

$$
\begin{aligned}
\bar{S}_n^{[3]}(2,-1) - 2\bar{S}_{n-1}^{[3]}(2,-1) &= 2^{2s-1} - \sum_{k=1}^{s} B_{2s-k,2s-2k}^{[3]} + 2\sum_{k=1}^{s-1} B_{2s-1-k,2s-1-2k}^{[3]} \\
&= 2^{2s-1} - \underbrace{B_{s,0}^{[3]}}_{1} - \sum_{k=1}^{s-1}\left(B_{2s-k,2s-2k}^{[3]} - 2B_{2s-1-k,2s-1-2k}^{[3]}\right).
\end{aligned}
$$

From Definition 3 and Relation (2), we have, for $m \geqslant 2$,

$$
\forall p, q \geqslant 1,\ B_{p,q}^{[m]} - 2B_{p-1,q-1}^{[m]} = B_{p-1,q}^{[m]} - B_{p-1,q-1}^{[m]} = B_{p-1,q}^{[m-1]},
$$

hence,

$$
\bar{S}_n^{[3]}(2,-1) - 2\bar{S}_{n-1}^{[3]}(2,-1) = 2^{2s-1} - 1 - \sum_{k=1}^{s-1} B_{2s-1-k,2s-2k}.
$$

Moreover,

$$\begin{aligned}
\sum_{k=1}^{s-1} B_{2s-1-k,2s-2k} &= \sum_{k=2}^{s} B_{2s-k,2s-2k+2} \\
&= \sum_{k=2}^{s} \left(B_{2s-k,2s-2k} + \binom{2s-k}{2s-2k+1} + \binom{2s-k}{2s-2k+2} \right) \\
&= \sum_{k=2}^{s} B_{2s-k,2s-2k} + \sum_{k=2}^{s} \binom{2s-k+1}{2s-2k+2} \\
&= S_{2s}^{[2]}(2,-1) - \underbrace{B_{2s,2s}}_{2^{2s}} - B_{2s-1,2s-2} + \sum_{k=2}^{s} \binom{2s-k+1}{k-1}.
\end{aligned}$$

Theorem 7 states that $S_{2s}^{[2]}(2,-1) = 2^{2s+1} - F_{2s+2}$.

We also have $B_{2s-1,2s-2} = B_{2s-1,2s-1} - \binom{2s-1}{2s-1} = 2^{2s-1} - 1$. Furthermore,

$$\begin{aligned}
\sum_{k=2}^{s} \binom{2s+1-k}{k-1} &= \sum_{k=1}^{s} \binom{2s-k+1}{k-1} - \binom{2s}{0} \\
&= \sum_{k=0}^{s-1} \binom{2s-k}{k} - 1 \\
&= \sum_{k=0}^{s} \binom{2s-k}{k} - \binom{s}{s} - 1 \\
&= F_{2s+1} - 2.
\end{aligned}$$

Finally,

$$\begin{aligned}
\bar{S}_{n}^{[3]}(2,-1) - 2\bar{S}_{n-1}^{[3]}(2,-1) &= 2^{2s-1} - 1 - \left(2^{2s+1} - F_{2s+2} - 2^{2s} - (2^{2s-1} - 1) + F_{2s+1} - 2\right) \\
&= F_{2s+2} - F_{2s+1} \\
&= F_{2s} \\
&= F_n.
\end{aligned}$$

$\square$

Lemma 6 and Relation (3) lead to an explicit expression of $\bar{S}_n^{[3]}(2,-1)$, as follows:

$$\begin{aligned}
\bar{S}_n^{[3]}(2,-1) &= 2^n \bar{S}_0^{[3]}(2,-1) + \sum_{k=1}^{n} 2^{n-k} F_k \\
&= 2^n + \sum_{k=1}^{n+1} 2^{n+1-k} F_{k-1} \\
&= 2^n + \left(2^{n+1} - F_{n+3}\right) \\
&= 3 \cdot 2^n - F_{n+3}.
\end{aligned}$$

From the explicit form of $\bar{S}_n^{[3]}(2,-1)$, we deduce that $S_n^{[3]}(2,-1) = 2B_{n,n}^{[3]} + F_{n+3} - 3 \cdot 2^n$ and obtain the following theorem.

Theorem 13.

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{q=0}^{n-2k} \sum_{r=0}^{q} \binom{n-k}{r} = F_{n+3} + (n-1)2^n.$$

4.2 The path following direction $(-1,-1)$ from $(n,0)$

In $\mathcal{B}^{[3]}$, the path following direction $(-1,-1)$ from $(n,0)$ does not appear at first glance to have any obvious connection to the Fibonacci sequence; however, the differences of consecutive terms of $T_n^{[3]}(-1,-1)$ follow a pattern similar to $T_n^{[2]}(-1,-1)$. Indeed, the odd subsequence of the difference sequence has the same behavior as in $\mathcal{B}^{[2]}$, while the even subsequence has a connection to the $T_n^{[2]}(-1,-1)$ numbers of even n (see Figure 8).

More precisely, we notice that $\forall p \geqslant 1,\ T_{2p}^{[3]}(-1,-1) = T_{2p-1}^{[3]}(-1,-1) + T_{2p}^{[2]}(-1,-1)$ and $T_{2p+1}^{[3]}(-1,-1) = T_{2p}^{[3]}(-1,-1) + T_{2p-1}^{[3]}(-1,-1)$.

Proof. We prove these two relations directly at order m, using Relation (2) and proceeding as in the derivation of Theorem 11 (section 3.3). For odd terms, we have

$$\begin{aligned}
\forall m \geqslant 2, \forall p \geqslant 1,\ T_{2p+1}^{[m]}(-1,-1) &= \sum_{k=0}^{p} B_{2p+1-k,k}^{[m]} \\
&= B_{2p+1,0}^{[m]} + \sum_{k=1}^{p} \left(B_{2p-k,k-1}^{[m]} + B_{2p-k,k}^{[m]}\right) \\
&= B_{2p,0}^{[m]} + \sum_{k=1}^{p} B_{2p-k,k}^{[m]} + \sum_{k=0}^{p-1} B_{2p-1-k,k}^{[m]} \\
&= \sum_{k=0}^{p} B_{2p-k,k}^{[m]} + \sum_{k=0}^{p-1} B_{2p-1-k,k}^{[m]}, \\
T_{2p+1}^{[m]}(-1,-1) &= T_{2p}^{[m]}(-1,-1) + T_{2p-1}^{[m]}(-1,-1). \qquad (8)
\end{aligned}$$

$$
\begin{array}{cccccccc|l|l}
1 & & & & & & & & 1 = 1 & \\
1 & 3 & & & & & & & 1 = 1 & \\
1 & \boxed{4} & \boxed{8} & & & & & & 4 = 1 + 3 & = 1 + T_2^{[2]}(-1,-1) \\
\boxed{1} & \boxed{5} & \boxed{12} & \boxed{20} & & & & & \boxed{5 = 1 + 4} & = 4 + 1 \\
\boxed{1} & \boxed{6} & \boxed{17} & 32 & 48 & & & & \boxed{14 = 1 + 5 + 8} & = 5 + T_4^{[2]}(-1,-1) \\
\boxed{1} & \boxed{7} & 23 & 49 & 80 & 112 & & & \boxed{19 = 1 + 6 + 12} & = 14 + 5 \\
\boxed{1} & 8 & 30 & 72 & 129 & 192 & 256 & & \boxed{45 = 1 + 7 + 17 + 20} & = 19 + T_6^{[2]}(-1,-1) \\
1 & 9 & 38 & 102 & 201 & 321 & 448 & 576 & 64 = 1 + 8 + 23 + 32 & = 45 + 19
\end{array}
$$

Figure 8: Differences between successive terms of $T_n^{[3]}(-1,-1)$ in Bernoulli's third-order triangle.

For even terms,

$$
\begin{aligned}
\forall m \geqslant 2, \forall p \geqslant 1,\ T_{2p}^{[m]}(-1,-1) &= \sum_{k=0}^{p} B_{2p-k,k}^{[m]} \\
&= \sum_{k=0}^{p} \sum_{q=0}^{k} B_{2p-k,q}^{[m-1]} \\
&= \sum_{k=0}^{p} \left(\sum_{q=0}^{k-1} B_{2p-k,q}^{[m-1]} + B_{2p-k,k}^{[m-1]} \right) \\
&= \sum_{k=0}^{p-1} \sum_{q=0}^{k} B_{2p-1-k,q}^{[m-1]} + \sum_{k=0}^{p} B_{2p-k,k}^{[m-1]} \\
&= \sum_{k=0}^{p-1} B_{2p-1-k,k}^{[m]} + T_{2p}^{[m-1]}(-1,-1), \\
T_{2p}^{[m]}(-1,-1) &= T_{2p-1}^{[m]}(-1,-1) + T_{2p}^{[m-1]}(-1,-1). \qquad (9)
\end{aligned}
$$

□

From Relations (8) and (9), we derive a general expression of $T_n^{[3]}(-1,-1)$ for $n \in \mathbb{N}_0$. Isolating the odd subsequence leads to

$$\forall p \geqslant 1,\ T_{2p+1}^{[3]}(-1,-1) - 2T_{2p-1}^{[3]}(-1,-1) = T_{2p}^{[2]}(-1,-1).$$

Therefore, using Lemma 6 and Relation (4), we obtain

$$\begin{aligned}
\forall p \geqslant 1,\ T_{2p+1}^{[3]}(-1,-1) &= 2^p \underbrace{T_1^{[3]}(-1,-1)}_{1} + \sum_{k=1}^{p} 2^{p-k} \underbrace{T_{2k}^{[2]}(-1,-1)}_{F_{2k+3}-2^k} \\
&= 2^p + \sum_{k=1}^{p} 2^{p-k} F_{2k+3} - \sum_{k=1}^{p} 2^p \\
&= 2^p + F_{2p+6} - 2^{p+3} - p2^p \\
&= F_{2p+6} - (p+7)2^p.
\end{aligned}$$

Hence,

$$\begin{aligned}
\forall p \geqslant 1,\ T_{2p}^{[3]}(-1,-1) &= T_{2p-1}^{[3]}(-1,-1) + T_{2p}^{[2]}(-1,-1) \\
&= F_{2p+4} - (p+6)2^{p-1} + F_{2p+3} - 2^p \\
&= F_{2p+5} - (p+8)2^{p-1}.
\end{aligned}$$

Finally, we reach the following relation:

$$\forall n \in \mathbb{N}_0,\ T_n^{[3]}(-1,-1) = F_{n+5} - 2^{\lfloor (n+1)/2 \rfloor} \left(\frac{1}{2} \lfloor (n+1)/2 \rfloor + \frac{7}{2} + \frac{(-1)^n}{2} \right).$$

Theorem 14.

$$\forall n \in \mathbb{N}_0,\ \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{q=0}^{k} \sum_{r=0}^{q} \binom{n-k}{r} = F_{n+5} - 2^p \left(\frac{1}{2}p + \frac{7}{2} + \frac{(-1)^n}{2} \right),$$

where $p = \lfloor (n+1)/2 \rfloor$.

Remark 15. Paths in $\mathcal{B}^{[3]}$ following other directions yield interesting sequences in a similar manner to those in $\mathcal{B}^{[2]}$. For example, $\left(\bar{S}_n^{[3]}(3,-1) - 2\bar{S}_{n-1}^{[3]}(3,-1) \right)_{n \in \mathbb{N}^*}$ corresponds to A005314 [7]. Further work should be able to uncover additional sequences by following other paths in $\mathcal{B}^{[3]}$.

5 Higher-order triangles

5.1 The path following direction $(-1,-1)$ from $(n,0)$

Using Relations (8) and (9) for $m = 4$ and $m = 5$, we proceed as we did for $T_n^{[3]}(-1,-1)$ to determine the sums over the path following direction $(-1,-1)$ from $(n,0)$ in $\mathcal{B}^{[4]}$ and $\mathcal{B}^{[5]}$.

We obtain the following identities:

$$\forall n \in \mathbb{N}_0,\ \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{q=0}^{k} \sum_{r=0}^{q} \sum_{s=0}^{r} \binom{n-k}{s} = F_{n+7} - 2^p \left(\frac{1}{8}p^2 + \frac{17}{8}p + 10 + (-1)^n \left(\frac{1}{4}p + 2 \right) \right),$$

$$\text{and } \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{q=0}^{k} \sum_{r=0}^{q} \sum_{s=0}^{r} \sum_{t=0}^{s} \binom{n-k}{t} = F_{n+9} - 2^p \left(\frac{1}{48}p^3 + \frac{5}{8}p^2 + \frac{317}{48}p + 27 + (-1)^n \left(\frac{1}{16}p^2 + \frac{19}{16}p + 6 \right) \right),$$

where $p = \left\lfloor \frac{n+1}{2} \right\rfloor$.

The expressions of $T_n^{[m]}(-1,-1)$ for $m \in [\![2,5]\!]$ contain a Fibonacci term and a second term composed of the product of a power of 2 by a polynomial, which suggests a general connection between $T_n^{[m]}(-1,-1)$ and the Fibonacci sequence. We write a general expression of $T_n^{[m]}(-1,-1)$ in the following theorem.

Theorem 16.

$$\forall m \in \mathbb{N}^*, \forall n \in \mathbb{N}_0,\ \sum_{i_1=0}^{\lfloor n/2 \rfloor} \sum_{i_2=0}^{i_1} \cdots \sum_{i_m=0}^{i_{m-1}} \binom{n-i_1}{i_m} = F_{n+2m-1} - 2^p \left(Q^{[m]}(p) + (-1)^n R^{[m]}(p) \right),$$

where $p = \left\lfloor \frac{n+1}{2} \right\rfloor$, $Q^{[m]}$ *and* $R^{[m]}$ *are polynomials with coefficients in* $\mathbb{Q}$ *and of degree* $(m-2)^+$ *and* $(m-3)^+$ *respectively.*

Proof. We seek to prove that $\forall m \geqslant 1$,

$$\forall n \in \mathbb{N}_0,\ T_n^{[m]}(-1,-1) = F_{n+2m-1} - 2^p \left(Q^{[m]}(p) + (-1)^n R^{[m]}(p) \right),$$

where $p = \left\lfloor \frac{n+1}{2} \right\rfloor$, $Q^{[m]}$ and $R^{[m]}$ are polynomials with coefficients in $\mathbb{Q}$ and of degree $(m-2)^+$ and $(m-3)^+$ respectively.

We proceed by induction on m. For $m = 1$, we have [8, Id. 4]

$$\forall n \in \mathbb{N}_0,\ T_n^{[1]}(-1,-1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F_{n+1},$$

which is in line with the above formulation with $Q^{[1]} = R^{[1]} = 0$.

Similarly, we already have the result for $m = 2$ with $Q^{[2]} = 1$ and $R^{[2]} = 0$ from Theorem 11:

$$\forall n \in \mathbb{N}_0,\ T_n^{[2]}(-1,-1) = F_{n+3} - 2^{\lfloor (n+1)/2 \rfloor}.$$

Assume that the result is true for a given $m \geqslant 2$, i.e., there exist two polynomials $Q^{[m]}$ and $R^{[m]}$ with coefficients in $\mathbb{Q}$ and respectively of degree $(m-2)^+$ and $(m-3)^+$, such that

$\forall n \in \mathbb{N}_0,\ T_n^{[m]}(-1,-1) = F_{n+2m-1} - 2^p \left(Q^{[m]}(p) + (-1)^n R^{[m]}(p)\right)$, where $p = \left\lfloor \frac{n+1}{2} \right\rfloor$. We will now prove the result for $m+1$.

First, from Relations (8) and (9) at order $m+1$, we have

$$\forall s \geqslant 1,\ T_{2s+1}^{[m+1]}(-1,-1) = 2T_{2s-1}^{[m+1]}(-1,-1) + T_{2s}^{[m]}(-1,-1).$$

Using Lemma 6 applied to $u_s = T_{2s+1}^{[m+1]}(-1,-1)$ (i.e., $v_s = T_{2s}^{[m]}(-1,-1)$), we obtain

$$\forall s \geqslant 1,\ T_{2s+1}^{[m+1]}(-1,-1) = 2^s \underbrace{T_1^{[m+1]}(-1,-1)}_{1} + \sum_{k=1}^{s} 2^{s-k} T_{2k}^{[m]}(-1,-1).$$

Since $T_1^{[m]}(-1,-1) = 1$, we may write that

$$\forall s \geqslant 1,\ T_{2s-1}^{[m+1]}(-1,-1) = 2^{s-1} + \sum_{k=1}^{s-1} 2^{s-1-k} T_{2k}^{[m]}(-1,-1).$$

Recall that $\forall k \in \mathbb{N}_0$, $T_{2k}^{[m]}(-1,-1) = F_{2k+2m-1} - 2^k \left(Q^{[m]}(k) + R^{[m]}(k)\right)$. Lemma 6 applied to $u_s = F_{2s+2m+2}$ (i.e., $v_s = F_{2s+2m-1}$) yields $\sum_{k=1}^{s} 2^{s-k} F_{2k+2m-1} = F_{2s+2m+2} - 2^s F_{2m+2}$, hence,

$$\sum_{k=1}^{s-1} 2^{s-1-k} F_{2k+2m-1} = \frac{1}{2}\left(F_{2s+2m+2} - 2^s F_{2m+2} - F_{2s+2m-1}\right) = F_{2s+2m} - 2^{s-1} F_{2m+2}.$$

Thus, we have

$$\begin{aligned}
\forall s \geqslant 1,\ T_{2s-1}^{[m+1]}(-1,-1) &= 2^{s-1} + \sum_{k=1}^{s-1} 2^{s-1-k} \left(F_{2k+2m-1} - 2^k \left(Q^{[m]}(k) + R^{[m]}(k)\right)\right) \\
&= 2^{s-1} + F_{2s+2m} - 2^{s-1} F_{2m+2} - 2^{s-1} \sum_{k=1}^{s-1} \left(Q^{[m]}(k) + R^{[m]}(k)\right), \\
T_{2s-1}^{[m+1]}(-1,-1) &= F_{(2s-1)+2(m+1)-1} - 2^{s-1}\left(F_{2m+2} - 1 + A(s)\right), \qquad (10)
\end{aligned}$$

where $A(X) = \sum_{k=1}^{X-1} \left(Q^{[m]}(k) + R^{[m]}(k)\right)$. For any $\alpha \in \mathbb{N}_0$, the polynomial $\sum_{k=1}^{X-1} k^\alpha$ is of degree $\alpha + 1$ in X. Therefore, since $Q^{[m]} + R^{[m]}$ is of degree $(m-2)^+$ and $m \geqslant 2$, A is of degree $(m-1)^+$.

Relation (10) gives the expression of $T_n^{[m+1]}(-1,-1)$ for all odd n. We may thus retrieve the even subsequence from Relation (9) to obtain, for $s \geqslant 1$,

$$\begin{aligned}
T_{2s}^{[m+1]}(-1,-1) &= T_{2s-1}^{[m+1]}(-1,-1) + T_{2s}^{[m]}(-1,-1) \\
&= F_{(2s-1)+2(m+1)-1} - 2^{s-1}\left(F_{2m+2} - 1 + A(s)\right) \\
&\quad + F_{2s+2m-1} - 2^s \left(Q^{[m]}(s) + R^{[m]}(s)\right), \\
T_{2s}^{[m+1]}(-1,-1) &= F_{2s+2(m+1)-1} - 2^{s-1}\left(F_{2m+2} - 1 + A(s) + 2Q^{[m]}(s) + 2R^{[m]}(s)\right). \qquad (11)
\end{aligned}$$

We introduce the polynomials $Q^{[m+1]}, R^{[m+1]}$ as follows:

$$Q^{[m+1]} = \frac{1}{2}\left(A + F_{2m+2} - 1 + Q^{[m]} + R^{[m]}\right) \text{ and } R^{[m+1]} = \frac{1}{2}\left(Q^{[m]} + R^{[m]}\right).$$

It is clear that $Q^{[m+1]}, R^{[m+1]} \in \mathbb{Q}[X]$. In addition, $Q^{[m+1]}$ is of the same degree as A, i.e., $(m-1)^+$ and $R^{[m+1]}$ is of the same degree as $Q^{[m]}$, i.e., $(m-2)^+$.

Therefore, we can synthesize Relations (10) and (11) as follows:

$$T_n^{[m+1]} = F_{n+2(m+1)-1} - 2^p\left(Q^{[m+1]}(p) + (-1)^n R^{[m+1]}(p)\right),$$

where $p = \left\lfloor \frac{n+1}{2} \right\rfloor$, $Q^{[m+1]}$ and $R^{[m+1]}$ are respectively $((m+1)-2)^+$th and $((m+1)-3)^+$th degree polynomials with coefficients in $\mathbb{Q}$. This concludes the proof by induction. $\square$

5.2 Towards additional formulae

The present work has reviewed Bernoulli's second and third-order triangles and found multiple connections to the Fibonacci sequence, expressed in Theorems 7 and 11 to 14. Theorem 16 generalizes Theorems 11 to 14 to higher orders. Furthermore, Theorem 9 uncovers a relation between partial sums of binomial coefficients in Bernoulli's triangle and sequences satisfying an additive recurrence relation that generalizes the Fibonacci sequence.

The methodology followed in this article (i.e., the study of paths over Bernoulli's triangles) reveals sequences that are formed by the partial sums of binomial coefficients. More generally, these paths could yield further fruit by mapping various functions f over the elements of Bernoulli's triangles in order to study the corresponding sequences $\sum_k f(B^{[m]}_{n_0+kl,k_0-kc})$.

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